Impromptu Updating of MST and ST in a Distributed Dynamic Graph

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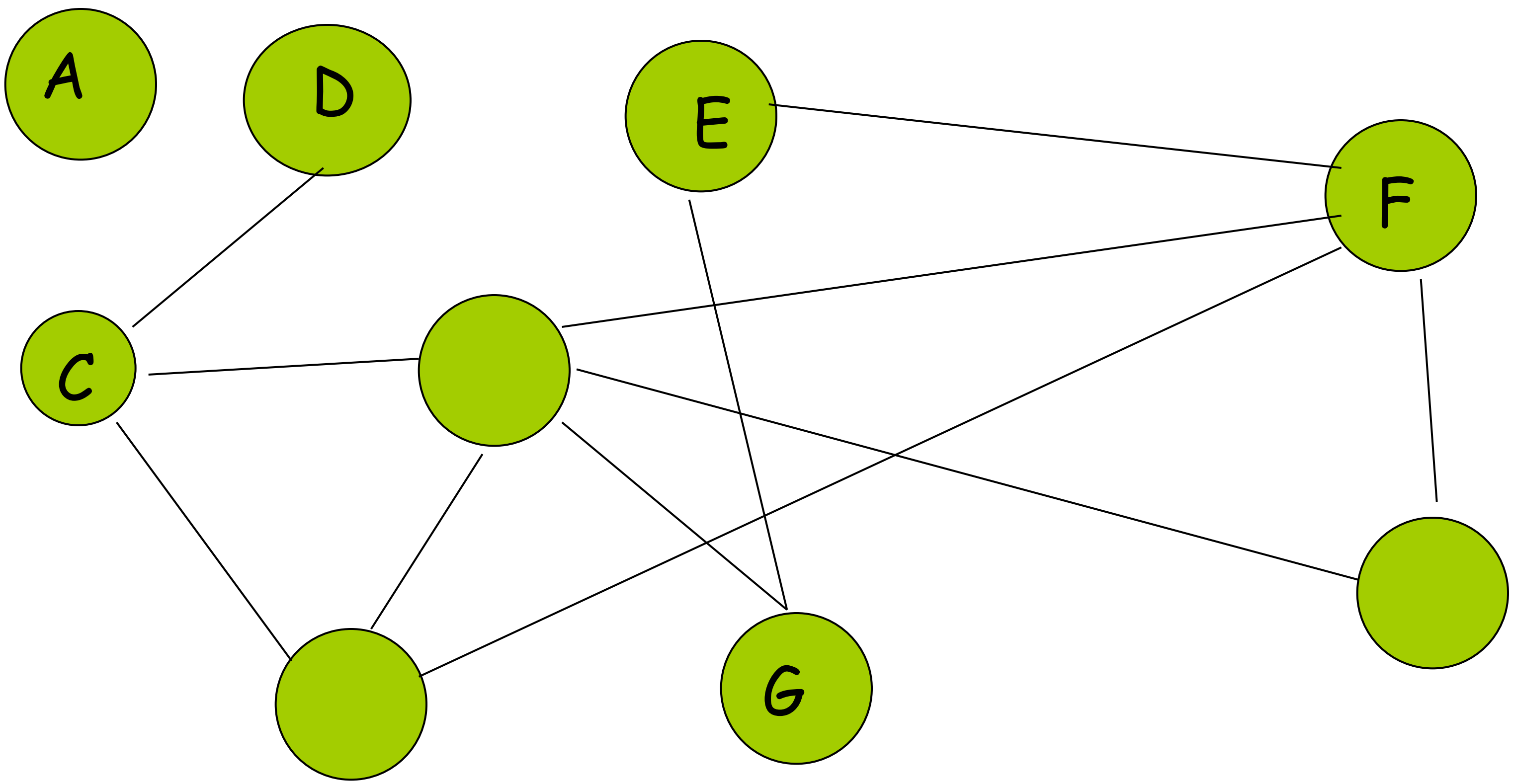
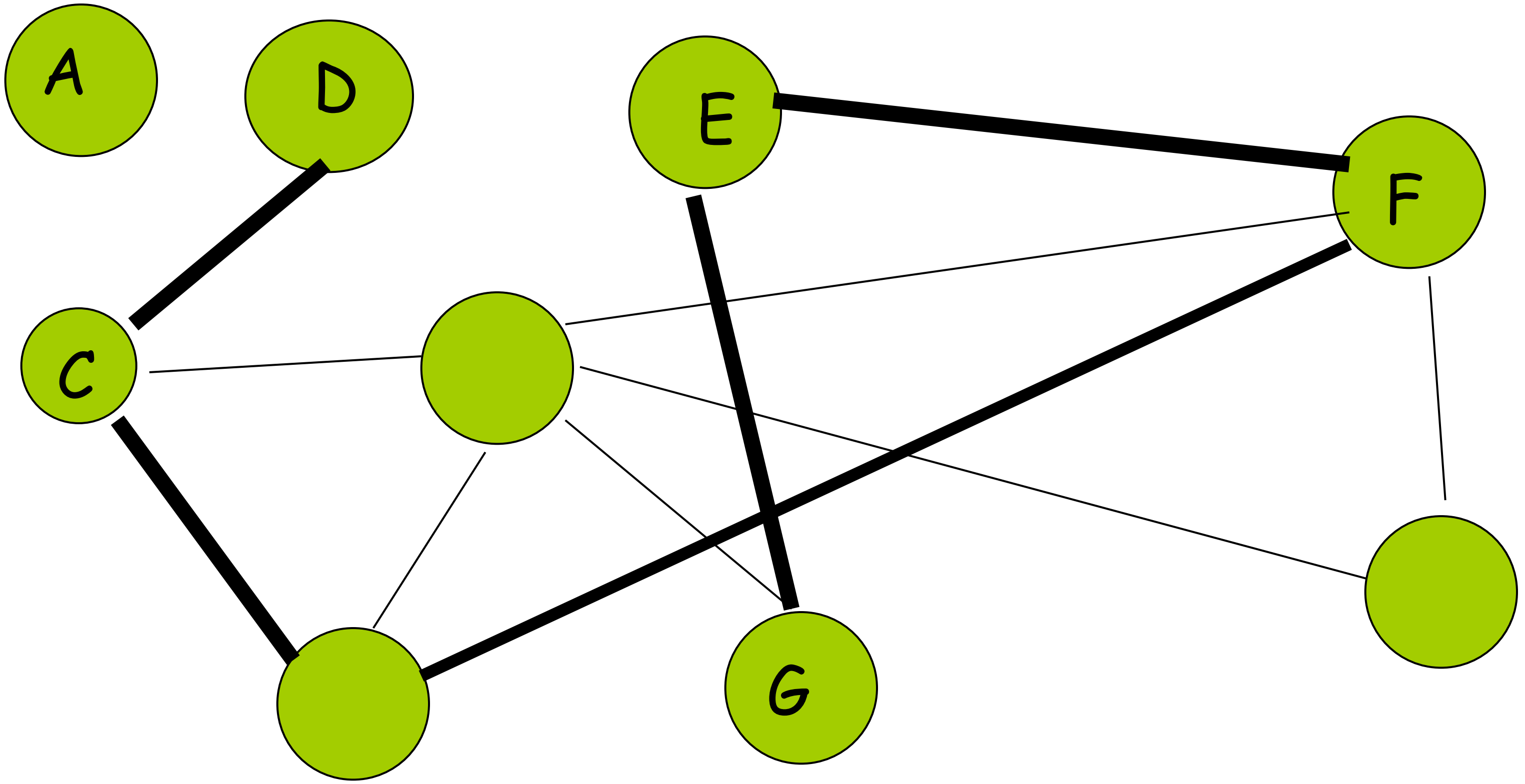
Network with n nodes, m edges
Each node has list of incident edges and edge weights. Nodes have distinct IDs

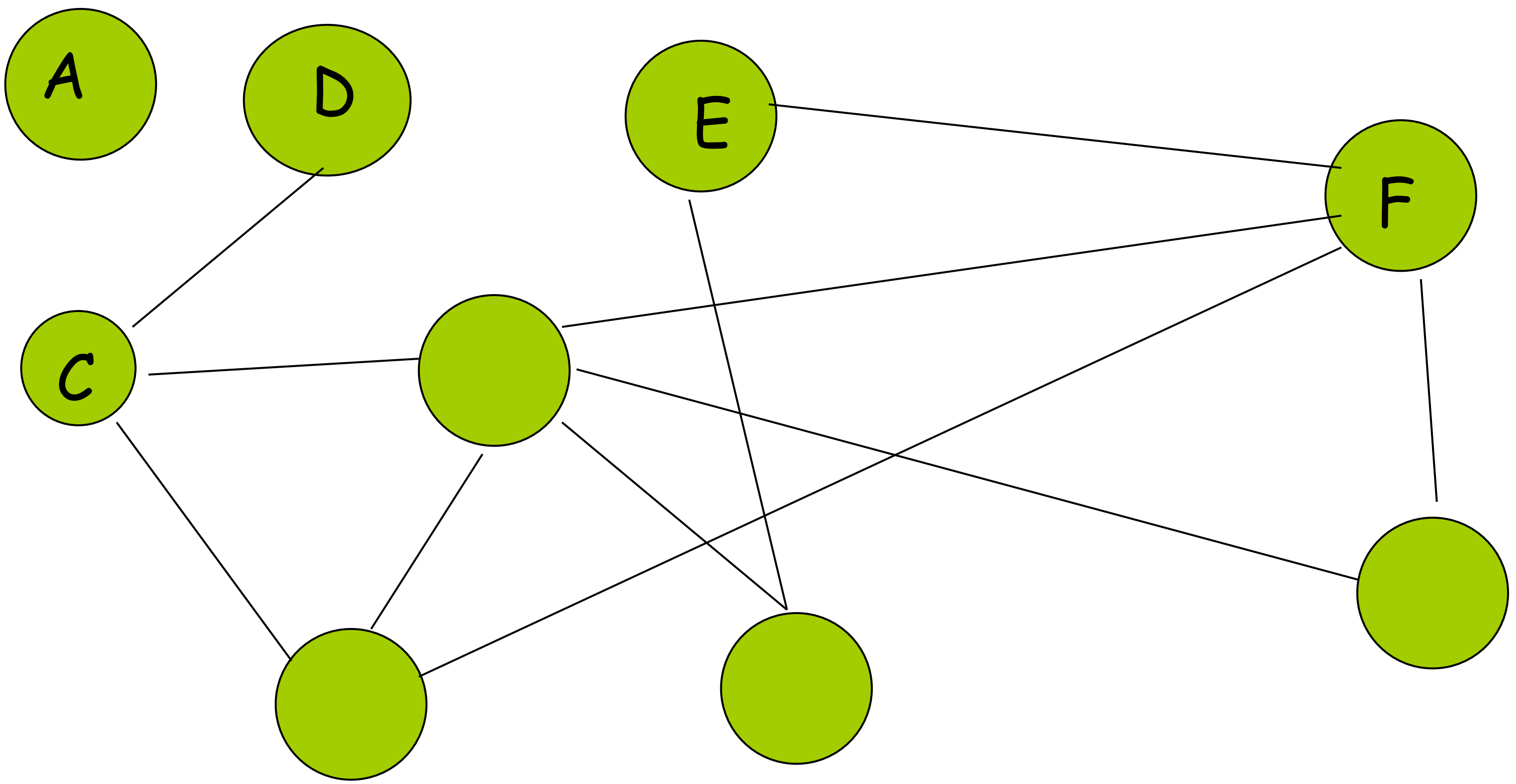


A network maintains a subgraph if its edges are marked by their endpoints

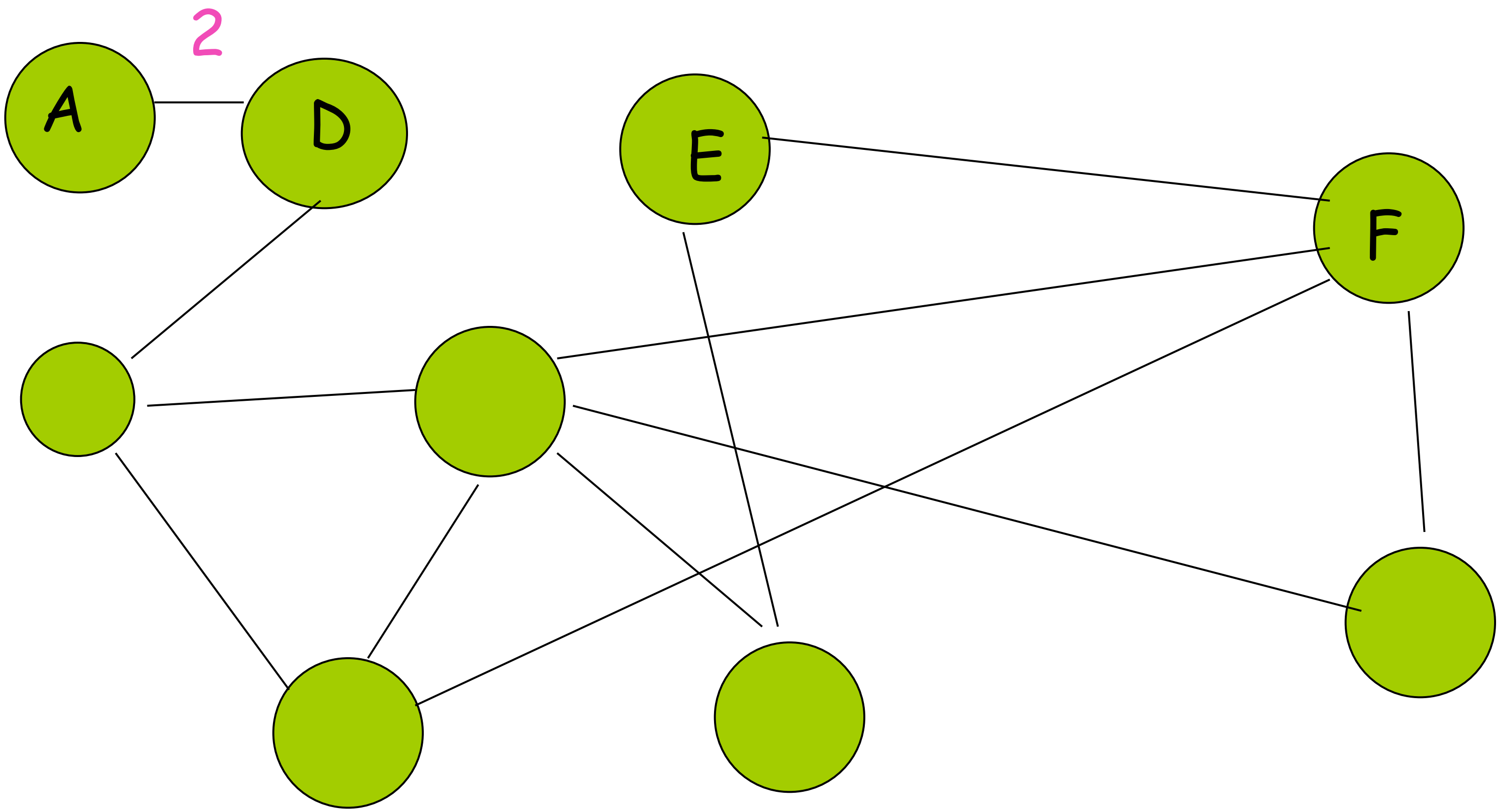


Communication: Each node may send messages of size O(log n) to all its neighbors in a single step.

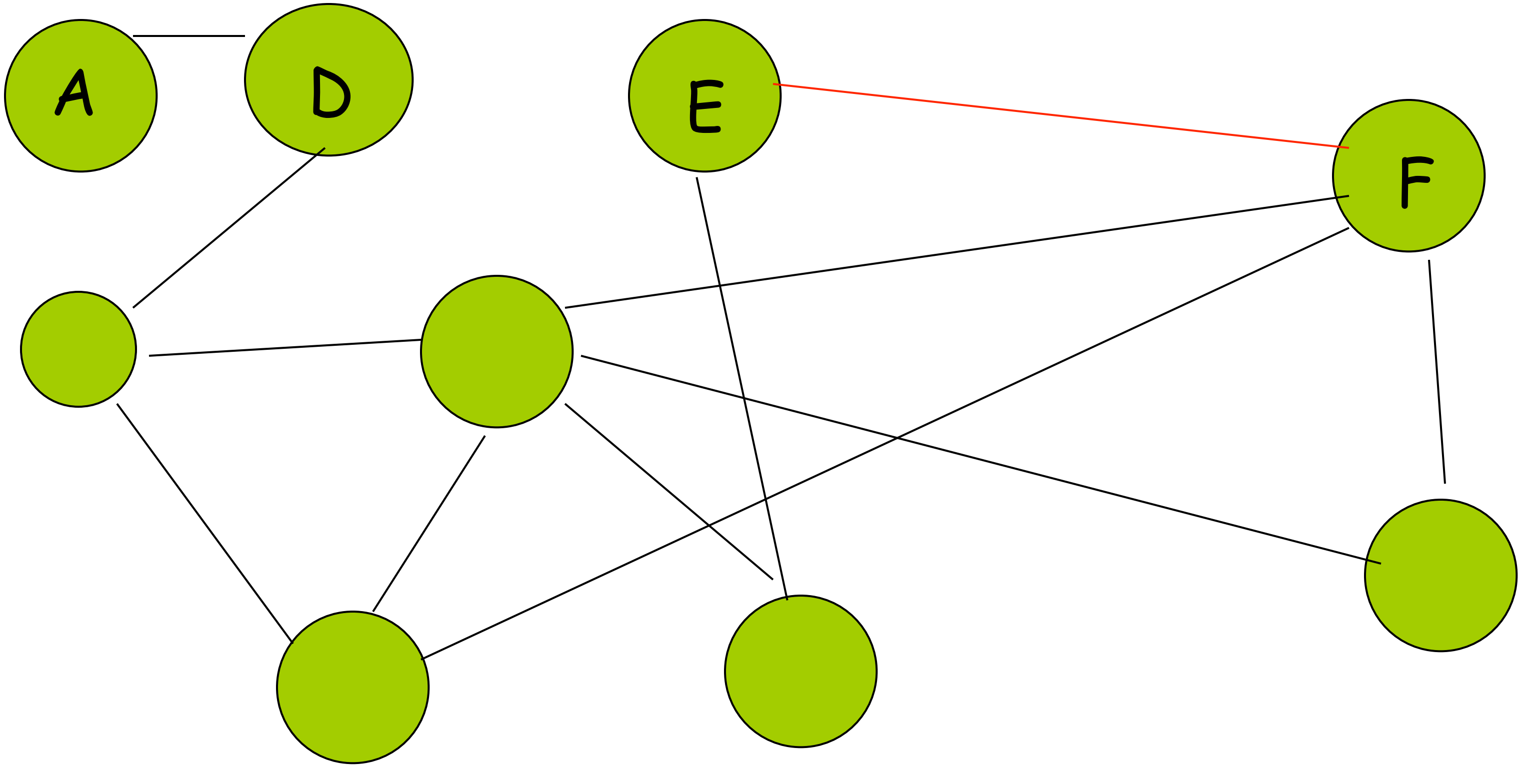
Synchronous vs. Asynchronous



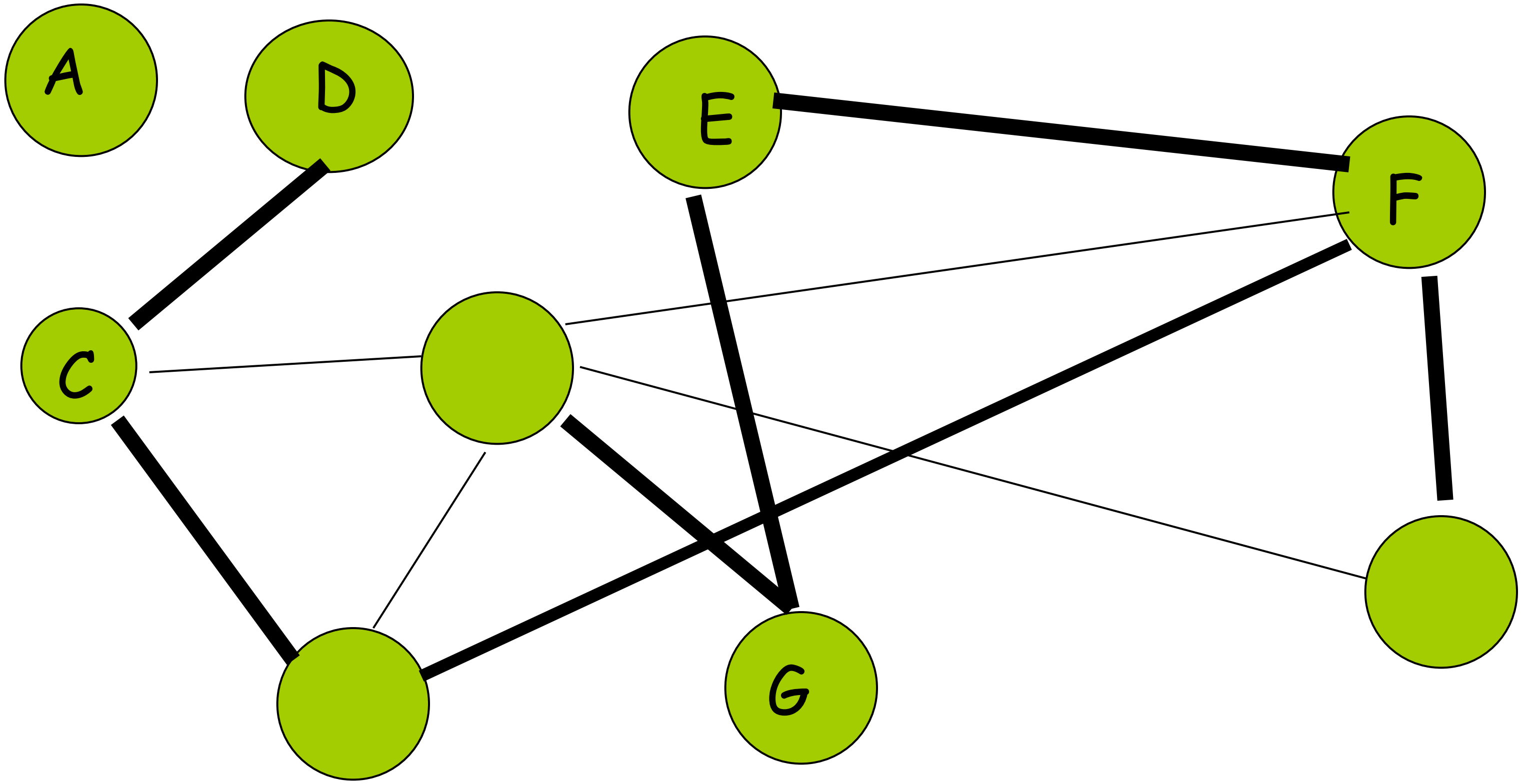
UPDATES:
Insert ({A,D}, edge_weight)



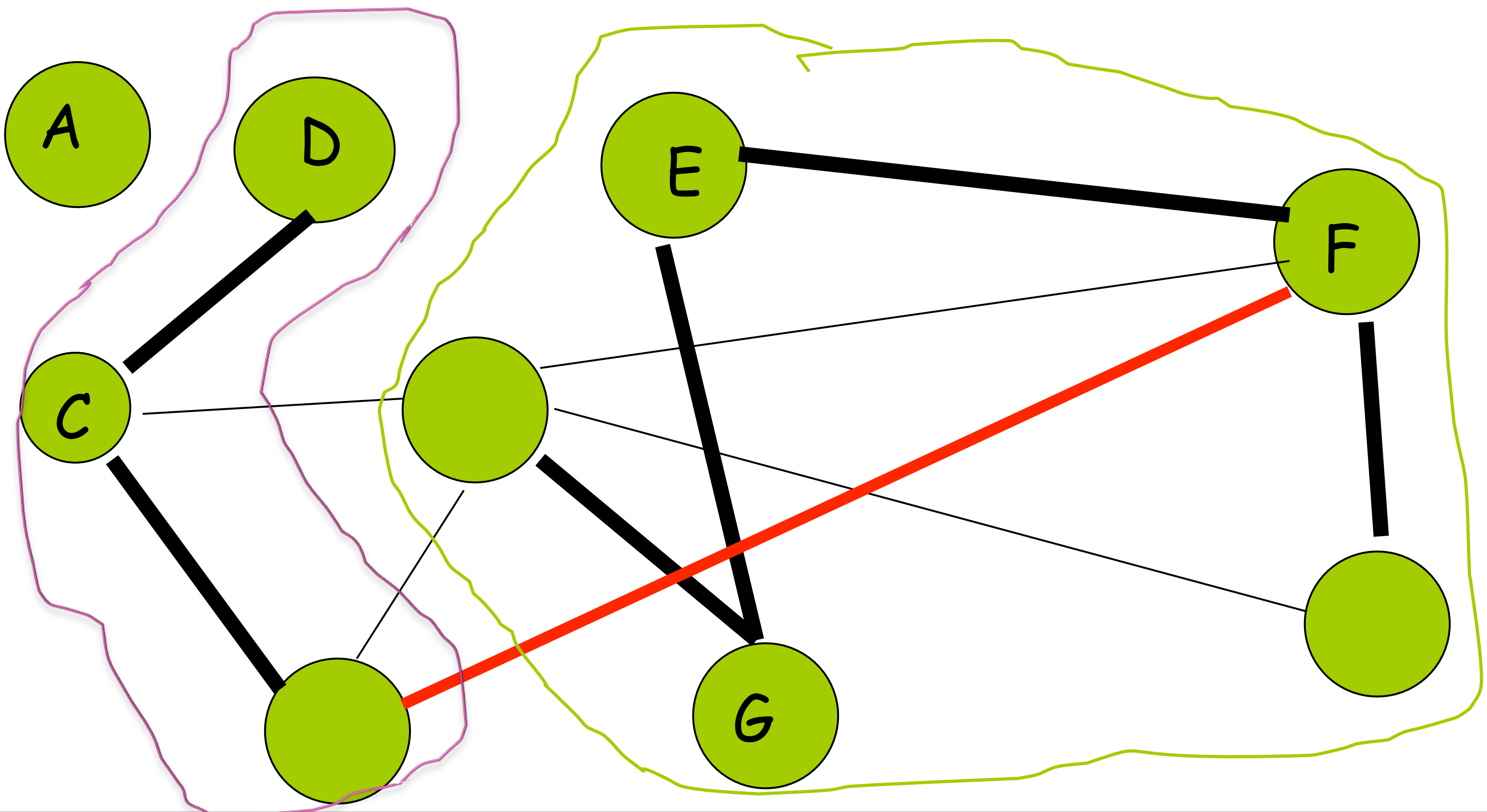
Delete edge {E,F}



MST (resp. ST) Problem: Maintain a minimum spanning forest (resp., spanning forest) in a dynamic network



Main difficulty:
How to find a replacement edge when a tree edge is deleted



Our contribution:

A New Tool
 for finding, impromptu, a replacement edge w.h.p.

for MST: an edge of min cost leaving a tree in a graph

for ST: any edge leaving a tree in a graph

Costs: MST / ST

Message complexity O(n log n) , O(n) (expected)

Preprocessing Time NONE

Update Time: O(diam(tree)*log n) , O(diam(tree)) expected.

Local memory needed between updates NONE

previous distributed dynamic MST:

Awerbuch, Cidon, Kutten:1990, 2008
O(n) messages--First dynamic updating
in o(m) messages per update.

But local memory=
O(n* degree of node*logn)
Stores the forest in each node ;

Static MST/ST thought to require m messages!

Gallagher, Humblet and Spira(1983)
O(m+ n log n) messages for building one from scratch (asynchronously)

Our method yields $O(n \log^2 n)$ messages for constructing an MST in the synchronous model.

NOT KNOWN if m can be avoided for the asynchronous model.

Talk outline:

1. KEY IDEA
2. The Odd hash function
3. Updating MST
4. Static MST
5. Updating ST
6. OPEN problems for distributed and sequential dynamic graphs.

KEY IDEA:

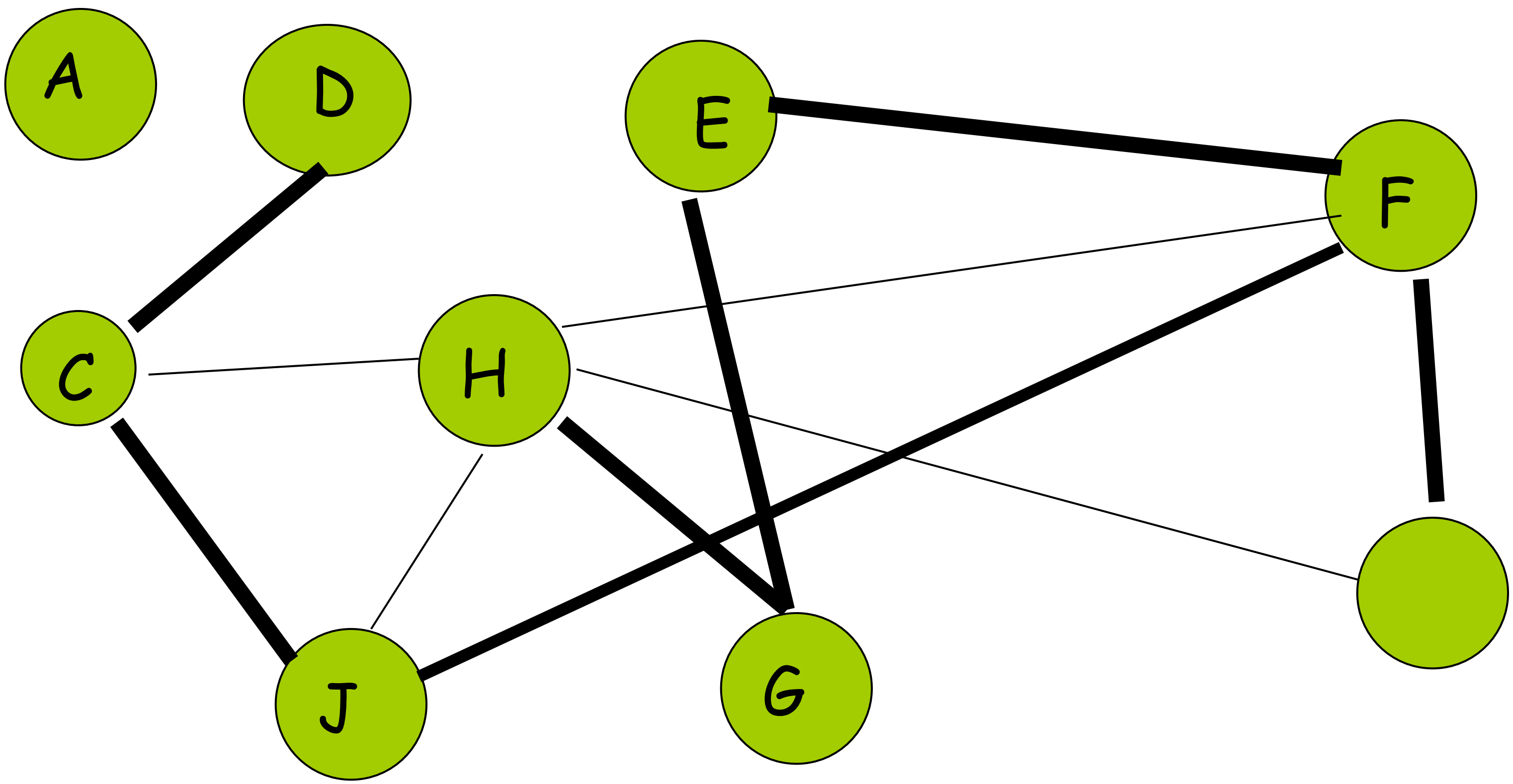
- C a maximally connected component of a graph.

→the sum of the degrees of the nodes in C is **even**, since every edge incident to a node in C contributes 2.

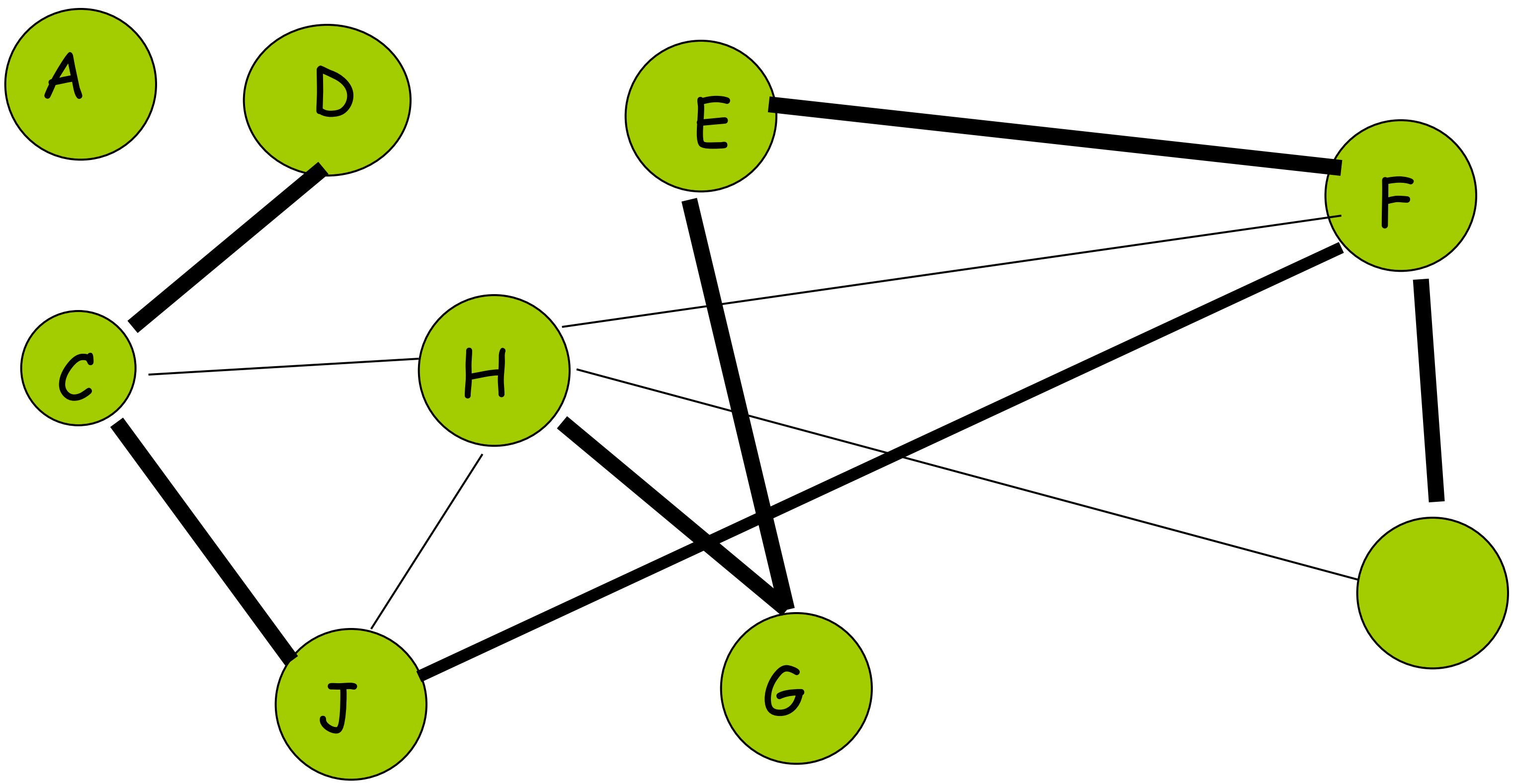
- If C is not maximally connected,

→ the sum of degrees of the nodes in C of a random subset of edges is **odd** with prob 1/2.

basic communication step: broadcast and return



How do we randomly sample and report results efficiently?



Recall:
Goal is to find (min weight) edge with only one endpoint in the tree

Odd hash function F

For any set S, we design hash
F:{weights}→{0,1}
s.t. there is a constant probability (1/36) that an ODD number of its elements hash to 1, iff S is non empty. Else it is 0.

Applying the Odd hash function F

Let E'={edges incident to nodes in tree T_x }

XOR F(e) (over all e is incident to a node in T_x

=XOR F(e) (over all e with one endpoint in T_x)

=1 with prob. 1/36 unless the cut is empty.

Using the ODD Hash function: TEST if there is a Replacement edge

- When a tree edge {X,Y} is deleted, if X<Y
- X becomes leader, broadcasts Odd hash F to other nodes in tree T_{x}.
- Each node applies F to their set of incident edges and computes the XOR;
- XOR is taken over all nodes in T_x
- Repeat in parallel O(log n) times to get prob error $1/n^c$
- Output 1 iff any one XOR =1

Find min wt replacement edge
(assuming distinct wts)

- Use binary search over the range of possible weights, testing w.h.p each time if there is a replacemt edge in that wt range and narrowing the range.
- Return weight when only one is left.

Analysis

- lg (Weight range) tests* cost of test
- Cost of test = initial cost of sending log n hash functions, +

  + 1 broadcast and return for each phase of the binary search
- Total =O(n $\log^2$ n) messages

Constructing the Odd hash function

- Let U be the universe of elements.
- S a subset of U.
- F(x)→ {0,1}
- We want:

$$XOR_{\{y \text{ in } S\}} F(y) = 1$$

iff S is non empty

Odd hash function F

Obvious approach takes O(log n) hashes

F has two parts,

- a 2-wise independent hash function

  $h: U \rightarrow U$

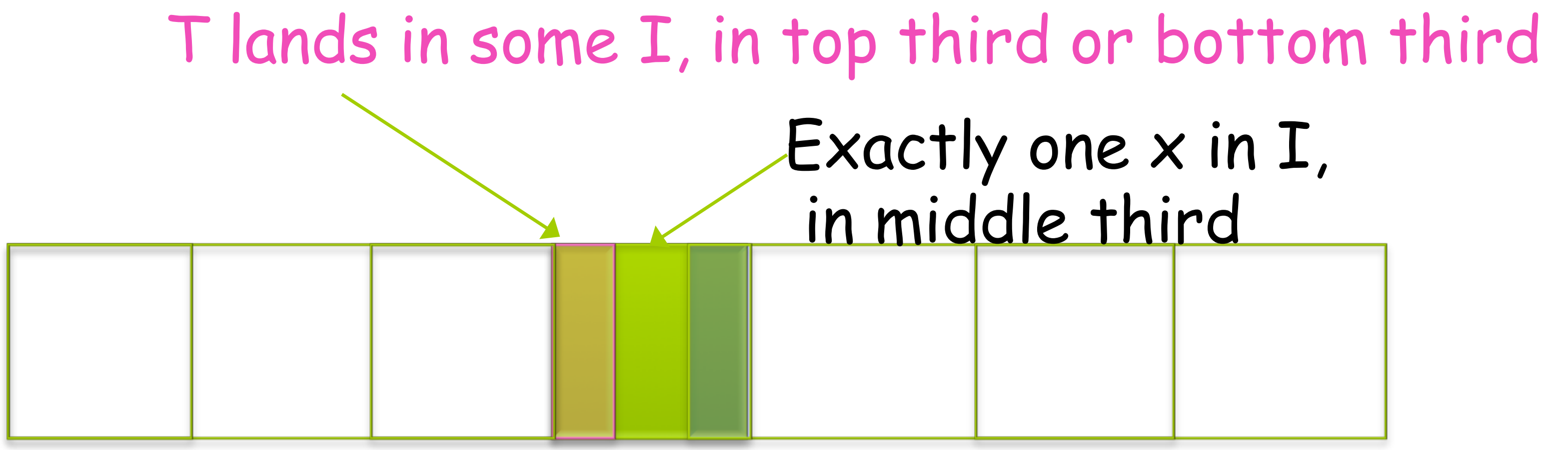
- t, a random element of U

DEF: $F(s)=1$ iff $h(s) < t$

Note: F can be described in O(log n) bits.

Why F is an Odd hash function

- h hashes U→U
- Imagine 2|S| equal sized intervals.

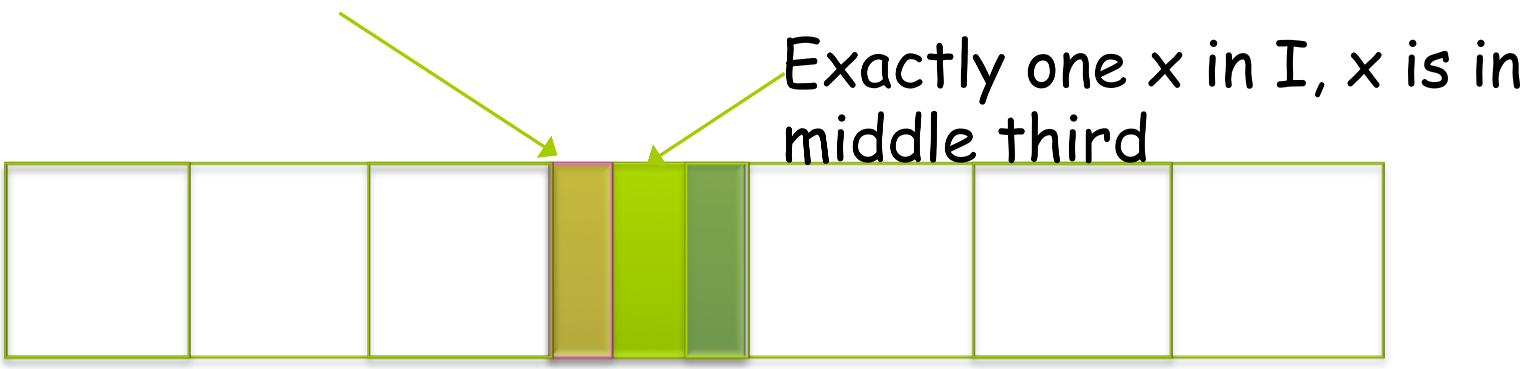


CASE: F works if

Parity of elements hashed to intervals left of I is

- Odd and t is in bottom third or
- Even and t is in top third

T lands in some I, and either top third or bottom third of I



Static synchronous MST alg

- While I < log n
- Repeat:
- Each component finds min wt edge incident to it, sends messgage to other endpoint,and waits n time steps. Then the found edges are inserted to form larger components.

Log n phases, each takes log n broadcasts and returns, for a total of $O(n \log^2 n)$ expected message communication.

Find *any* edge in expected O(1) broadcasts and returns

STEP 1: (IF step) Determine *if* there is a replacement edge w.h.p.

- Use deterministic amplification to send out O(log n) bits which can be used by individual nodes to deterministically generate log n Odd Hash functions s.t one is good w.h.p.
- Return log n outputs using ONE return

STEP 2: (find) If there is a replacement edge, find it

- Broadcast a single 2-wise independent hash function h
- For i=0,…, 2lg n, every node x computes one word whose i^{th} bit =

$$XOR_{y \text{ incindent to } x}\, h(y) \le 2^i$$

- If XOR over tree ≠ 0, min ← first i ≠0
- Test if there is exactly one edge with

$h(y) \le 2^{min.}$ If so, return it.

- Else Repeat find.

Open problem and discussion

- Can we avoid the O(m) communication costs of Gallagher for the asynchronous static model?
- Why this method is more complicated for sequential dynamic graph problem
- How the sequential dynamic graph method is not fully understood

Open problems for Sequential dynamic ST

- How to apply it to MST
- How to bring it down to $O(\log^3 n)$?
- Can we remove the tiers?