Aperiodic tilings (tutorial)

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February 12, 2015, ICERM
Plan of the talk

1. Introduction.
2. Tiling definitions, tiling spaces, tiling dynamical systems.
3. Spectral theory.
I. Aperiodic tilings: some references


I. Aperiodic tilings

A tiling (or tesselation) of $\mathbb{R}^d$ is a collection of sets, called tiles, which have nonempty disjoint interiors and whose union is the entire $\mathbb{R}^d$.

Aperiodic set of tiles can tile the space, but only non-periodically.
I. Aperiodic tilings

A tiling (or tessellation) of $\mathbb{R}^d$ is a collection of sets, called tiles, which have nonempty disjoint interiors and whose union is the entire $\mathbb{R}^d$.

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Origins in Logic: Hao Wang (1960’s) asked if it is decidable whether a given set of tiles (square tiles with marked edges) can tile the plane?

R. Berger (1966) proved undecidability, and in the process constructed an aperiodic set of 20,426 Wang tiles.

R. Robinson (1971) found an aperiodic set of 6 tiles (up to isometries).
One of the most interesting aperiodic sets is the set of Penrose tiles, discovered by Roger Penrose (1974). Penrose tilings play a central role in the theory because they can be generated by any of the three main methods:

1. local matching rules ("jigsaw puzzle");
2. tiling substitutions;
3. projection method (projecting a "slab" of a periodic structure in a higher-dimensional space to the plane).
I. Penrose and his tiles

Figure: Sir Roger Penrose

Figure: Penrose rhombi
Figure: A patch of the Penrose tiling
1. Penrose tiling (kites and darts)
I. Penrose tiling: basic properties

- Non-periodic: no translational symmetries.

- Hierarchical structure, “self-similarity,” or “composition”; can be obtained by a simple “inflate-and-subdivide” process. This is how one can show that the tiling of the entire plane exists.

- “Repetitivity” and uniform pattern frequency: every pattern that appears somewhere in the tiling appears throughout the plane, in a relatively dense set of locations, even with uniform frequency.

- 5-fold (even 10-fold) rotational symmetry: every pattern that appears somewhere in the tiling also appears rotated by 36 degrees, and with the same frequency (impossible for a periodic tiling).
I. Quasicrystals

Figure: Dani Schechtman (2011 Chemistry Nobel Prize); quasicrystal diffraction pattern (below)
Quasicrystals were discovered by D. Schechtman (1982). A quasicrystal is a solid (usually, metallic alloy) which, like a crystal, has a sharp X-ray diffraction pattern, but unlike a crystal, has an aperiodic atomic structure. Aperiodicity was inferred from a “forbidden” 10-fold symmetry of the diffraction picture.
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Other types of quasicrystals have been discovered by **T. Ishimasa, H. U. Nissen, Y. Fukano (1985)** and others (Al-Mn alloy, 10-fold symmetry, Ni-Cr alloy, 12-fold symmetry; V-Ni-Si alloy, 8-fold symmetry)
I. Other quasicrystal diffraction patterns

Figure: From the web site of Uwe Grimm
(http://mcs.open.ac.uk/ugg2/quasi.shtml)
Symbolic substitutions have been studied in Dynamics (coding of geodesics), Number Theory, Automata Theory, and Combinatorics of Words for a long time.
I. Substitutions

Symbolic substitutions have been studied in Dynamics (coding of geodesics), Number Theory, Automata Theory, and Combinatorics of Words for a long time.

Symbolic substitution is a map $\zeta$ from a finite “alphabet” $\{0, \ldots, m - 1\}$ into the set of “words” in this alphabet.

- **Thue-Morse**: $\zeta(0) = 01$, $\zeta(1) = 10$. Iterate (by concatenation):

  $\begin{align*}
  0 & \to 01 \\
  01 & \to 0110 \\
  0110 & \to 01101001 \\
  01101001 & \to \ldots \\
  
  u = u_0u_1u_2\ldots &= \lim_{n \to \infty} \zeta^n(0) \in \{0, 1\}^\mathbb{N}, \quad u = \zeta(u)
  \end{align*}$

- **Fibonacci**: $\zeta(0) = 01$, $\zeta(1) = 0$. Iterate (by concatenation):

  $\begin{align*}
  0 & \to 01 \\
  01 & \to 010 \\
  010 & \to 01001 \\
  01001 & \to \ldots \\
  u = 01001010010010100100100100100100\ldots
  \end{align*}$
Symbolic substitutions have been generalized to higher dimensions. One can just consider higher-dimensional symbolic arrays, e.g.

\[
\begin{array}{ccc}
0 & \rightarrow & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
1 & \rightarrow & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\end{array}
\]
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\begin{align*}
0 & \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & 1 & \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\end{align*}
\]

More interestingly, one can consider “geometric” substitutions, with the symbols replaced by tiles. Penrose tilings can be obtained in such a way.
I. Example: chair tiling

Examples are taken from "Tiling Encyclopedia", see http://tilings.math.uni-bielefeld.de/

**Figure**: tile-substitution, real expansion constant $\lambda = 2$
I. Example: chair tiling
I. Example: Ammann-Beenker rhomb-triangle tiling

Figure: tile-substitution, real expansion constant $\lambda = 1 + \sqrt{2}$
I. Example: Ammann-Beenker rhomb-triangle tiling
I. Substitution tilings in $\mathbb{R}^d$

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In fact, such tilings arise naturally in connection with

- Markov partitions for hyperbolic toral automorphisms in dimensions 3 and larger
- Numeration systems with a complex base.
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In fact, such tilings arise naturally in connection with

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- Numeration systems with a complex base.

**Rauzy tiling** is a famous example.

Let $\lambda$ be the complex root of $1 - z - z^2 - z^3 = 0$ with positive imaginary part, $z \approx -0.771845 + 1.11514i$. Then (Lebesgue) almost every $\zeta \in \mathbb{C}$ has a unique representation

$$\zeta = \sum_{n=-N}^{\infty} a_n \lambda^{-n},$$

where $a_n \in \{0, 1\}$, $a_n a_{n+1} a_{n+2} = 0$ for all $n$. 
Rauzy tiles
Gerard Rauzy

Figure: Gerard Rauzy (1938-2010)
II. Tiling definitions

- **Prototile set:** $\mathcal{A} = \{A_1, \ldots, A_N\}$, compact sets in $\mathbb{R}^d$, which are closures of its interior; interior is connected. (May have “colors” or “labels”.)

Remark. Often prototiles are assumed to be polyhedral, or at least topological balls.

A tiling of $\mathbb{R}^d$ with the prototile set $\mathcal{A}$: collection of tiles whose union is $\mathbb{R}^d$ and interiors are disjoint. All tiles are isometric copies of the prototiles.

A patch is a finite set of tiles. $\mathcal{B}$ denotes the set of patches with tiles from $\mathcal{A}$. 
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- A **patch** is a finite set of tiles. $\mathcal{A}^+$ denotes the set of patches with tiles from $\mathcal{A}$. 

Finite local complexity

• Several options: identify tiles (patches) up to (a) translation; (b) orientation-preserving isometry (Euclidean motion); (c) isometry. Let $G$ be the relevant group of transformations.
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**Definition.** A tiling $\mathcal{T}$ is said to have **finite local complexity** (FLC) with respect to the group $G$ if for any $R > 0$ there are finitely many $\mathcal{T}$-patches of diameter $\leq R$, up to the action of $G$. 
II. Tile-substitutions in $\mathbb{R}^d$

Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear map, that is, all its eigenvalues are greater than 1 in modulus. (Often $\phi$ is assumed to be a similitude or even a pure dilation $\phi(x) = \lambda x$.)

**Definition.** Let $\{A_1, \ldots, A_m\}$ be a finite prototile set. A **tile-substitution** with expansion $\phi$ is a map $\omega : A \to A^+$, where each $\omega(A_i)$ is a patch $\{g(A_j)\}_{g \in D_{ij}}$, where $D_{ij}$ is a finite subset of $G$, such that

$$\text{supp}(\omega(A_i)) = \phi(A_i), \ i \leq m.$$ 

The substitution is extended to patches and tilings in a natural way.

**Substitution matrix** counts the number of tiles of each type in the substitution of prototiles: $M = (M_{ij})_{i,j \leq m}$, where $M_{ij} = \# D_{ij} = \#$ tiles of type $i$ in $\omega(A_j)$. 
II. Substitution tiling space

**Definition.** Given a tiling substitution $\omega$ on the prototile set $\mathcal{A}$, the substitution tiling space $X_\omega$ is the set of all tilings whose every patch appears as a “subpatch” of $\omega^k(\mathcal{A})$, for some $A \in \mathcal{A}$ and $k \in \mathbb{N}$.
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- **Tile-substitution is primitive** if the substitution matrix is primitive, that is, some power of $M$ has only positive entries (equivalently, $\exists k \in \mathbb{N}, \forall i \leq m$, the patch $\omega^k(A_i)$ contains tiles of all types). For a primitive tile-substitution, $X_\omega \neq \emptyset$ (in fact, there exists an $\omega$-periodic tiling in $X_\omega$; that is, $\omega^\ell(T) = T$ for some $\ell$).

- The tile-substitution $\omega$ and the space $X_\omega$ are said to have **finite local complexity** (FLC) with respect to the group $G$ if for any $R > 0$ there are finitely many patches of diameter $\leq R$ in tilings of $X_\omega$, up to the action of $G$. 

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II. Example: Kenyon’s non-FLC tiling space

Figure: non-FLC tiling

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II. Example: Kenyon’s non-FLC substitution tiling space

Figure: non-FLC self-similar tiling
The prototiles appear in infinitely many orientations; the tiling is FLC with $G = \text{Euclidean group}$.
Work on non-FLC tilings

- Non-FLC substitution tilings with a translationally-finite prototile set have been studied by L. Danzer, R. Kenyon, N. P. Frank-E. A. Robinson, Jr., N. P. Frank-L. Sadun, and others.
Work on non-FLC tilings

Non-FLC substitution tilings with a translationally-finite prototile set have been studied by L. Danzer, R. Kenyon, N. P. Frank-E. A. Robinson, Jr., N. P. Frank-L. Sadun, and others.

Pinwheel-like tilings have been studied by C. Radin, L. Sadun, N. Ormes, and others.
II. Tiling spaces (not just substitutions)

**Tiling metric in the $G$-finite setting:** two tilings are close if after a transformation by small $g \in G$ they agree on a large ball around the origin.

More precisely (in the translationally-finite setting):

$$\tilde{\varrho}(\mathcal{T}_1, \mathcal{T}_2) := \inf \{ r \in (0, 2^{-1/2}) : \exists g \in B_r(0) : \mathcal{T}_1 - g \text{ and } \mathcal{T}_2 \text{ agree on } B_{1/r}(0) \}.$$  

Then $\varrho(\mathcal{T}_1, \mathcal{T}_2) := \min\{2^{-1/2}, \tilde{\varrho}(\mathcal{T}_1, \mathcal{T}_2)\}$ is a metric.
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**Tiling metric in the $G$-finite setting:** two tilings are close if after a transformation by small $g \in G$ they agree on a large ball around the origin.

More precisely (in the translationally-finite setting):

$$\bar{\varrho}(\mathcal{T}_1, \mathcal{T}_2) := \inf \{ r \in (0, 2^{-1/2}) : \exists g \in B_r(0) : \mathcal{T}_1 - g \text{ and } \mathcal{T}_2 \text{ agree on } B_{1/r}(0) \}.$$  

Then $\varrho(\mathcal{T}_1, \mathcal{T}_2) := \min \{ 2^{-1/2}, \varrho(\mathcal{T}_1, \mathcal{T}_2) \}$ is a metric.

**Tiling space:** a set of tilings which is (i) closed under the translation action and (ii) complete in the tiling metric. The **hull** of $\mathcal{T}$, denoted by $X_\mathcal{T}$, is the closure of the $\mathbb{R}^d$-orbit $\{ \mathcal{T} - x : x \in \mathbb{R}^d \}$ in the tiling metric.
II. Local topology of the tiling space

Unless stated otherwise, we will assume that the tilings are translationally finite and $G = \mathbb{R}^d$. 

There is a lot of interesting work on the algebraic topology of tiling spaces: e.g. [J. Anderson and I. Putnam '98], [F. Gähler '02], [A. Forrest, J. Hunton, J. Kellendonk '02], [J. Kellendonk '03], [L. Sadun '03, '07, AMS Lecture Series '08], [L. Sadun and R. Williams '03], [M. Barge and B. Diamond '08], . . .
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Assume that $\mathcal{T}$ is non-periodic, that is, $\mathcal{T} - \mathbf{t} \neq \mathcal{T}$ for $\mathbf{t} \neq 0$. Then a small neighborhood in $X_\mathcal{T}$ is homeomorphic to $\mathbb{R}^d \times \Gamma$, where $\Gamma$ (the “transversal”) is a Cantor set.
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Assume that $\mathcal{T}$ is non-periodic, that is, $\mathcal{T} - t \neq \mathcal{T}$ for $t \neq 0$. Then a small neighborhood in $X_{\mathcal{T}}$ is homeomorphic to $\mathbb{R}^d \times \Gamma$, where $\Gamma$ (the “transversal”) is a Cantor set.

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II. Tiling dynamical system

Important properties of a tiling are reflected in the properties of the tiling space and the associated dynamical system:

Theorem. $T$ has finite local complexity (FLC) $\iff X_T$ is compact.

$\mathbb{R}^d$ acts by translations: $T_t(S) = S - t$. Topological dynamical system (action of $\mathbb{R}^d$ by homeomorphisms): $(X_T, T_t)_{t \in \mathbb{R}^d} = (X_T, \mathbb{R}^d)$

Definition. A topological dynamical system is minimal if every orbit is dense (equivalently, if it has no nontrivial closed invariant subsets).

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II. Tiling dynamical system

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**Theorem.** \( \mathcal{T} \) has finite local complexity (FLC) \( \iff \) \( X_\mathcal{T} \) is compact.

\( \mathbb{R}^d \) acts by translations: \( T^t(S) = S - t \). Topological dynamical system (action of \( \mathbb{R}^d \) by homeomorphisms):

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(X_\mathcal{T}, T^t)_{t \in \mathbb{R}^d} = (X_\mathcal{T}, \mathbb{R}^d)
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**Definition.** A topological dynamical system is *minimal* if every orbit is dense (equivalently, if it has no nontrivial closed invariant subsets).

**Theorem.** \( \mathcal{T} \) is repetitive \( \iff (X_\mathcal{T}, \mathbb{R}^d) \) is minimal.
II. Substitution action and non-periodicity

Recall that the substitution $\omega$ extends to **tilings**, so we get a map

$$\omega : X_T \to X_T,$$

which is always surjective.
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Recall that the substitution $\omega$ extends to \textit{tilings}, so we get a map

$$\omega : \mathcal{X}_\mathcal{T} \rightarrow \mathcal{X}_\mathcal{T},$$

which is always surjective.

\textbf{Theorem} [B. Mossé '92], [B. Sol. '98] \textit{The map $\omega : \mathcal{X}_\mathcal{T} \rightarrow \mathcal{X}_\mathcal{T}$ is invertible iff $\mathcal{T}$ is non-periodic.}
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**Theorem** [B. Mossé ’92], [B. Sol. ’98] *The map* $\omega : X_T \rightarrow X_T$ *is invertible iff* $T$ *is non-periodic.*

**Useful analogy:**

- substitution $\mathbb{Z}$-action by $\omega \sim$ geodesic flow
- translation $\mathbb{R}^d$-action $\sim$ horocycle flow
II. Uniform patch frequencies

For a patch $P \subset T$ consider

$$N_P(T, A) := \# \{ t \in \mathbb{R}^d : -t + P \text{ is a patch of } T \text{ contained in } A \},$$

the number of $T$-patches equivalent to $P$ that are contained in $A$.

**Definition.** A tiling $T$ has *uniform patch frequencies* (UPF) if for any non-empty patch $P$, the limit

$$\text{freq}(P, T) := \lim_{R \to \infty} \frac{N_P(T, t + Q_R)}{R^d} \geq 0$$

exists uniformly in $t \in \mathbb{R}^d$. Here $Q_R = [-\frac{R}{2}, \frac{R}{2}]^d$. 
II. Unique ergodicity for tiling systems

**Theorem.** Let $\mathcal{T}$ be a tiling with FLC. Then the dynamical system $(X_\mathcal{T}, \mathbb{R}^d)$ is uniquely ergodic, i.e. has a unique invariant probability measure, if and only if $\mathcal{T}$ has UPF.

**Theorem.** Let $\mathcal{T}$ be a self-affine tiling, for a primitive FLC tile-substitution. Then the dynamical system $(X_\mathcal{T}, \mathbb{R}^d)$ is uniquely ergodic.

Denote by $\mu$ the unique invariant measure.
Constructions of substitution tilings (and cut-and-project tilings) are non-local...

Note: if $X_S$ is nonempty, but every tiling in $X_S$ is non-periodic, we say that $S$ is an aperiodic set.

S. Mozes (1989): for any primitive aperiodic substitution $\omega$ in $\mathbb{R}^2$, with square tiles, there exists a set $S$ and a factor map $\Phi : X_S \rightarrow X_\omega$ which is 1-to-1 outside a set of measure zero (for all translation-invariant measures).

This was extended by C. Radin (1994) to the pinwheel tiling, and by C. Goodman-Strauss (1998) to all substitution tilings.
II. Local matching rules

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Let $S$ be a set of prototiles, together with the rules how two tiles can fit together (can usually be implemented by ”bumps” and ”dents”). Let $X_S$ be the set of all tilings of $\mathbb{R}^d$ which satisfy the rules.

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III. Diffraction spectrum of a tiling

Pick a point in each prototile. This yields a discrete point set (separated net, or Delone set) $\Lambda$, which models a configuration of atoms.
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For simplicity, assume that all atoms are modeled by $\delta$-functions:

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$$
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$$

For self-affine tilings, the **autocorrelation measure** is well-defined and equals

$$
\gamma = \sum_{z \in \Lambda - \Lambda} \nu(z) \delta_z,
$$

where $\nu(z)$ is the frequency of the cluster $\{x, x + z\}$ in $\Lambda$. 

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III. Diffraction spectrum and dynamical spectrum

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**Dynamical spectrum** of the tiling $\mathcal{T}$ is the spectral measure of the tiling dynamical system, or equivalently, of the group of unitary operators $\{U_t\}_{t \in \mathbb{R}^d}$, where $U_t f(S) = f(S - t)$ for $S \in X_{\mathcal{T}}$ and $f \in L^2(X_{\mathcal{T}}, \mu)$. Remarkably, there is a connection! [S. Dworkin '93]: the diffraction is (essentially) a “part” of the dynamical spectrum.

**Theorem** [J.-Y. Lee, R. V. Moody, B. S. '02]: The diffraction is pure point if and only if the dynamical spectrum is pure discrete, i.e. there is a basis for $L^2(X_{\mathcal{T}}, \mu)$ consisting of eigenfunctions.
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III. Eigenvalues and Eigenfunctions

**Definition.** $\alpha \in \mathbb{R}^d$ is an eigenvalue for the measure-preserving $\mathbb{R}^d$-action $(X, T^t, \mu)_{t \in \mathbb{R}^d}$ if $\exists$ eigenfunction $f_\alpha \in L^2(X, \mu)$, i.e., $f_\alpha$ is not $0$ in $L^2$ and for $\mu$-a.e. $x \in X$

$$f_\alpha(T^t x) = e^{2\pi i \langle t, \alpha \rangle} f_\alpha(x), \quad t \in \mathbb{R}^d.$$ 

Here $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$.

**Warning:** eigenvalue is a vector! (like wave vector in physics)
III. Characterization of eigenvalues

Return vectors for the tiling:

\[ \mathcal{Z}(\mathcal{T}) := \{ z \in \mathbb{R}^d : \exists T, T' \in \mathcal{T}, T' = T + z \} \].

**Theorem** [S. 1997] Let \( \mathcal{T} \) be a non-periodic self-affine tiling with expansion map \( \phi \). Then the following are equivalent for \( \alpha \in \mathbb{R}^d \):

(i) \( \alpha \) is an eigenvalue for the measure-preserving system \((X_\mathcal{T}, \mathbb{R}^d, \mu)\);

(ii) \( \alpha \) satisfies the condition:

\[
\lim_{n \to \infty} \langle \phi^n z, \alpha \rangle \pmod{1} = 0 \quad \text{for all } z \in \mathcal{Z}(\mathcal{T}).
\]
III. When is there a discrete component of the spectrum?

**Theorem [S.’07]** Let $T$ be a self-similar tiling of $\mathbb{R}^d$ with a pure dilation expansion map $t \mapsto \lambda t$. Then the associated tiling dynamical system has non-trivial eigenvalues (equivalently, is not weakly mixing) iff $\lambda$ is a Pisot number. Moreover, in this case the set of eigenvalues is relatively dense in $\mathbb{R}^d$.

**Definition** An algebraic integer $\lambda > 1$ is a Pisot number if all of its algebraic conjugates lie inside the unit circle.

The role of Pisot numbers in the study of “mathematical quasicrystals” was already pointed out by [E. Bombieri and J. Taylor '87]. [F. Gähler and R. Klitzing '97] have a result similar to the theorem above, in the framework of diffraction spectrum.
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The necessity of the Pisot condition for existence of non-trivial eigenvalues, when the expansion is pure dilation by $\lambda$, follows from the characterization of eigenvalues and the classical theorem of Pisot:

$$\langle \phi^n z, \alpha \rangle = \lambda^n \langle z, \alpha \rangle \to 0 \ (\text{mod } 1), \text{ as } n \to \infty,$$

and we can always find a return vector $z$ such that $\langle z, \alpha \rangle \neq 0$ if $\alpha \neq 0$. 
III. When is there a large discrete component of the spectrum?

**Theorem** Let $T$ be self-affine with a diagonalizable over $\mathbb{C}$ expansion map $\phi$. Suppose that all the eigenvalues of $\phi$ are algebraic conjugates with the same multiplicity. Then the following are equivalent:

(i) the set of eigenvalues of the tiling dynamical system associated with $T$ is relatively dense in $\mathbb{R}^d$;

(ii) the spectrum of $\phi$ is a **Pisot family**: for every eigenvalue $\lambda$ of $\phi$ and its conjugate $\gamma$, either $|\gamma| < 1$, or $\gamma$ is also an eigenvalue of $\phi$.

(i) $\Rightarrow$ (ii) was proved by [E. A. Robinson '04], using the criterion for eigenvalues in [S. '97].

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III. Pure discrete spectrum

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Pisot discrete spectrum conjecture: A primitive irreducible symbolic substitution $\mathbb{Z}$-action (or $\mathbb{R}$-action) of Pisot type has pure discrete spectrum.

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Those interested in the topic should read


This is a comprehensive account of the knowledge up to 2000, with a large bibliography and many open questions. Some of the open questions have been resolved in

[J.-Y. Lee and B. Solomyak, Pure point diffractive Delone sets have the Meyer property], Discrete Comput. Geom. (2008), and

[N. Lev and A. Olevskii, Quasicrystals and Poisson’s summation formula], math. arXiv:1312.6884