Extremality and dynamically defined measures

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References

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Very well approximable vectors

Definition

A vector $\mathbf{x} \in \mathbb{R}^d$ is *very well approximable* if there exists $\varepsilon > 0$ such that for infinitely many $\mathbf{p}/q \in \mathbb{Q}^d$,

$$\left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| \leq \frac{1}{q^{1+1/d+\varepsilon}}.$$

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Example

Roth's theorem states that no algebraic irrational number in $\mathbb{R}$ is very well approximable. Its higher-dimensional generalization (a corollary of Schmidt's subspace theorem) says that an algebraic vector in $\mathbb{R}^d$ is very well approximable if and only if it is contained in an affine rational subspace of $\mathbb{R}^d$.

Dynamical interpretation

Theorem (Kleinbock–Margulis '99)

Let

$$g_t = \begin{bmatrix} e^{t/d} I_d & \\ & e^{-t} \end{bmatrix}, \qquad u_{\mathbf{x}} = \begin{bmatrix} I_d & -\mathbf{x} \\ & 1 \end{bmatrix},$$

$$\Lambda_* = \mathbb{Z}^{d+1} \in \Omega_{d+1} = \{\text{unimodular lattices in } \mathbb{R}^{d+1}\}.$$

Then $\mathbf{x}$ *is very well approximable if and only if*

$$\limsup_{t\to\infty} \frac{1}{t} \mathrm{dist}_{\Omega_{d+1}}(\Lambda_*, g_t u_{\mathbf{x}} \Lambda_*) > 0.$$

Extremal measures

A measure on $\mathbb{R}^d$ is called *extremal* if it gives full measure to the set of not very well approximable vectors.

Example (Corollary of Borel–Cantelli)

Lebesgue measure on $\mathbb{R}^d$ is extremal.

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Conjecture (Sprindžuk '80, proven by Kleinbock–Margulis '98)

Lebesgue measure on any real-analytic manifold not contained in an affine hyperplane is extremal.

Extremality and dynamically defined measures: First results

Theorem (Klenbock–Lindenstrauss–Weiss '04)

Let Λ be the limit set of a finite iterated function system generated by similarities and satisfying the open set condition, and let $\delta = \dim_H(\Lambda)$. Suppose that Λ is not contained in any affine hyperplane. Then $\mathcal{H}^\delta \upharpoonright \Lambda$ is extremal.

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Theorem (Urbański '05)

Same is true if "similarities" is replaced by "conformal maps", and if $\mathcal{H}^\delta \upharpoonright \Lambda$ is replaced by "the Gibbs measure of a Hölder continuous potential function".

Extremality and dynamically defined measures: First results

Theorem (Stratmann–Urbański '06)

Let G be a convex-cocompact Kleinian group whose limit set is not contained in any affine hyperplane. Then the Patterson–Sullivan measure of G is extremal.

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Theorem (Urbański '05 + Markov partition argument)

Let $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a hyperbolic (i.e. expansive on its Julia set) rational function, let $\phi : \widehat{\mathbb{C}} \to \mathbb{R}$ be a Hölder continuous potential function, and let μ_ϕ be the corresponding Gibbs measure. If $\mathrm{Supp}(\mu_\phi)$ is not contained in an affine hyperplane, then μ_ϕ is extremal.

Friendly and absolutely friendly measures

These theorems in fact all prove a stronger condition than extremality, namely *friendliness*.

Definition (Kleinbock–Lindenstrauss–Weiss '04)

A measure μ is called *friendly* (resp. *absolutely friendly*) if:

- μ is doubling and gives zero measure to every hyperplane.
- There exist $C_1, \alpha > 0$ such that for every ball $B = B(\mathbf{x}, \rho)$ with $\mathbf{x} \in \mathrm{Supp}(\mu)$, for every $0 < \beta \leq 1$, and for every hyperplane $\mathcal{L} \subseteq \mathbb{R}^d$,

  $$\mu\big(\mathcal{N}(\mathcal{L}, \beta \operatorname*{ess\,sup}_B d(\cdot, \mathcal{L})) \cap B\big) \leq C_1 \beta^\alpha \mu(B) \text{ (decaying)}$$

  resp.

  $$\mu\big(\mathcal{N}(\mathcal{L}, \beta\rho) \cap B\big) \leq C_1 \beta^\alpha \mu(B) \text{ (absolutely decaying)}$$

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Theorem (Kleinbock–Lindenstraus–Weiss '04)

Every friendly measure is extremal.

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If $\Phi : \mathbb{R}^k \to \mathbb{R}^d$ is a real-analytic embedding whose image is not contained in any affine hyperplane, then Φ sends absolutely friendly measures to friendly measures.

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Theorem (Folklore)

If $\delta > d - 1$, then every Ahlfors δ-regular measure on $\mathbb{R}^d$ is absolutely friendly.

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Philosophical meta-theorem: Every Ahlfors regular "nonplanar" measure is absolutely friendly.

Philosophical issues with friendliness/absolute friendliness

Although we have seen several measures from dynamics which are friendly or absolutely friendly, it seems that "most" such measures are not friendly. Intuitively, this is because the friendliness condition compares the measures of sets on similar length scales, while for any given dynamical system, the behavior of a measure at a given length scale may be heavily dependent on location.

Philosophical issues with friendliness/absolute friendliness

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Extremal measures which are not necessarily friendly

All subsequent results are from Das–Fishman–S.–Urbański (preprint 2015) unless otherwise noted.

Theorem

If $\delta > d - 1$, then every exact dimensional measure on $\mathbb{R}^d$ of dimension δ is extremal.

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Definition

A measure μ is called *exact dimensional of dimension δ* if for μ-a.e. $\mathbf{x} \in \mathbb{R}^d$,

$$\lim_{\rho \searrow 0} \frac{\log \mu(B(\mathbf{x}, \rho))}{\log \rho} = \delta.$$

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Example (Barreira–Pesin–Schmeling '99)

Any measure ergodic, invariant, and hyperbolic with respect to a diffeomorphism is exact dimensional.

Invariant measures of one-dimensional dynamical systems

When $d = 1$, $d - 1 = 0$, so every exact dimensional measure on $\mathbb{R}$ of positive dimension is extremal.

Theorem (Hofbauer '95)

Let $T : [0,1] \to [0,1]$ be a piecewise monotonic transformation whose derivative has bounded p-variation for some $p > 0$. Let μ be a measure on $[0,1]$ which is ergodic and invariant with respect to T. Let $h(\mu)$ and $\chi(\mu)$ denote the entropy and Lyapunov exponent of μ, respectively. If $\chi(\mu) > 0$, then μ is exact dimensional of dimension

$$\delta(\mu) = \frac{h(\mu)}{\chi(\mu)}.$$

So if $h(\mu) > 0$, then μ is extremal.

Positive entropy assumption

The positive entropy assumption is necessary, as shown by the following example:

Theorem

Let $T : X \to X$ be a hyperbolic toral endomorphism, where $X = \mathbb{R}^d/\mathbb{Z}^d$ (e.g. $Tx = nx \pmod 1$ for some $n \geq 2$). Let $\mathbb{M}_T(X)$ be the space of T-invariant probability measures on X. Then the set of non-extremal measures is comeager in $\mathbb{M}_T(X)$.

Gibbs states of CIFSes

Theorem

Fix $d \in \mathbb{N}$, and let $(u_a)_{a\in A}$ be an irreducible CIFS on $\mathbb{R}^d$. Let $\phi : A^{\mathbb{N}} \to \mathbb{R}$ be a summable locally Hölder continuous potential function, let μ_ϕ be a Gibbs measure of ϕ, and let $\pi : A^{\mathbb{N}} \to \mathbb{R}^d$ be the coding map. Suppose that the Lyapunov exponent

$$\chi_{\mu_\phi} := \int \log(1/|u'_{\omega_1}(\pi \circ \sigma(\omega))|)\, \mathrm{d}\mu_\phi(\omega) \tag{1}$$

is finite. Then $\pi_*[\mu_\phi]$ is quasi-decaying.

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The improvements on Urbański '05 are twofold:

- The CIFS can be infinite, as long as the Lyapunov exponent is finite.
- The open set condition is no longer needed.

Finite Lyapunov exponent assumption

The necessity of the finite Lyapunov exponent assumption is demonstrated by the following example:

Theorem (Fishman–S.–Urbański '14)

There exists a set $I \subseteq \mathbb{N}$ such that if μ is the conformal measure of the CIFS $(u_n(x) = \frac{1}{n+x})_{n\in I}$, then μ is not extremal.

Finite Lyapunov exponent assumption

The necessity of the finite Lyapunov exponent assumption is demonstrated by the following example:

Theorem (Fishman–S.–Urbański '14)

There exists a set $I \subseteq \mathbb{N}$ such that if μ is the conformal measure of the CIFS $(u_n(x) = \frac{1}{n+x})_{n \in I}$, then μ is not extremal.

Another connection between the finite Lyapunov exponent condition and extremality appears in the following theorem:

Theorem (Fishman–S.–Urbański '14)

If μ is a probability measure on $[0,1] \setminus \mathbb{Q}$ invariant with finite Lyapunov exponent under the Gauss map, then μ is extremal.

Patterson–Sullivan measures

Theorem

Let G be a geometrically finite group of Möbius transformations of $\mathbb{R}^d$ which does not preserve any affine hyperplane. Then the Patterson–Sullivan measure of G is extremal. If G also does not preserve any sphere, then the Patterson–Sullivan measure is friendly, and is absolutely friendly if and only if all cusps have maximal rank.

Remark

The first part of this theorem (extremality) is easier to prove than the second part (friendliness).

Gibbs states of rational functions via inducing

Definition (Inoquio-Renteria + Rivera-Letelier, '12)

If $T : X \to X$ is a dynamical system, then a potential function $\phi : X \to \mathbb{R}$ is called *hyperbolic* if there exists $n \in \mathbb{N}$ such that $\sup(S_n\phi) < P(T^n, S_n\phi)$, where $P(T, \phi)$ is the pressure of ϕ with respect to T.

Theorem

Let $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function, let $\phi : \widehat{\mathbb{C}} \to \mathbb{R}$ be a Hölder continuous hyperbolic potential function, and let μ_ϕ be the Gibbs measure of (T, ϕ). If the Julia set of T is not contained in an affine hyperplane, then μ_ϕ is extremal.

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Proof uses the "fine inducing" technique of Szostakiewicz–Urbański–Zdunik (preprint 2011).

Quasi-decaying and weakly quasi-decaying measures

As before, all our theorems prove more than extremality:

Definition

A finite measure μ is called *weakly quasi-decaying* (resp. *quasi-decaying*) if for every $\varepsilon > 0$ there exists $E \subseteq \mathbb{R}^d$ with $\mu(\mathbb{R}^d \setminus E) \leq \varepsilon$ such that for all $\mathbf{x} \in E$ and $\gamma > 0$, there exist $C_1, \alpha > 0$ such that for all $0 < \rho \leq 1$, $0 < \beta \leq \rho^\gamma$, and affine hyperplane $\mathcal{L} \subseteq \mathbb{R}^d$, if $B = B(\mathbf{x}, \rho)$ then

$$\mu\left(\mathcal{N}(\mathcal{L}, \beta \operatorname*{ess\,sup}_{B} d(\cdot, \mathcal{L})) \cap B \cap E\right) \leq C_1 \beta^\alpha \mu(B) \quad \text{(weak QD)}$$

resp.

$$\mu\left(\mathcal{N}(\mathcal{L}, \beta\rho) \cap B \cap E\right) \leq C_1 \beta^\alpha \mu(B) \quad \text{(QD)}$$

Differences between (weak) quasi-decay and (absolute) friendliness

The main difference between our conditions and those of Kleinbock–Lindenstrauss–Weiss is the restriction $\beta \leq \rho^\gamma$, which makes our condition cover a larger class of measures. It makes precise the earlier intuitive notion that any criterion on a measure should consider "significantly different length scales".

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The main difference between our conditions and those of Kleinbock–Lindenstrauss–Weiss is the restriction $\beta \leq \rho^\gamma$, which makes our condition cover a larger class of measures. It makes precise the earlier intuitive notion that any criterion on a measure should consider "significantly different length scales". Other differences between our conditions and KLW's are that we consider measure-theoretically valid bounds rather than bounds that hold uniformly, and that we do not assume that our measures are doubling. The reason we do not need a doubling assumption is that we prove an "almost doubling" criterion that holds for all measures on $\mathbb{R}^d$.

Differences between (weak) quasi-decay and (absolute) friendliness

The following implications hold:

$$\begin{array}{ccc} \text{Absolutely friendly} & \Rightarrow & \text{Friendly} \\ \Downarrow & & \Downarrow \\ \text{Quasi-decaying} & \Rightarrow & \text{Weakly quasi-decaying} \end{array}$$

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Also, the image of an absolutely friendly (resp. quasi-decaying) measure under a nondegenerate embedding is friendly (resp. weakly quasi-decaying).

Examples of measures in various categories

	Absolutely friendly	Friendly but not absolutely friendly	Not friendly
QD	• Patterson–Sullivan measures of convex-cocompact groups • Gibbs measures of finite IFSes and hyperbolic rational functions	• Patterson–Sullivan measures of geometrically finite groups which satisfy $k_{\min} < d - 1$	• Gibbs measures of nonplanar infinite IFSes and rational functions
WQD\QD	Impossible	• Lebesgue measures of nondegenerate manifolds	• Conformal measures of infinite IFSes which have invariant spheres
Extr\WQD	Impossible	Impossible	• Measures with finite Lyapunov exponent and zero entropy under the Gauss map
Not Extr	Impossible	Impossible	• Generic invariant measures of hyperbolic toral endomorphisms • Certain measures with infinite Lyapunov exponent under the Gauss map

Sketch of a proof

Theorem

Let $T : X \to X$ be a hyperbolic toral endomorphism, where $X = \mathbb{R}^d/\mathbb{Z}^d$ (e.g. $Tx = nx \pmod 1$ for some $n \geq 2$). Let $\mathbb{M}_T(X)$ be the space of T-invariant probability measures on X. Then the set of non-extremal measures is comeager in $\mathbb{M}_T(X)$.

Proof. For each $n \in \mathbb{N}$, let

$$U_n = \bigcup_{\substack{\mathbf{p}/q \in \mathbb{Q} \\ q \geq n}} B\left(\frac{\mathbf{p}}{q}, \frac{1}{q^n}\right),$$

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and let $\mathcal{U}_n$ be the set of all measures $\mu \in \mathbb{M}_T(X)$ such that $\mu(U_n) > 1 - 2^{-n}$.

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and let $\mathcal{U}_n$ be the set of all measures $\mu \in \mathbb{M}_T(X)$ such that $\mu(U_n) > 1 - 2^{-n}$. The sets U_n and $\mathcal{U}_n$ are both open. By definition, the set $G := \bigcap_n U_n$ contains only very well approximable numbers.

Sketch of a proof

By definition, the set $G := \bigcap_n U_n$ contains only very well approximable numbers. Thus since every measure in $\mathcal{G} := \bigcap_n \mathcal{U}_n$ gives full measure to G, it follows that no measure in $\mathcal{G}$ is extremal.

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Remark

This argument gives another proof that the set of measures with entropy zero is comeager in $\mathbb{M}_T(X)$.

The end