On vector-valued singular perturbation problems involving potentials vanishing on curves

Nelly André and **Itai Shafrir**

Univ. Tours, **Technion**

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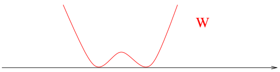

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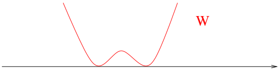

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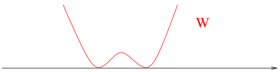

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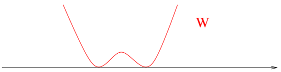

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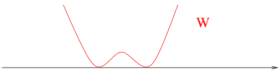

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- For each $\varepsilon > 0$ let u_ε be a minimizer for

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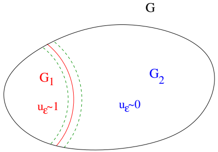

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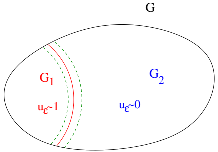

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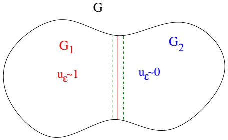

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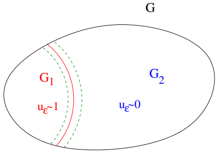

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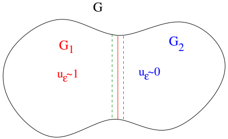

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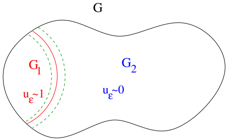

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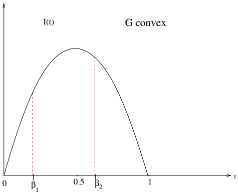

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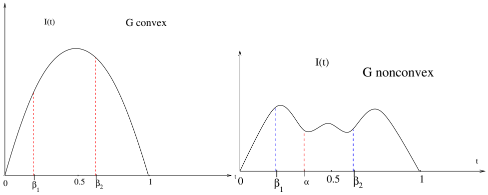

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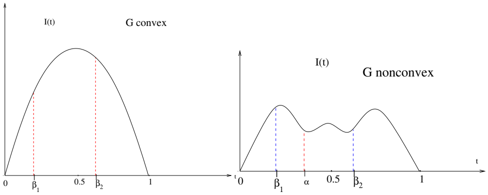

Note: When G is convex, $I(t)$ is concave.

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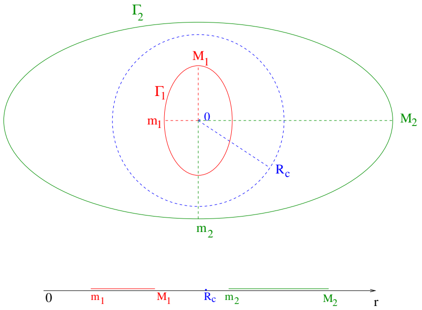

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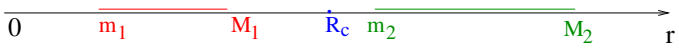

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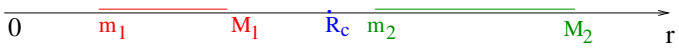

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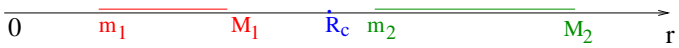

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$\mu(G_1) := \alpha$ satisfies $I(\alpha) = \min_{t \in \mathcal{I}_0} I(t)$.

The results

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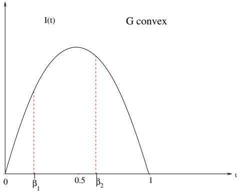

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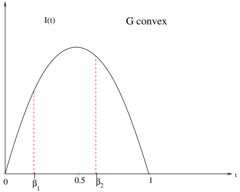

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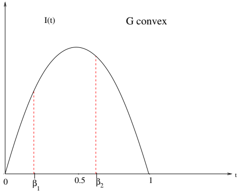

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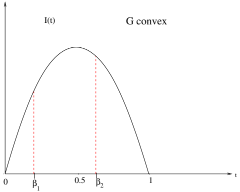

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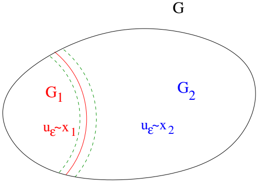

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- Using c_0 we can know which x_j is selected among several candidates.

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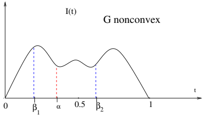

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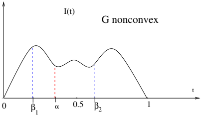

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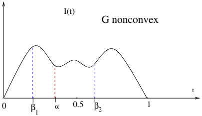

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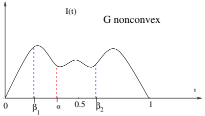

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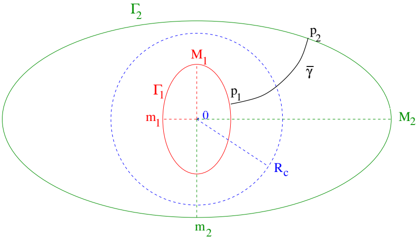

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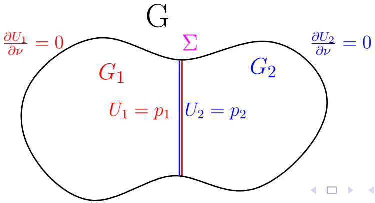

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