Limit shapes
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Limit shapes for interacting particle systems and their universal fluctuations

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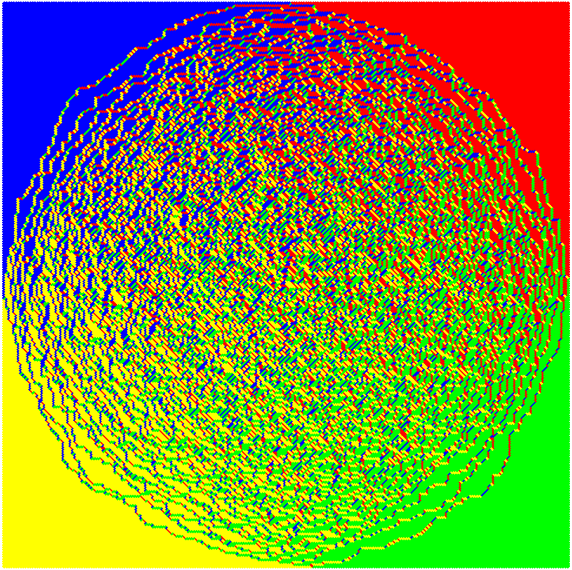
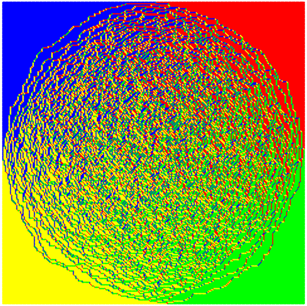
An example of random tilings



An Aztec diamond of size $N = 240$



The border of the four regular facets, as the size $N \to \infty$:

- has a circular limit shape (aka arctic circle)

  Jockush, Propp, Shor'98

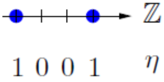
- the fluctuations of border of the four facets are $\mathcal{O}(N^{1/3})$ and (GUE) Tracy-Widom distributed
- As a process, it converges to the Airy_2 process on the $(N^{2/3}, N^{1/3})$ scale

  Johansson'03

- TASEP: Totally Asymmetric Simple Exclusion Process

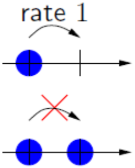
- **Configurations**

  $$\eta = \{\eta_j\}_{j\in\mathbb{Z}},\ \eta_j = \begin{cases} 1, & \text{if } j \text{ is occupied,} \\ 0, & \text{if } j \text{ is empty.} \end{cases}$$



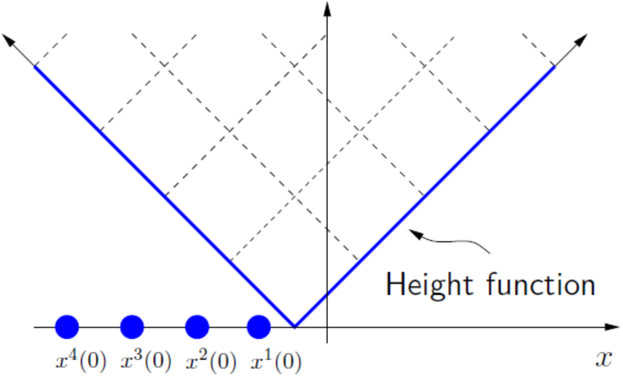
- **Dynamics**

  Independently, particles jump on the right site with rate 1, provided the right is empty.



⇒ Particles are ordered: position of particle n is $x^n(t)$

- Step initial condition is $x^n(0) = -n$, $n \geq 1$.



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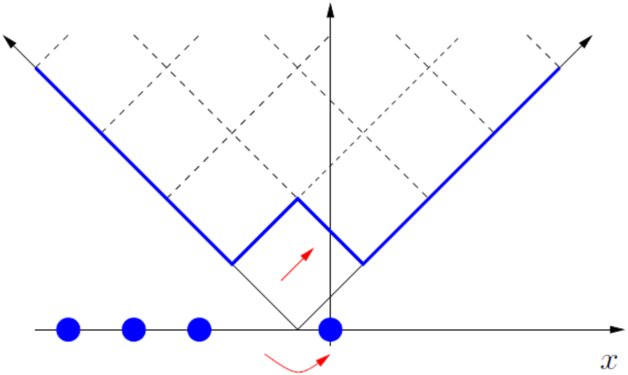

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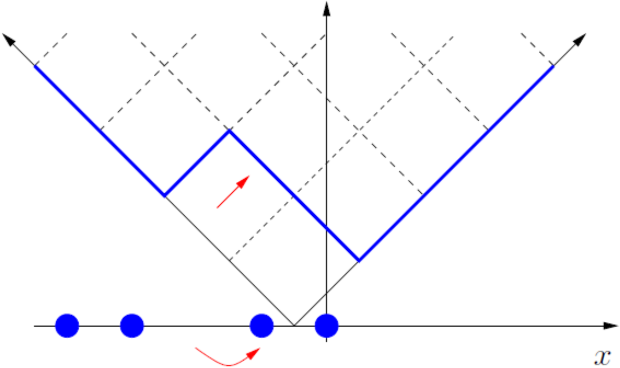

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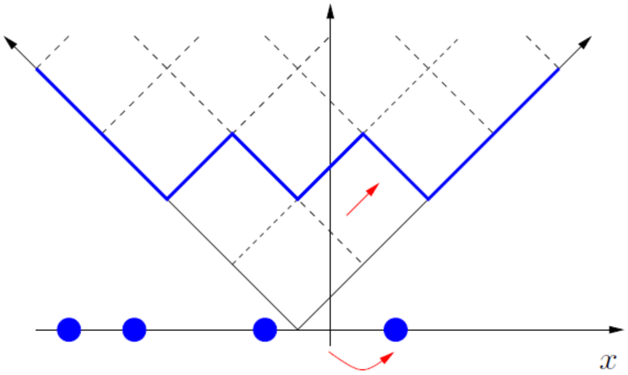


$\Rightarrow$ Particles are ordered: position of particle n is $x^n(t)$

- Step initial condition is $x^n(0) = -n$, $n \geq 1$.

Some known asymptotic results:

- law of large number: $\lim_{t\to\infty} x^{\eta t}(t)/t = 1 - 2\sqrt{\eta}$, $\eta \in [-1, 1]$
  Rost'81
- the fluctuations of particles are $\mathcal{O}(t^{1/3})$ and (GUE) Tracy-Widom distributed
- As a process in n, it converges to the Airy_2 process on the $(t^{2/3}, t^{1/3})$ scale
  Johansson'03 (LPP); Borodin,Ferrari'07 (TASEP)

KPZ scaling theory

- Given a height function of a model in the Kardar-Parisi-Zhang universality class in one-dimension: $x \mapsto h(x,t)$ (example: $n \mapsto x^n(t)$)
- Deterministic limit shape

$$h_{\rm ma}(\xi) = \lim_{t\to\infty} h(\xi t, t)/t$$

- Stationary spatial diffusivity

$$A = \lim_{x\to\infty} \frac{\lim_{t\to\infty} \mathrm{Var}(h(\xi t, t) - h(\xi t + x, t))}{|x|}$$

- Define further

$$\lambda = h''_{\rm ma}(\xi) \quad \text{and} \quad \Gamma = |\lambda| A^2$$

- Rescaled process

$$h_t^{\rm resc}(u) := \frac{h(\xi t + u t^{2/3}, t) - t h_{\rm ma}(\xi + u t^{-1/3})}{t^{1/3}}.$$

- Rescaled process

$$h_t^{\mathrm{resc}}(u) := \frac{h(\xi t + u t^{2/3}, t) - t h_{\mathrm{ma}}(\xi + u t^{-1/3})}{t^{1/3}}.$$

- If $h''_{\mathrm{ma}}(\xi) \neq 0$, one expects the following:

$$\lim_{t\to\infty} h_t^{\mathrm{resc}}(u) = \mathrm{sgn}(\lambda)(\Gamma/2)^{1/3} \mathcal{A}_2 \left(\frac{Au}{2\Gamma^{2/3}}\right)$$

where $\mathcal{A}_2$ is the Airy$_2$ process Prähofer,Spohn'02

- For flat interfaces (i.e., if $h''(\xi) = 0$) one has similar formulas but with either the Airy$_1$ process

Sasamoto'05; Borodin,Ferrari,Prähofer,Sasamoto'06

or the Airy$_{\mathrm{stat}}$ depending on the initial conditions

Baik,Ferrari,Péché'09

- With Alexei Borodin, in Anisotropic growth of random surfaces in 2+1 dimensions (arXiv:0804.3035), we introduced and studied a model of interacting particles in $2+1$-dimensions
- In discrete time, we have either *parallel update* or *sequential update*
- A discrete time *parallel update* includes (as different space-time projections) the Aztec diamond and the discrete time TASEP simultaneously

- The state space of our model is the Gelfand-Tsetlin pattern

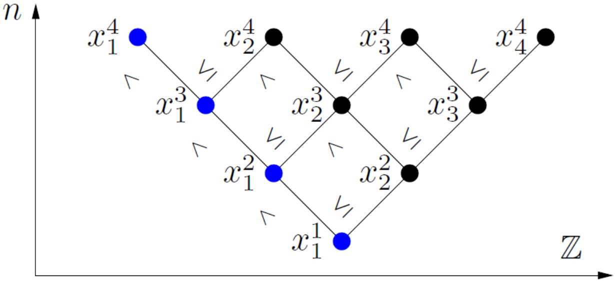
$$\mathrm{GT}_N = \{X^N = (x^1, \ldots, x^N); x^n = (x_1^n, \ldots, x_n^n) \,|\, x^n \prec x^{n+1}, \forall n\}$$

where

$$x^n \prec x^{n+1} \Leftrightarrow x_1^{n+1} < x_1^n \leq x_2^{n+1} < x_2^n \leq \ldots < x_n^n \leq x_{n+1}^{n+1}$$

means that x^n and x^{n+1} interlace.

- x^n is the called configuration at level n



- The Markov chain at level n (discrete time) is given by $x_1^n, \ldots, x_n^n$ being one-sided random walk conditioned to stay forever in

$$W_n = \{x^n \in \mathbb{Z}^n \,|\, x_1^n < x_2^n < \ldots < x_n^n\}.$$

- It is the Doob h-transform of the free walk with h function the Vandermonde determinant

$$\Delta_n(x^n) = \prod_{1 \leq i < j \leq n} (x_j^n - x_i^n),$$

i.e., it has the one-time transition probability given by

$$P_n(x^n, y^n) = \frac{\Delta_n(y^n)}{\Delta_n(x^n)} \det(P(x_i^n, y_j^n))_{i,j=1}^n$$

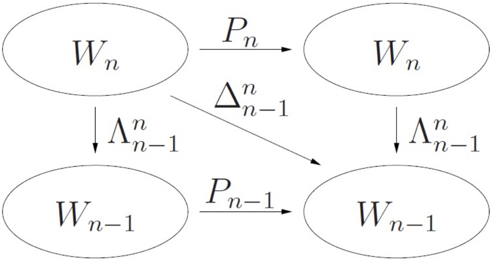
with $P(x,y) = p\delta_{y,x+1} + (1-p)\delta_{y,x}$.

- The chain at fixed time t is the one that, given x^N, it generates the uniform measure on the interlacing configurations:

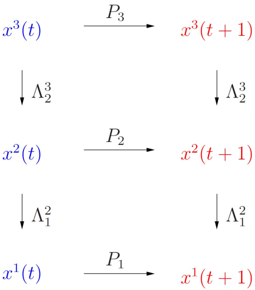
$$\begin{aligned}\Lambda^n_{n-1}(x^n,x^{n-1}) &:= \mathbb{P}(x^{n-1}|x^n) \\ &= \frac{\#\mathrm{GT}_{n-1} \text{ with given } x^{n-1}}{\#\mathrm{GT}_n \text{ with given } x^n} \mathbb{1}_{x^{n-1}\prec x^n} \\ &= (n-1)! \frac{\Delta_{n-1}(x^{n-1})}{\Delta_n(x^n)} \mathbb{1}_{x^{n-1}\prec x^n}\end{aligned}$$

- The key property used below is the intertwining property of the chains: Diaconis, Fill '90

$$\Delta^n_{n-1} := P_n \Lambda^n_{n-1} = \Lambda^n_{n-1} P_{n-1}$$



- The sequential update is the following:
  1. $x^1(t) \to x^1(t+1)$ according to $P_1(x^1(t), x^1(t+1))$,
  2. $x^2(t) \to x^2(t+1)$ to be the middle point of the chain $(P_2 \circ \Lambda_1^2)(x^2(t), x^1(t+1))$
  3. and so on



- Projection on $\{x_1^1, x_1^2, \ldots, x_1^N\}$ is TASEP in discrete time with sequential update

- There is a class of measure which form is invariant under P_Λ^N. Let $\mu_N(x^N)$ be a probability measure on W_N and define

$$M_N(X^N) := \mu_N(x^N)\Lambda_{N-1}^N(x^N,x^{N-1})\cdots\Lambda_1^2(x^2,x^1).$$

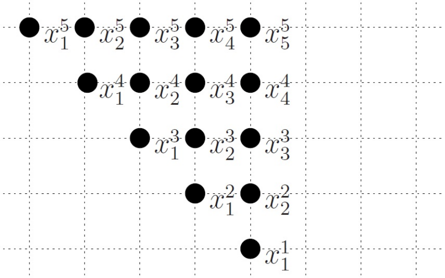
Then, applying t times P_Λ^N we have

$$(M_N(P_\Lambda^N)^t)(Y^N) = (\mu_N(P_N)^t)(y^N)\Lambda_{N-1}^N(y^N,y^{N-1})\cdots\Lambda_1^2(y^2,y^1)$$

- This is a consequence of the intertwining properties of the Markov chains!

- Consider further the "packed" initial condition: $x_k^n(0) = -n+k$, $1 \le k \le n \le N$. One can see that it can be written as above with μ_N of the form

$$\mu_N(x^N) = \Delta_N(x^N)\det(\Psi_j(x_i^N,0))_{i,j=1}^N.$$



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$$\mu_N(x^N) = \Delta_N(x^N)\det(\Psi_j(x_i^N,0))_{i,j=1}^N.$$

$\Rightarrow$ The measure at time t has the form

$$\prod_{n=1}^{N-1} \mathbb{1}_{[x^n \prec x^{n+1}]} \det(\Psi_j(x_i^N,t))_{i,j=1}^N$$

$\Rightarrow$ The measure at fixed level N and times $t_1 < \ldots < t_m$ has the form

$$\det(\Psi_j(x_i^N(t_1),t_1))_{i,j=1}^N \prod_{k=1}^{m-1} \det(P_{t_k,t_{k+1}}(x^N(t_k),x^N(t_{k+1}))\Delta_N(x^N(t_m))$$

- Measure of this form have determinantal correlations as they are conditional L-ensembles Borodin, Rains'06

Correlation structure of the blue lozenges / particles

Theorem (arXiv:0804.3035)

Consider any N triples (x_j, n_j, t_j) such that

$$t_1 \le t_2 \le \ldots \le t_N, \quad n_1 \ge n_2 \ge \ldots \ge n_N.$$

Then,

$$\mathbb{P}\Big(\text{at each } (x_j, n_j, t_j), j = 1, \ldots, N, \\ \text{there exists a blue lozenge / particle}\Big) \\ = \det[K(x_i, n_i, t_i; x_j, n_j, t_j)]_{1 \le i,j \le N}$$

for an explicit kernel K.

Correlation structure of the three types of lozenges

Theorem (arXiv:0804.3035)

Consider any N triples (x_j, n_j, t_j) such that

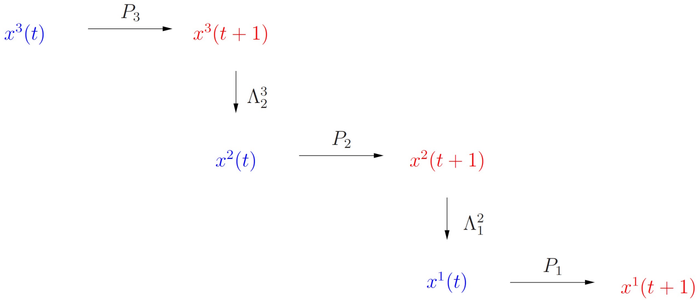
$$t_1 \leq t_2 \leq \ldots \leq t_N, \quad n_1 \geq n_2 \geq \ldots \geq n_N.$$

Then,

$$\mathbb{P}\Big(at\ each\ (x_j,n_j,t_j), j=1,\ldots,N, \\ there\ exists\ a\ lozenge\ of\ color\ c_j\Big) \\ = \det[\tilde{K}(x_i,n_i,t_i,c_i;x_j,n_j,t_j,c_j)]_{1\leq i,j\leq N}$$

for an explicit kernel $\tilde{K}$.

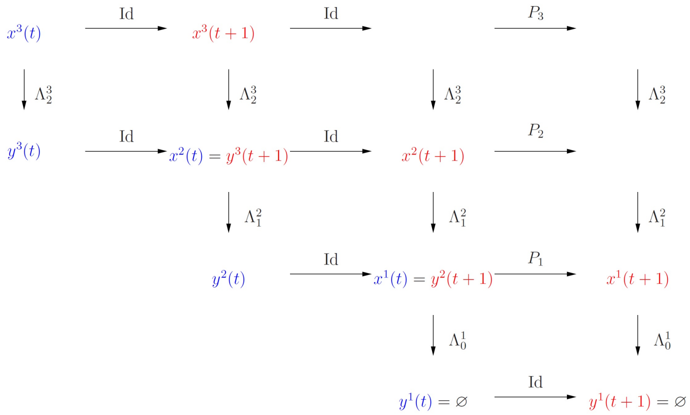
- The parallel update is the following $x^n(t) \to x^n(t+1)$ to be the middle point of the chain $(P_n \circ \Lambda^n_{n-1})(x^n(t), x^{n-1}(t))$



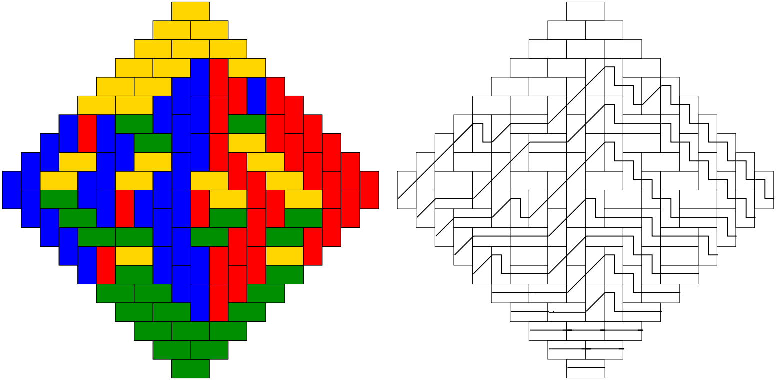
- Projection on $\{x^1_1, x^2_1, \ldots, x^N_1\}$ is TASEP in discrete time with parallel update

A $2+1$-dimensional model - parallel update

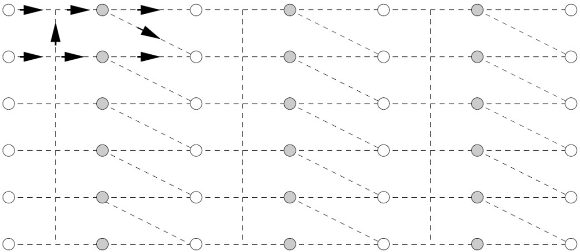
- This particle system is tightly related with the Aztec diamond:
  1. Start with packed initial condition:
  $$x^1 = (-1), \quad x^2 = (-2,-1), \quad x^3 = (-3,-2,-1).$$
  2. Extend our configuration space to:



- Aztec diamond and line ensembles:

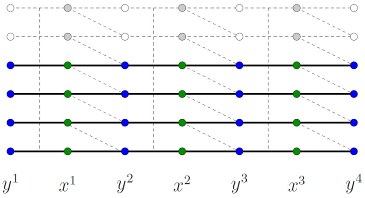


- A simple transformation of the Aztec line ensembles give a set of non-intersecting line ensembles on the following LGV graph with uniform weights



- Below we consider line ensembles with weight α on vertical segments

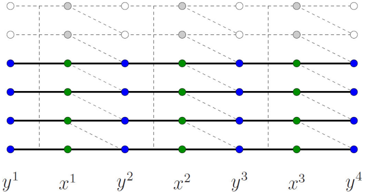
- Possible configurations with their weights.



Time $t=0$
Weight 1
Probability 1

- The red points are the only that are not fixed by the boundary conditions: they form the particle process described above.

- Possible configurations with their weights.



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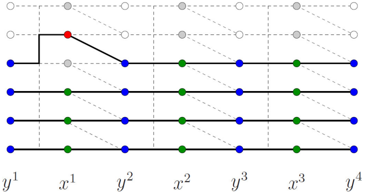

Time $t = 1$
Weight α
Probability p

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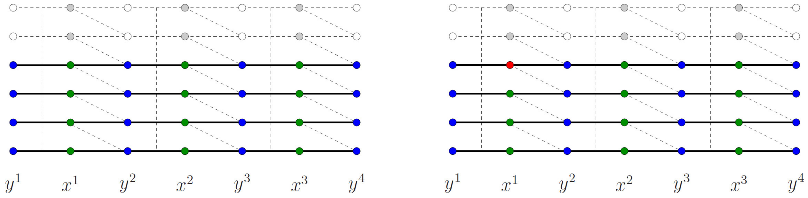
- Possible configurations with their weights.



Time $t = 0$
Weight 1
Probability 1

Time $t = 1$
Weight 1
Probability $1 - p$

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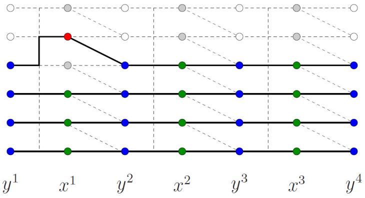
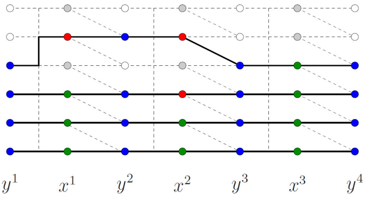
- Possible configurations with their weights.



Time $t = 1$
Weight α
Probability p



Time $t = 2$
Weight α
Probability $p \cdot (1-p)^2$

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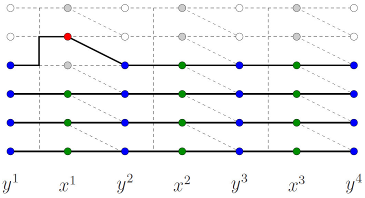
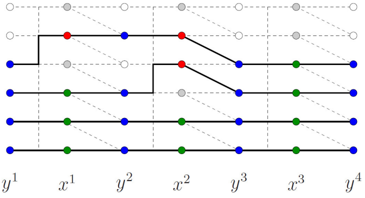
- Possible configurations with their weights.



Time $t=1$
Weight α
Probability p



Time $t=2$
Weight α^2
Probability $p\cdot(1-p)p$

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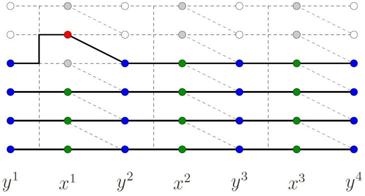
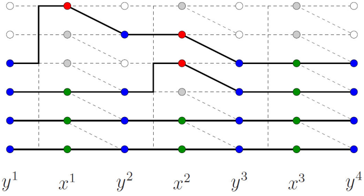
- Possible configurations with their weights.



Time $t=1$
Weight α
Probability p



Time $t=2$
Weight α^3
Probability $p \cdot p^2$

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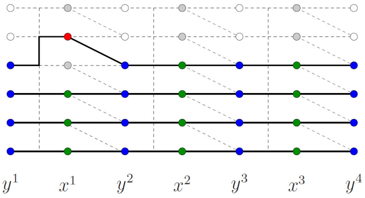
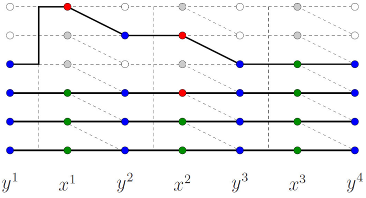
- Possible configurations with their weights.



Time $t=1$
Weight α
Probability p



Time $t=2$
Weight α^2
Probability $p \cdot p(1-p)$

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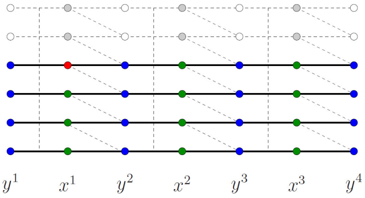
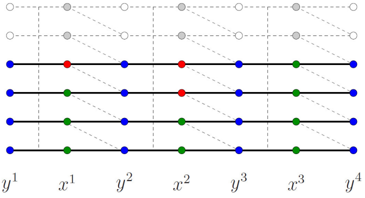
- Possible configurations with their weights.



Time $t=1$
Weight 1
Probability $1-p$



Time $t=2$
Weight 1
Probability $(1-p)\cdot(1-p)^2$

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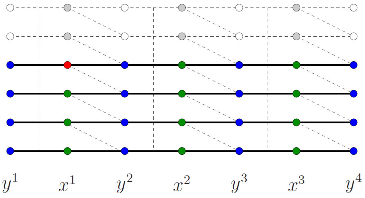
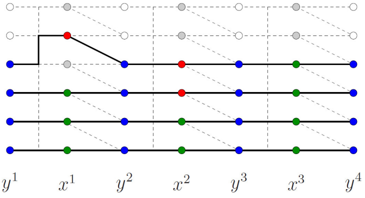
- Possible configurations with their weights.



Time $t=1$
Weight 1
Probability $1-p$



Time $t=2$
Weight α
Probability $(1-p)\cdot p(1-p)$

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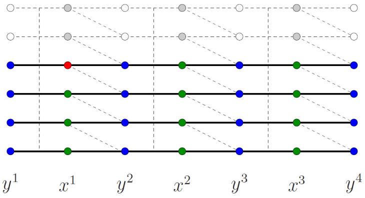
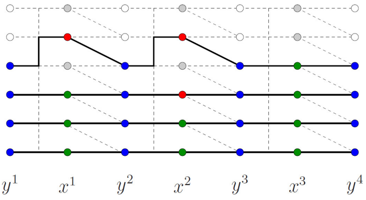
- Possible configurations with their weights.



Time $t=1$
Weight 1
Probability $1-p$



Time $t=2$
Weight α^2
Probability $(1-p)\cdot p^2$

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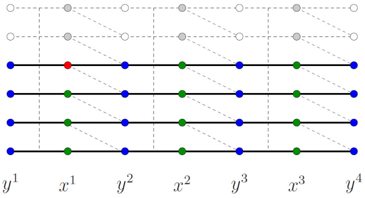
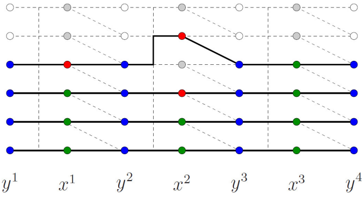
- Possible configurations with their weights.



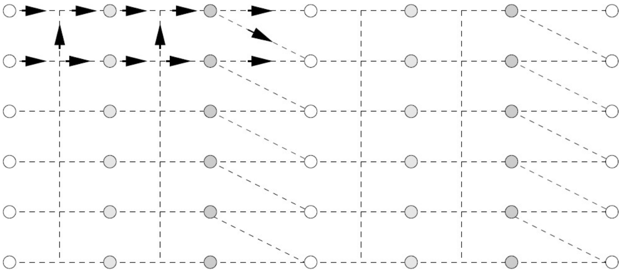
Time $t = 1$
Weight 1
Probability $1 - p$



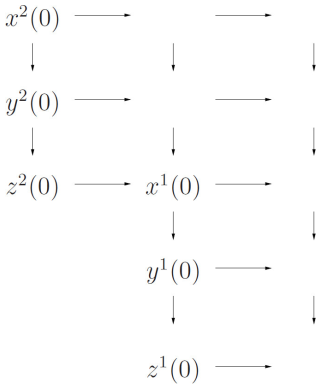
Time $t = 2$
Weight α
Probability $(1-p)\cdot(1-p)p$

- The red points are the only that are not fixed by the boundary conditions: they form the particle process described above.

- Consider a simple generalization of the line ensembles by staying this time to this LGV graph



- The previous example fits in a dynamics with the following scheme



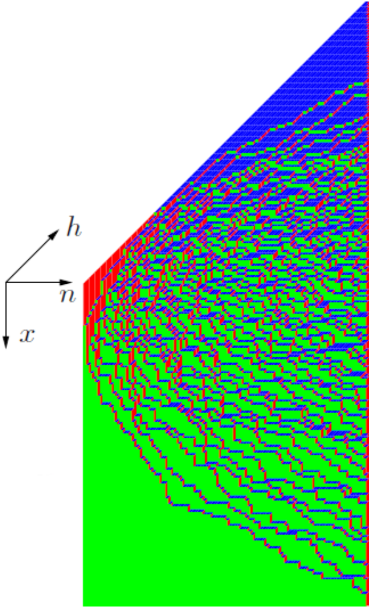
Macroscopic parametrization:

- $x = [-\eta L + \nu L]$
- $n = [\eta L]$
- $t = \tau L$

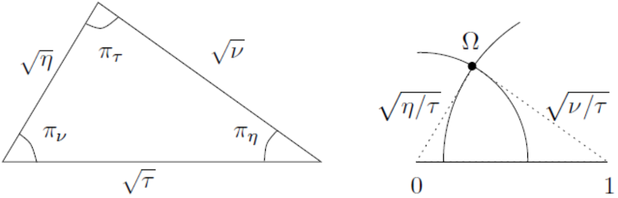
for a $L \gg 1$.

Asymptotic domain with "irregular" tiling (bordered by facets)

$$\mathcal{D} = \{(\nu,\eta,\tau), |\sqrt{\tau}-\sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau}+\sqrt{\eta}\}$$



Bulk: $\mathcal{D} = \{(\nu, \eta, \tau) \in \mathbb{R}^3_+, |\sqrt{\tau} - \sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau} + \sqrt{\eta}\}$



Map $\Omega : \mathcal{D} \to \mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$

Kenyon'04

- Ω is the critical point in the steep descent analysis of the correlation kernel!
- $(\pi_\nu/\pi, \pi_\eta/\pi, \pi_\tau/\pi)$ are the **frequencies** of the three types of lozenge tilings: Blue for π_η, Red for π_τ, Green for π_ν

- Limit shape:

$$\bar{h}(\nu,\eta,\tau) := \lim_{L\to\infty} \frac{\mathbb{E}(h((\nu-\eta)L,\eta L,\tau L))}{L} = \int_{\nu}^{(\sqrt{\tau}+\sqrt{\nu})^2} \frac{\pi_\eta(\nu',\eta,\tau)}{\pi}\mathrm{d}\nu'$$

- The slopes are

$$\frac{\partial \bar{h}}{\partial \nu} = -\frac{\pi_\eta}{\pi}, \quad \frac{\partial \bar{h}}{\partial \eta} = 1 - \frac{\pi_\nu}{\pi}$$

- Growth velocity:

$$\frac{\partial \bar{h}}{\partial \tau} = \frac{\sin(\pi_\nu)\sin(\pi_\eta)}{\pi \sin(\pi_\tau)} = \frac{\mathrm{Im}(\Omega)}{\pi}$$

Theorem (arXiv:0804.3035)

For all $(\nu, \eta, \tau) \in \mathcal{D}$, denote $\kappa = (\nu - \eta, \eta, \tau)$. We have moment convergence of

$$\lim_{L\to\infty} \frac{h(\kappa L) - \mathbb{E}(h(\kappa L))}{\sqrt{c \ln L}} = \xi \sim \mathcal{N}(0,1)$$

with $c = 1/(2\pi^2)$ is independent of the macroscopic position in $\mathcal{D}$.

Theorem (arXiv:0804.3035)

Consider any (disjoints) N triples $\kappa_j = (\nu_j - \eta_j, \eta_j, \tau_j)$, with $(\nu_j, \eta_j, \tau_j) \in \mathcal{D}$,

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_N \quad \eta_1 \geq \eta_2 \geq \ldots \geq \eta_N.$$

Set $H_L(\kappa) := \sqrt{\pi}\,(h(\kappa L) - \mathbb{E}(h(\kappa L)))$. Then,

$$\lim_{L\to\infty} \mathbb{E}(H_L(\kappa_1)\cdots H_L(\kappa_N)) = \begin{cases} 0, & \textit{odd } N, \\ \sum\limits_{\textit{pairings } \sigma} \prod\limits_{j=1}^{N/2} G(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}), & \textit{even } N, \end{cases}$$

with $G(z,w) = -(2\pi)^{-1} \ln |(z-w)/(z-\bar{w})|$ is the Green function of the Laplacian on $\mathbb{H}$ with Dirichlet boundary conditions.