Algorithms for Evolving Data Sets

Mohammad Mahdian

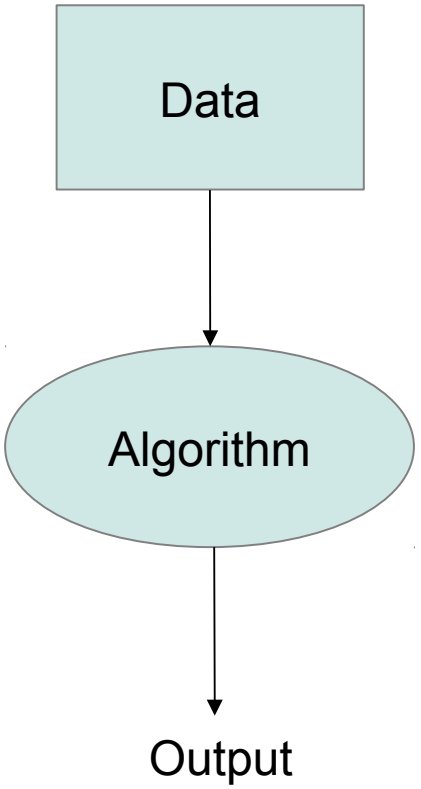
Google Research

Based on joint work with

Aris Anagnostopoulos, Bahman Bahmani, Ravi Kumar, Eli Upfal, and Fabio Vandin

Algorithm Design Paradigms

- Traditional paradigm:
  - stationary data set
  - algorithm has unrestricted access to data
- Alternative paradigms:
  - Online algorithms
    - Must make irrevocable decisions as data arrives
  - Streaming algorithms
    - Not enough space to store entire data set
  - Sublinear time algorithms
    - Not enough time to read entire data set
  - Algorithmic game theory, …
    - Feedback loop: choice of algo influences data



Evolving data: motivation

- Often data is a snapshot of the “nature”.
- The nature changes over time.
- Need to keep up with such changes by constantly observing the nature and adjusting the solution based on new observations.
- Example:
  - Computing PageRank, or other computations on the web graph
  - Polling public opinion
  - Finding paths to route traffic on a network

In this talk

- Define a general model for algorithm design on “evolving data”.
- Argue that the model is practically useful and mathematically interesting through three examples:
  - Sorting evolving data (ICALP 2009)
  - Basic graph algorithms (ITCS 2012)
  - PageRank computation (KDD 2012)

General Model

- At time t, real input U_t
- Need $v_t = f(U_t)$
- Input changes slowly stochastically (or adversarially):

$$d(U_{t+1}, U_t) = \text{small}$$

- Algorithm can make limited queries in each time step
- Must return approximate solution $\tilde{v}_t$
- Goal: Maintain $\tilde{v}_t \approx v_t$

Related Models

- Dynamic Data Structures
  - Similar models of gradual change
  - The algorithm immediately observes the change, has to update a data structure
  - Should be able to answer queries fast with the DS
- Property Testing
  - Solve a problem without reading the entire input

Sorting Dynamic Data

"Sort Me If You Can", Aris Anagnostopoulos, Ravi Kumar, Mohammad Mahdian and Eli Upfal, ICALP 2009.

- Want to keep track of a sorted list of objects, whose natural ordering changes over time.
- Can compare a pair of objects at a time.
- Motivated by applications in public opinion polling on websites like bix or youtube slam

bix a YAHOO! company

New User? Sign Up | Sign In

contests | help

search

Create, Enter, and Judge

what's Bix?

featured faceoff

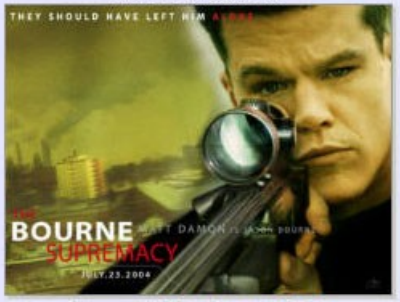
What's the best Trilogy ever made?

The Godfather Trilogy

pick The Godfather Trilogy by: rodypires

VS

The Bourne Trilogy



pick The Bourne Trilogy by: rodypires

» view contest

recent winners

2008 Hot Mom Contest

Proud Mom
by Julianna prada

The Spore Creature Contest - Finals

Tide Runner
by dolphinvisionsart

Cutest Dog

UNO
by mugupo

Top Summer Movie of 2008

The Dark Knight
by optimus

browse contests

- all categories
- a cappella
- art
- beauty
- comedy
- photography
- other

new contest

hottest debates

Top 10 Dogs Battle

Golden Retriever

VS

Bulldog

Who is the Sexiest Woman Alive?

Aishwarya Rai

VS

Jessica Alba

All Time Greatest Superhero

Superman

VS

Batman

popular faceoffs

Qual mulher brasileira é a mais bonita e atraente?

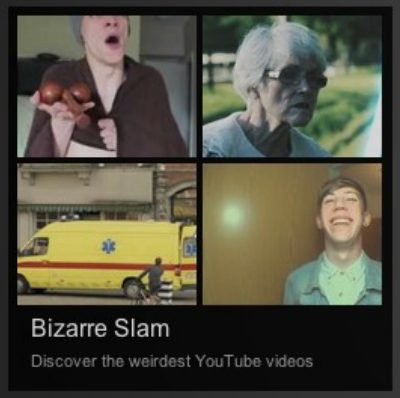
YouTube

Browse | Movies | Upload

Create Account | Sign In

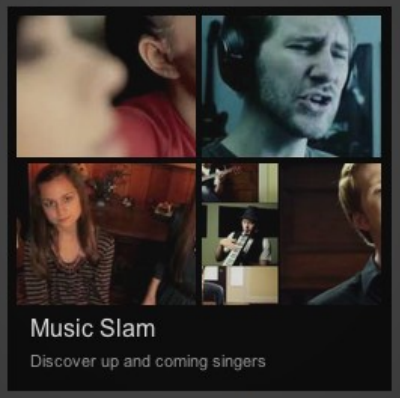
Feedback

Slam › Pick a game and start playing!



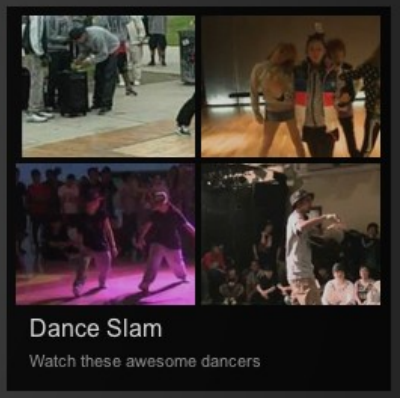
Bizarre Slam

Discover the weirdest YouTube videos



Music Slam

Discover up and coming singers



Dance Slam

Watch these awesome dancers

New to Slam?

This is the place for discovering talented amateur singers, the most adorable clips and the craziest videos on YouTube. Watch pairs of videos and vote for your favorite. Videos are scored based on your votes, and the best videos are featured on the slam leaderboard.

Start voting today and be the first to discover the best videos on YouTube!

Do you have an interesting idea for a Slam? Create one!

Slam Leaderboards

Bizarre Slam	View the winners »
Music Slam	View the winners »
Dance Slam	View the winners »
Cute Slam	View the winners »
Comedy Slam	View the winners »

Aggregating the public opinion

- Every time a user visits the site, she is asked to compare two options.
- Need to compute the aggregated “public opinion ranking” over time.
- The public opinion changes over time.
- Non-trivial, even assuming that each user correctly compares the given pair according to the public opinion.

Tracking the public opinion

Challenges:

- Public opinion changes over time
- limited access to public opinion through polling

Theoretical Problem:

- Maintain a sorted order of a set of elements
- True ordering changes slowly over time
- Objective: Maintain approximate order subject to bound on comparisons in every time step

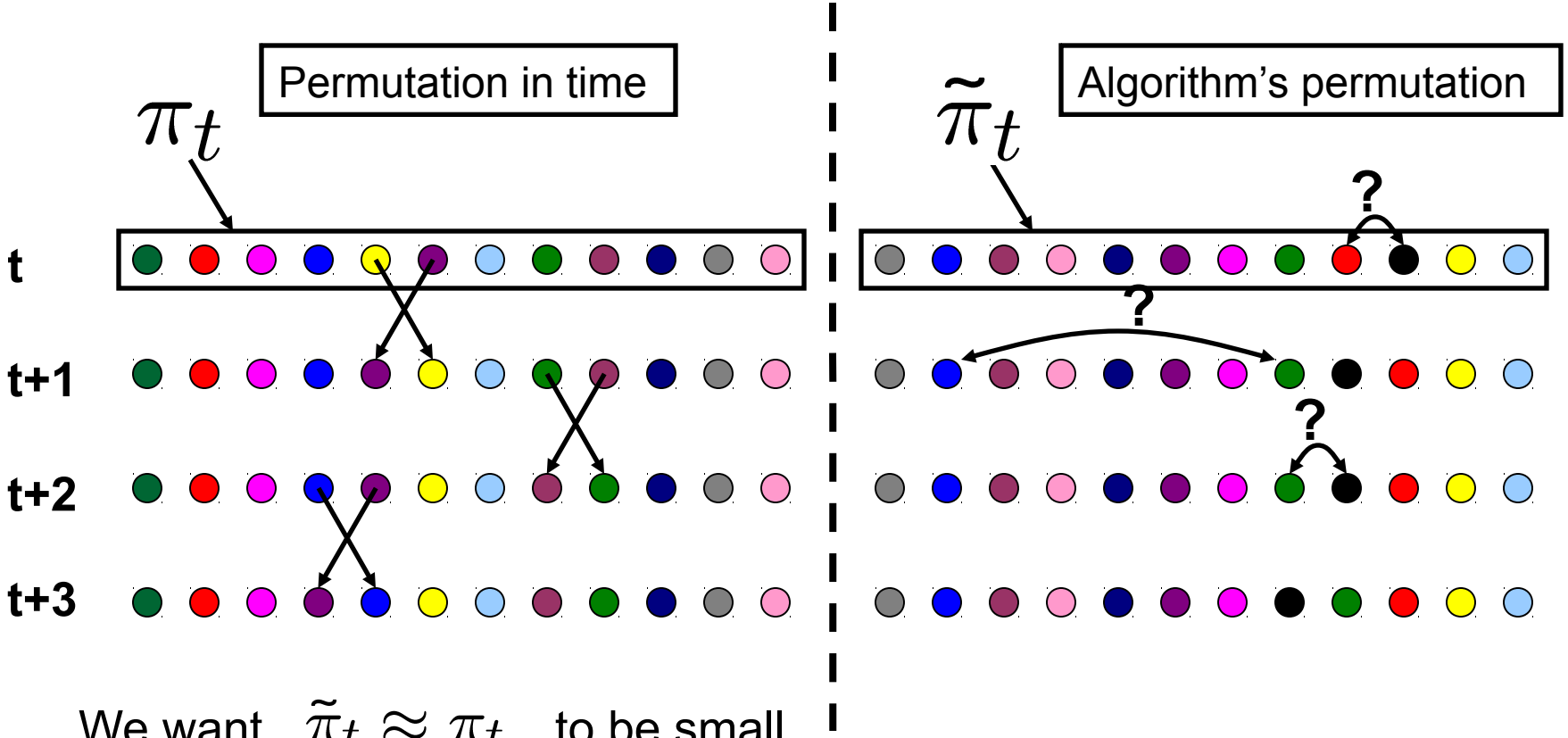
Stochastic Permutation Model

- Permutation of n elements evolving over time
- At time t, true permutation π_t
- At every time step a random **consecutive** pair swaps order
- **Goal:** Output a permutation $\tilde{\pi}_t \approx \pi_t$
- Algorithm can query one pair at every step

Kendall-Tau Distance

$$\begin{aligned} \mathrm{KT}(\pi_t, \tilde{\pi}_t) &= \#\ \text{incorrect pairs} = O(n^2) \\ &= \#\ \text{swaps of consecutive elements} \\ &\quad \text{for } \pi_t \to \tilde{\pi}_t \end{aligned}$$

Sorting Dynamic Data



We want $\tilde{\pi}_t \approx \pi_t$ to be small.

Kendall-Tau distance:

$$d(\pi_1, \tilde{\pi}_t) = \mathsf{KT}(\pi_t, \tilde{\pi}_t) = \#\ \text{incorrect pairs} = O(n^2)$$

Results

Sorting

- Lower bound: $\Omega(n)$
- Algorithm giving error: $O(n \ln \ln n)$
  - Based on a simpler algorithm giving error $O(n \ln n)$

Selection

- Algorithm returning element of rank $k + o(1)$

Lower Bound

Theorem

Any algorithm returns a permutation $\tilde{\pi}_t$ s.t.

$$\mathbf{E}[\mathsf{KT}(\pi_t, \tilde{\pi}_t)] = \Omega(n)$$

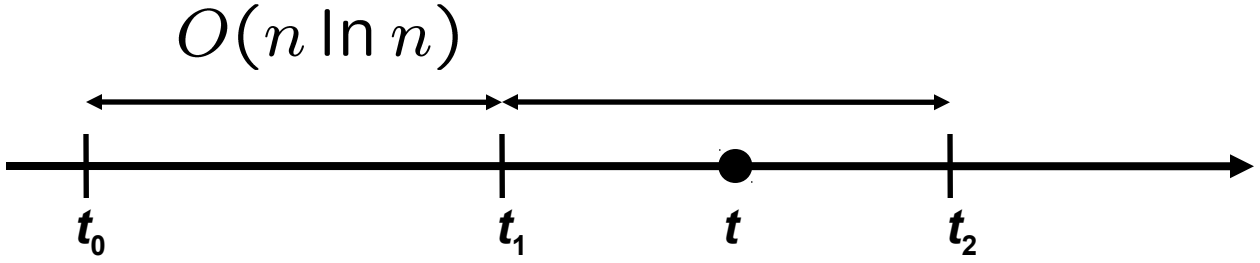
Proof idea

- Consider [*t* - *n*/8, *t*]
- We can query ≤ *n*/8 pairs = *n*/4 elements
- Those are adjacent to ≤ *n*/2 elements
- There are n/4 adjacent elements we know nothing about
- Each swaps with constant probability in [*t* - *n*/8, *t*]

O(n ln n) Algorithm

SimpleAlgorithm:

- Repeatedly run quicksort
- Return latest finished permutation $\tilde{\pi}_t = \tilde{\pi}_{t_1}$



Theorem. SimpleAlgorithm satisfies for all t :

$$\mathrm{KT}(\pi_t, \tilde{\pi}_t) = O(n \ln n).$$

Analysis

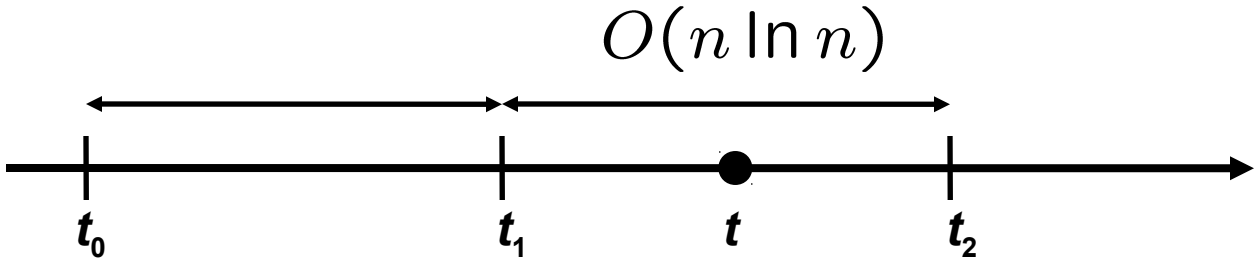
- Easy (wrong) proof: it takes O(n ln n) steps to sort, in each step at most one pair is swapped, so the distance between the permutations at the beginning and the end of each phase is at most O(n ln n).
- Wrong: the sorting algorithm needs to work with incorrect, sometimes even inconsistent data. This can create a cascading sequence of errors.
- Quicksort is special!

Quicksort - reminder

- Quicksort(A):
  - Pick a random element x of A as the “pivot”
  - Compare this element against other elements of A
  - Recursively sort elements that are less than x and those that are greater than x.

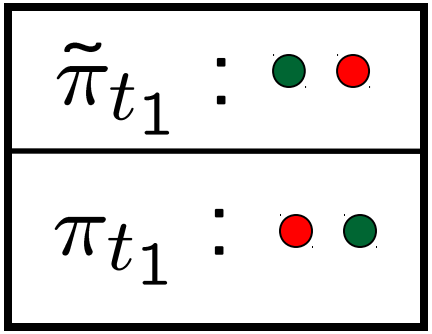
- A property of quicksort:
  - *if a is placed before b in the sorted order, either a is compared to b, or there’s an x such that a is compared to x and x is compared to b.*

Analysis



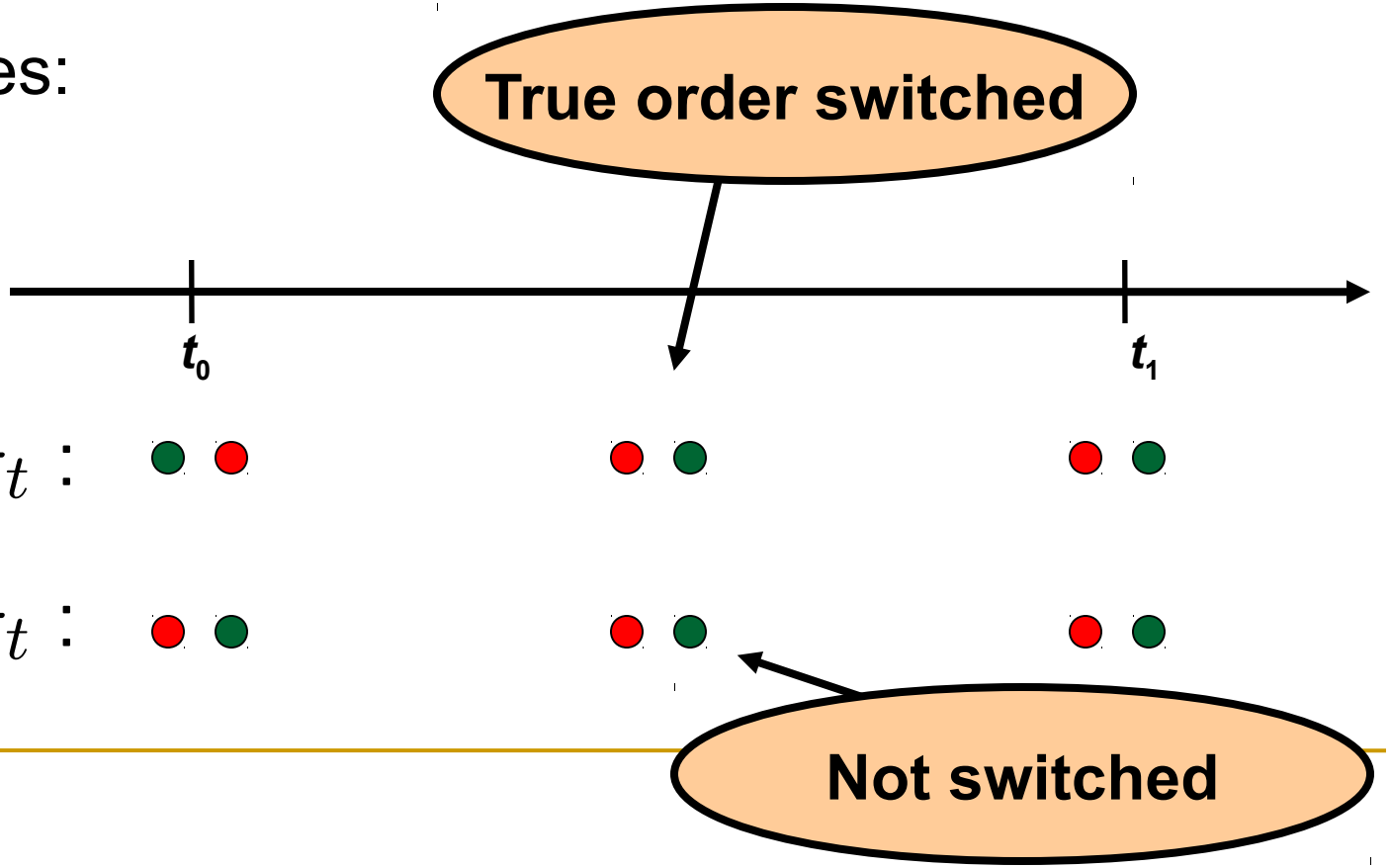
- Error: $\mathrm{KT}(\tilde{\pi}_t, \pi_t) = \mathrm{KT}(\tilde{\pi}_{t_1}, \pi_t)$
  $\leq \mathrm{KT}(\tilde{\pi}_{t_1}, \pi_{t_1}) + \mathrm{KT}(\pi_{t_1}, \pi_t)$
  $\leq \mathrm{KT}(\tilde{\pi}_{t_1}, \pi_{t_1}) + O(n \ln n)$
- Study error at t_1
- Error = # pairs where $\tilde{\pi}_{t_1}$:  π_{t_1} : 

Analysis

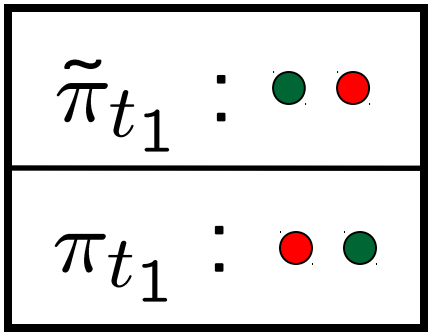


How did we end up with error?

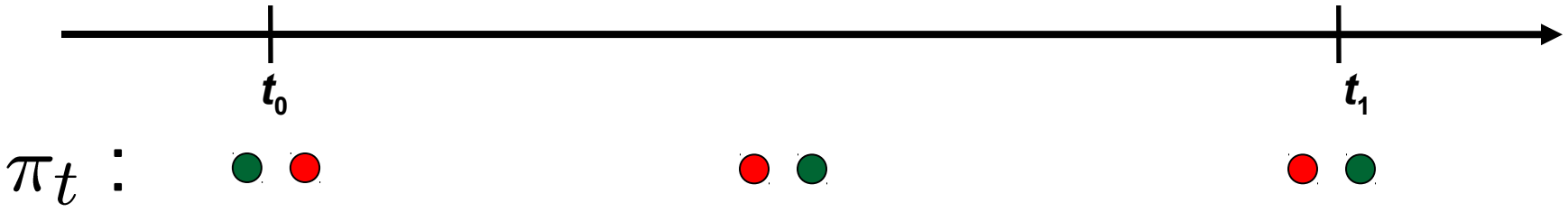
Two cases:



Analysis



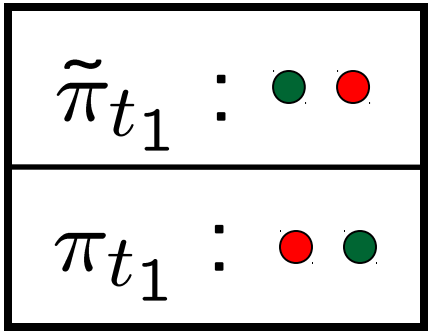
Case 1: True order switched



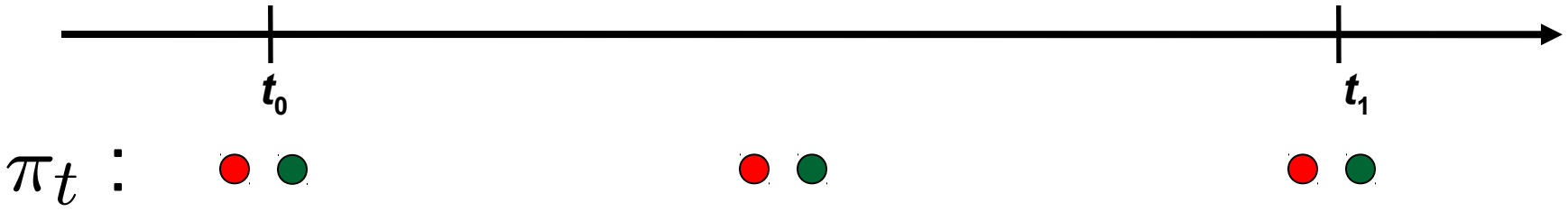
- Total steps in $[t_0, t_1]$ = O(n ln n)
- One pair swaps per step

Total Case-1 pairs = O(n ln n)

Analysis

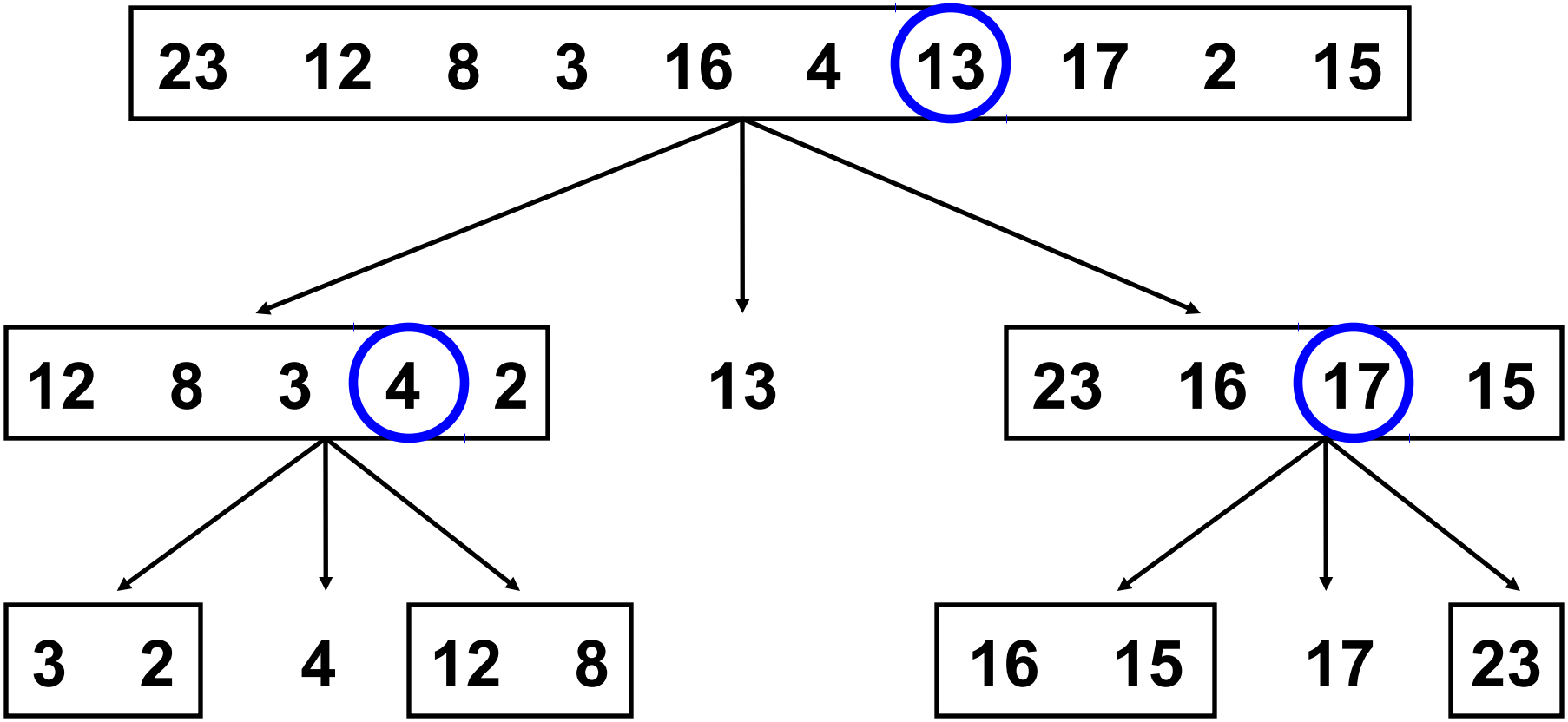


Case 2: True order never switched

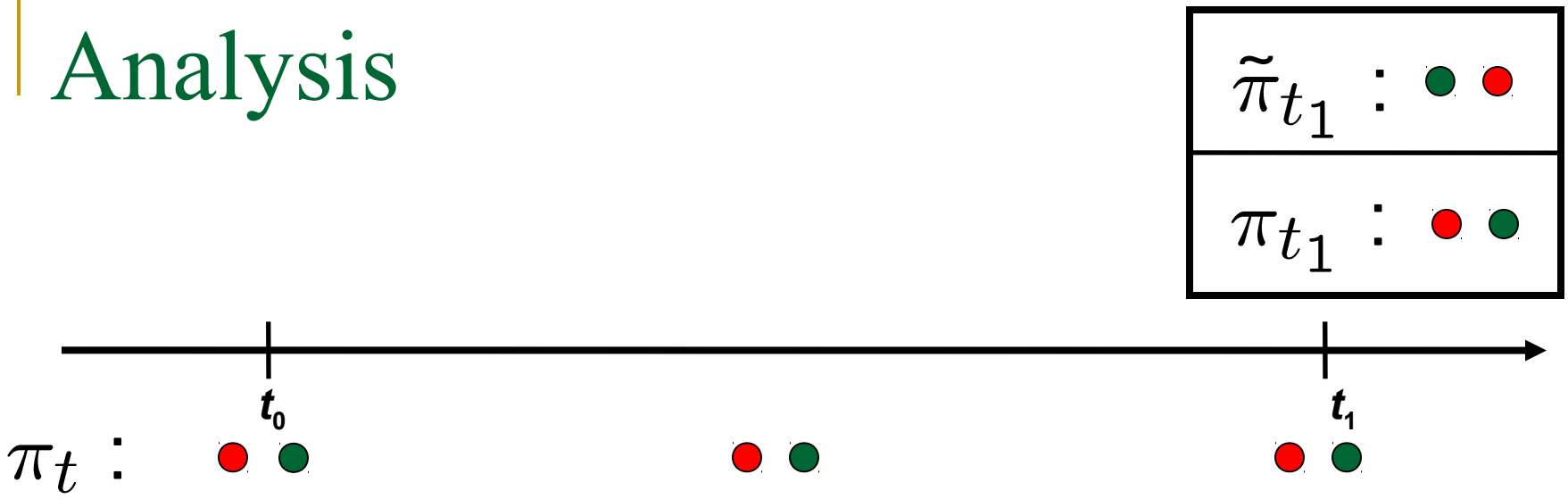


There is another (pivot) element ● that caused the error

Quicksort

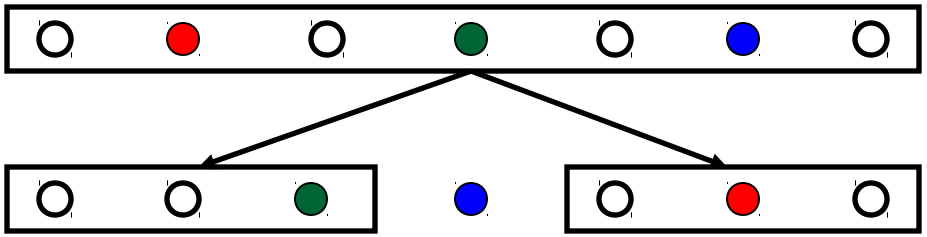


Analysis



There is a pivot element ● that caused the error

At some point, in true order:



● is pivot and we end up:

● was chosen to swap with **each** of the two elements ● , ●

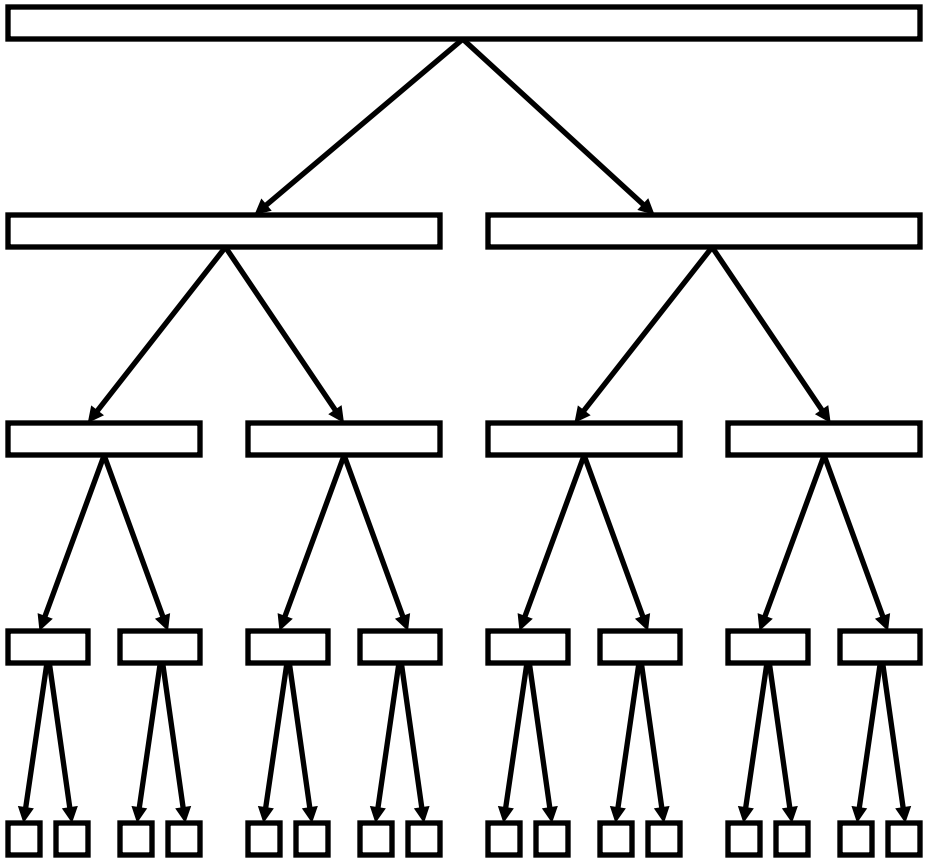
We charge the cost of the pair to the pivot

Analysis – Counting

$\tilde{\pi}_{t_1}$: ● ●

π_{t_1} : ● ●

Quicksort tree



E[pivot swaps]

$$cn \cdot \frac{2}{n} = 2c$$

$$2 \cdot c\frac{n}{2} \cdot \frac{2}{n} = 2c$$

$$4 \cdot c\frac{n}{4} \cdot \frac{2}{n} = 2c$$

E[pairs]

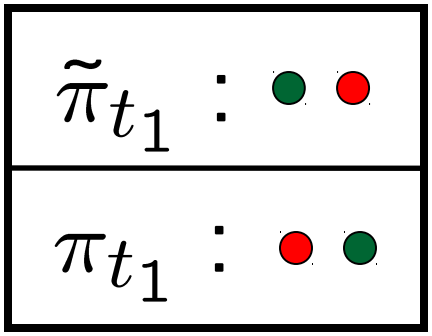
$$\binom{2c}{2} < (2c)^2$$

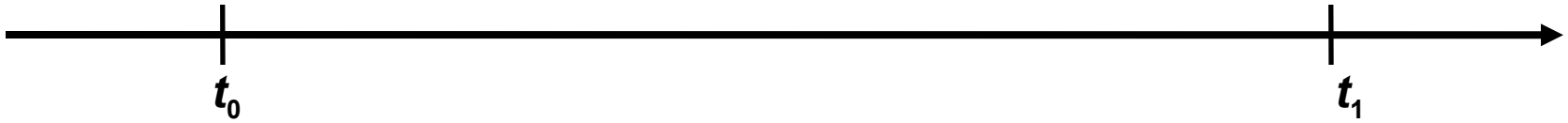
$$(2c)^2$$

$$(2c)^2$$

 # pairs = O(ln *n*)

Putting Together



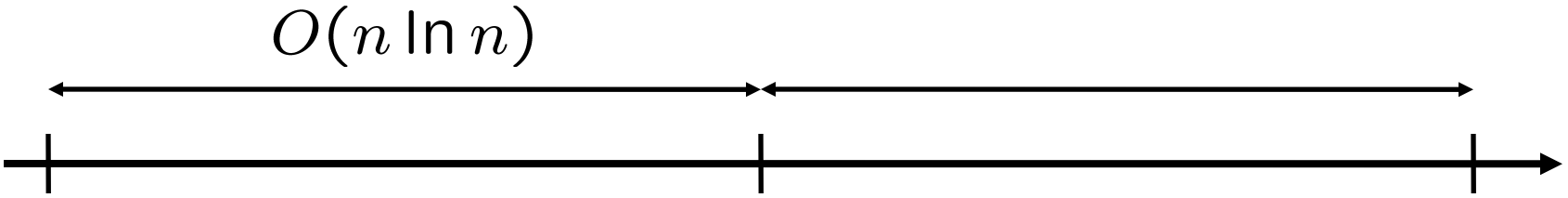


$$\mathrm{KT}(\pi_{t_1}, \tilde{\pi}_{t_1}) = \#\ \text{incorrect pairs}$$

- Case 1: True order has switched – O(*n* ln *n*)
- Case 2: True order not switched – O(ln *n*)

Total = O(*n* ln *n*)

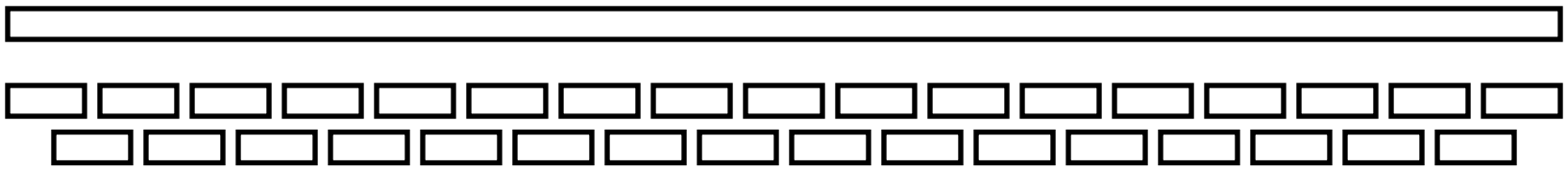
O(n ln ln n) Algorithm



- Quicksort runtime = O(n ln n) error = O(n ln n)
- No sorting algorithm can sort an arbitrary array with a runtime o(n ln n).
- However, at the end of Quicksort, *each* element is only O(ln n) from its correct rank.
- Such “almost sorted” arrays can be sorted faster!

Sorting for the almost-sorted

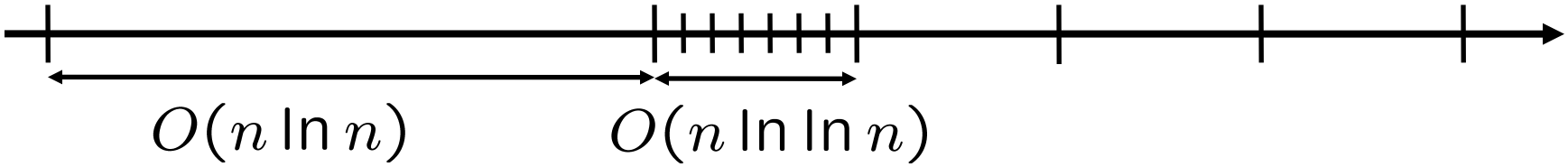
- Assume each element is within ln(n) of its correct rank.
- Divide the array into n/ln(n) blocks of length ln(n).
- Run Quicksort on each block, and also on blocks shifted by ln(n)/2 positions:



- Running time: $\frac{n}{\ln n} \times O(\ln n \cdot \ln \ln n) = O(n \ln \ln n)$
- What remains:
  1. analyzing this algorithm in the dynamic model
  2. Dealing with *accumulating errors*

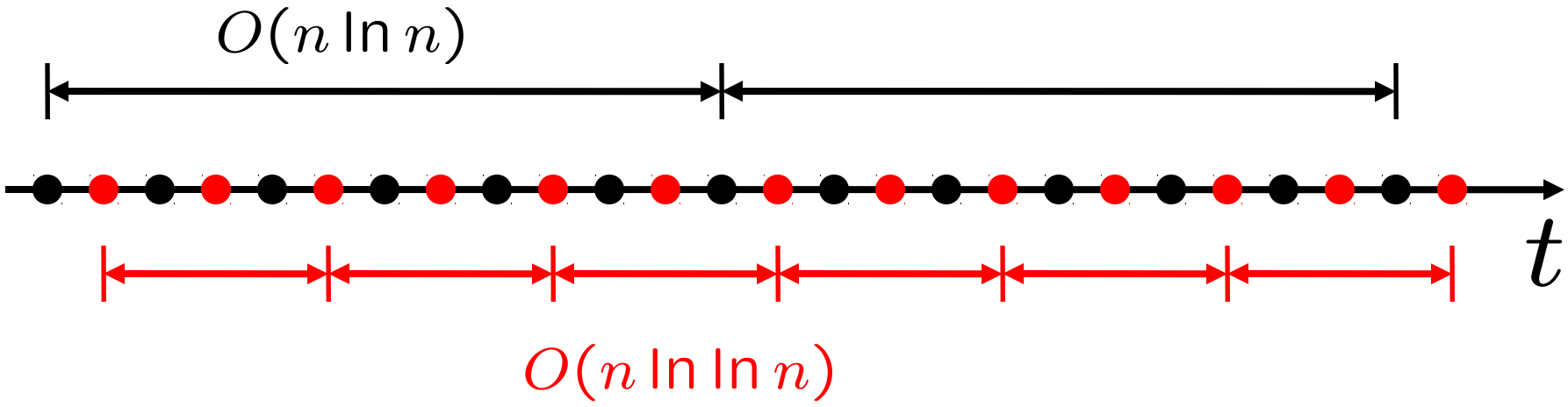
Dealing with Time

- Ideally we run a global quicksort and then a series of small quicksorts one after another:



- Eventually elements will drift away so we reset with a global quicksort
- But while running it error becomes O(*n* ln *n*)
- **Trick:** Execute both independently in parallel
  - **Odd steps:** Regular quicksort
  - **Even steps:** Series of small quicksorts

Parallel Execution



- The output of the algorithm is always the output of the O(n ln ln n) sort.
- The output of the O(n ln n) sort is used as the input to the faster sort.

Sorting – Recap

Model

- Real permutation π_t swaps a random consecutive pair each time step
- Algorithm can query 1 pair in every step
- Returns a permutation $\tilde{\pi}_t$ close to π_t
- Kendall tau distance: $\mathrm{KT}(\pi_t, \tilde{\pi}_t) = \#\ \text{incorrect pairs} = O(n^2)$

Results

- Lower bound: $\mathrm{KT}(\pi_t, \tilde{\pi}_t) = \Omega(n)$
- Simple algorithm: $\mathrm{KT}(\pi_t, \tilde{\pi}_t) = O(n \ln n)$
- More complicated algorithm: $\mathrm{KT}(\pi_t, \tilde{\pi}_t) = O(n \ln \ln n)$

Finding Element at Rank k

Same model

- Real permutation π_t swaps a random pair each time step
- Algorithm can query 1 pair in every step
- **Goal:** Return an element **e** and minimize $|\mathrm{rank}(e) - k|$

Results

- The Sorting algorithm gives a bound of O(ln ln n).
- Special case k = 1 (finding minimum):
  - Simpler algorithm: compare min with a random element and replace if that element is smaller
  - Defines a Markov chain on the rank of the output. Simple MC analysis shows rank is at most 2 in exp.

Finding Element at Rank k

- Algorithm with: $|\mathrm{rank}(e) - k| = o(1)$
- Based on the Motwani-Raghavan median algorithm:
  - R = n/ln(n) random elements
  - Quicksort(R).
  - C = elements between |R|/2 – $n^{1/2}$'th and |R|/2 + $n^{1/2}$'th element of R
  - Quicksort(C). Median is the L'th element of C, for some L.
- This can be adapted to the dynamic setting using the odd-even time steps trick:
  - In odd steps, sort R and compute C and L
  - In even steps, continuously sort C.

Algorithms on Evolving Graphs

- Model:
  - Input: graph G with n vertices and m edges
  - Change: in each step,
    - a random edge of G is removed, and
    - an edge is added between a random pair of vertices
  - Query: can query the neighborhood of a vertex
- Problem:
  - Maintain a path between two given nodes u and v, such that the probability that the path is invalid at any point is small.

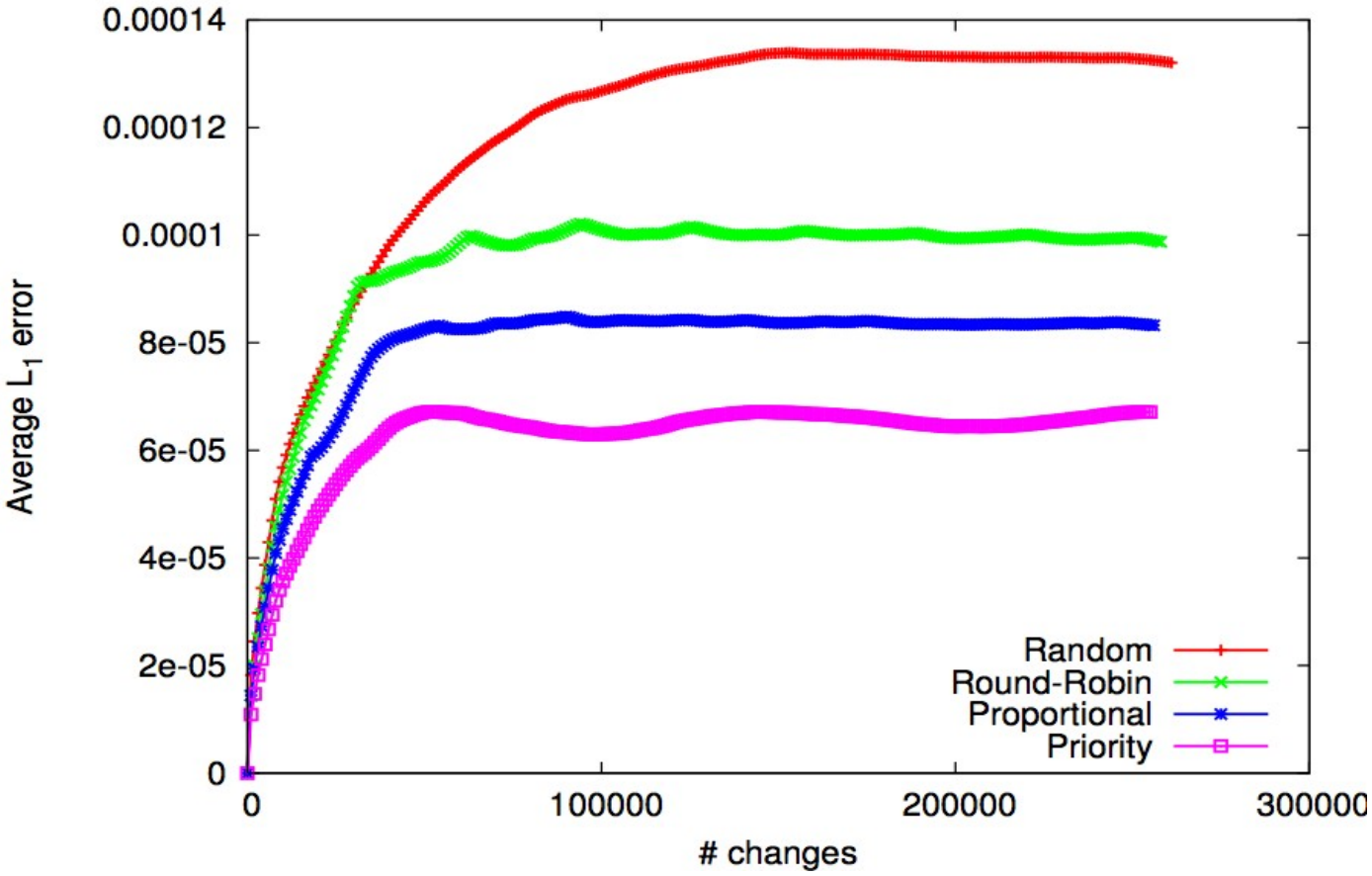
Algorithms on Evolving Graphs

- It is possible to achieve an error probability of O(log n / n).
- Almost matching lower bound, within a factor of (log log n)^2.
- Also, minimum spanning tree and page rank.

Evolving PageRank

- Change model: pick a random edge, move its head to a new vertex, chosen with probability proportional to current PR.
- Probe model: probe a node, see all outgoing links.
- Want a vector with small l_1 dist to true PR.
- Result: can get O(1/m) using Proportional Probing.

Experimental evaluation



Conclusion

- Evolving data sets is an interesting and useful model of computation.
- Open problems:
  - Finding an O(*n*) algorithm for sorting
    - Conjecture: the randomized algorithm that compares a *random* consecutive pair at each time step achieves this bound.
  - Other problems:
    - clustering/community finding in social networks
  - Imposing continuity constraints on the output

Thanks!