Universal Behavior near Erdős-Rényi

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Joint work with Rick Kenyon, Charles Radin and Kui Ren

Outline

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1. Recap of graphs and graphons
2. Results
3. Strategy
4. First goal
5. Second goal

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Graphons and Densities

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- Graphon entropy $s(g) = -\iint I_0(g(x,y))dx\,dy$.
- $I_0(u) = \frac{1}{2}[u\ln(u) + (1-u)\ln(1-u)]$.

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- $S(e, e^3) = -I_0(e)$.
- What happens when t is only *close* to e^3?

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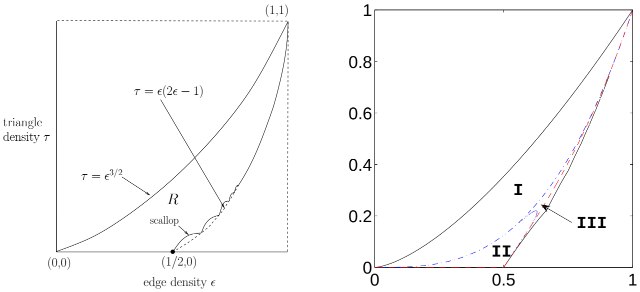
Phase portrait for edge-triangle model



Schematic Profile and Phase Portrait

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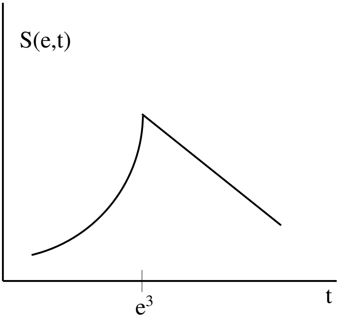
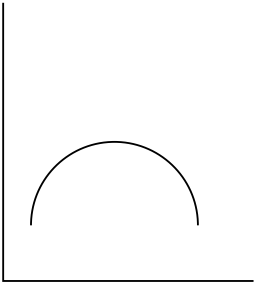
- Below ER, $S(e, e^3) - S(e, t)$ always goes as $(e^3 - t)^{2/n}$ for some $n > 2$. (Generically $n = 3$, but not always.)
- Caveat: for some graphs, density is minimized at ER. In those models, results below ER are moot.

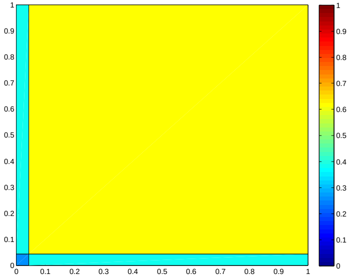
$S(e,t)$ for fixed e



Not



Graphon just above ER curve



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  - Get the most $\|\Delta g\|$ for the least $\|\Delta S\|$.
- Rewrite sub-goals as:
  - Maximize $|\Delta t|$ for fixed $\|\Delta g\|$.
  - Minimize $|\Delta S|$ for fixed $\|\Delta g\|$.

Use the L^2 norm

Treat g and Δg as integral kernels of operators on $L^2([0,1])$.

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$$\begin{aligned} \|\Delta g\|^2 &= \iint (\Delta g(x,y))^2 \, dx \, dy \\ &= \iint \Delta g(x,y) \Delta g(y,x) dx \, dy \\ &= Tr(\Delta g^2). \end{aligned}$$

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Where $h(x) := \int_0^1 \Delta g(x,y) dy$.

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- For $\Delta t > 0$, optimal Δg is rank 2 to lowest order, $\|\Delta g\|$ goes as $|\Delta t|^{1/2}$.
- Optimal solutions involve arbitrary functions $\alpha(x)$ or $h(x)$. Picking the right function is at the heart of second goal.

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- Exception is harmless, but requires a little more work.

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- Same as maximizing $s(g)$ for fixed $\|g_{1/2}\|^2$, where $g_{1/2}(x,y) = g(x,y) - 1/2$,
- Claim: That's achieved when $g_{1/2}(x,y)^2$ is constant.

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- Since $\int (g - \frac{1}{2})^2$ is fixed, $s(g)$ is maximized by taking $(g - \frac{1}{2})^2$ constant.

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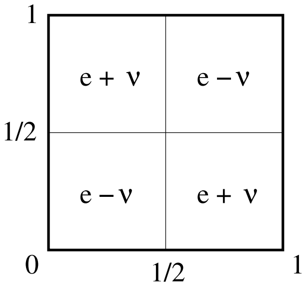

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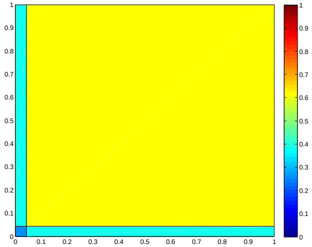

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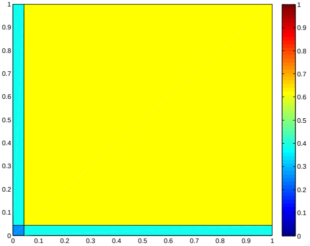

Failure is only on tiny square of area $\sim \|\Delta g\|^4$.

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- For edge-triangle, the optimizing graphon takes a specific form just above the ER curve. Highly asymmetric bipodal. $\Delta g(x, y) \approx h(x) + h(y)$.
- The results above ER are *universal*, and apply to any model with edge density and one other graph density.
- Below ER, $|\Delta S|$ always goes as $|\Delta t|^{2/n}$ for some $n > 2$, since negative terms in $|\Delta t|$ are cubic or higher in Δg.