Quantum chaos and the thermodynamical formalism

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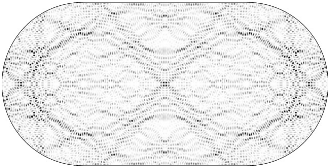
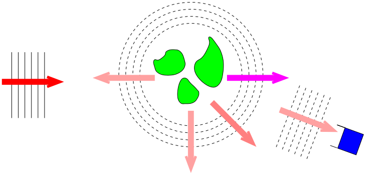
Outline

- quantum chaos on a compact manifold: structure of the high-frequency eigenstates
  - quantum ergodicity
  - a lower bound on the metric entropy (with N.Anantharaman)
- open quantum chaos: quantum scattering
  - quantum resonances, in the semiclassical regime
  - hyperbolic trapped sets (Axiom A)
  - "gap" in the resonance spectrum, in terms of a topological pressure (with M.Zworski)

In both problems, crucial role played by the hyperbolic dispersion of wavepackets.





Structure of chaotic eigenmodes

Quantum (unique?) ergodicity

Spectral geometry: spatial structure of vibration modes

Quantum particle propagating on (X, g) compact manifold, possibly with (piecewise smooth) boundary:

- Schrödinger equation $ih\partial_t\psi(t,x) = P_h\psi(t,x)$, with $P_h \stackrel{\rm def}{=} -h^2\Delta_X$.

Linear $\Longrightarrow$ relevant to consider the **spectrum** of the Laplacian: discrete spectrum $(\Delta_X + k_n^2)\psi_n = 0$ ($\Longleftrightarrow$ $(-h_n^2\Delta_X - 1)\psi_n = 0$)

> *What can we say about the spectrum $\{k_n\}$ and eigenmodes $\{\psi_n\}$ in the high-frequency limit $k_n \to \infty$? ($\Longleftrightarrow$ semiclassical limit $h_n \to 0$)*

Local Weyl's law: for any test function $f \in C^\infty(X)$,

$$\sum_{k_n \leq K} \int_X f(x)\,|\psi_n(x)|^2\,dx = C_d\,K^d \int_X f(x)\,dx + o(K^d),$$

On average, the eigenstates become equidistributed on X.

How about individual eigenstates?

Semiclassical analysis makes the connection with the underlying Hamiltonian dynamics: (broken) geodesic flow $\Phi^t : S^*X \to S^*X$.

Chaotic dynamics: Quantum Ergodicity

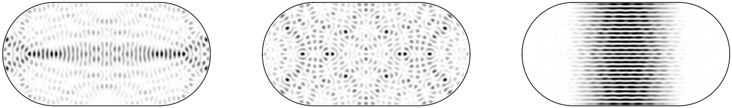
Quantum Chaos: preferably consider (X, g) s.t. the geodesic flow Φ^t has chaotic features.

Theorem (Quant. Ergod. [SCHNIRELMAN, ZELDITCH, COLIN DE VERDIÈRE...])

If Φ^t is ergodic on S^*X w.r.t. the Liouville measure, almost all the eigenmodes ψ_n become asymptotically equidistributed on X:

$$\langle \psi_{n_j}, f\psi_{n_j} \rangle_{L^2} \stackrel{j\to\infty}{\to} \frac{1}{\mathrm{Vol}(X)} \int_X f(x)\, dx$$

along subsequence of density 1.



k= 39.04516063 k= 39.22984274 k= 39.29160821

Qu: Can there be exceptional modes, for instance localizing along certain periodic geodesics?

[LINDENSTRAUSS'06]: X arithmetic surface of const. negative curvature and (ψ_n) “Hecke" eigenmodes: **Quantum Unique Ergodicity**.

[HASSELL'10]: for X a generic stadium billiard, $\exists$ bouncing-ball modes

Localization of high-frequency eigenstates: Semiclassical measures

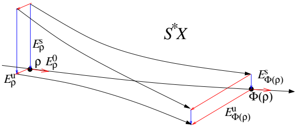
To connect with classical dynamics, lift the localization to phase space T^*X.

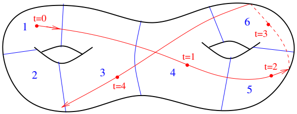
- $F(x,\xi) \in C_c^\infty(T^*X) \mapsto F(x, hD)$, pseudodiff. operator on X. Allows to test the localization of $\psi_n(x)$ both in position space and in Fourier space at the scale h^{-1} (microlocalization).
- Ex: the local plane wave $\psi_h(x) = a(x)\, e^{i\xi_0\cdot x/h}$ is microlocalized on the Lagrangian plane $\Lambda_{\xi_0} = \{(x,\xi_0),\ x \in \operatorname{supp} a\}$.
- Adapt "Planck's constant" h to ψ_n: $(-{h_n}^2\Delta - 1)\psi_n = 0$, so that ψ_n is microlocalized on $S^*X = \{(x,\xi)\ :\ |\xi| = 1\}$.
- Extracting subsequences, $\langle \psi_{n_j}, F(x, h_{n_j}D)\psi_{n_j}\rangle \stackrel{j\to\infty}{\to} \int_{T^*X} F\, d\mu_{sc}$, where μ_{sc} is called a semiclassical measure.
- Each μ_{sc} is a probability measure supported on S^*X, and is invariant through Φ^t. It represents the asymptotic phase space distribution of the subsequence (ψ_{n_j}).

$\Longrightarrow$ JOB FOR DYN. SYS.: describe the possible invariant measures of Φ^t.

Anosov flows: Entropy of semiclassical measures

Choose (X, g) with Anosov geodesic flow, e.g. with negative sectional curvature. Important quantity: unstable Jacobian $J_t^u(\rho) = |\det(d\Phi^t \restriction_{E_\rho^u})|$





Attempt to characterize the localization properties of eigenstates: study the metric entropy of the semiclassical measure μ_{sc}.

- partition of unity on S^*X: $1\!\!1_{S^*X} = \sum_{j=1}^J \pi_j$, $\pi_j = 1\!\!1_{V_j}$.
- Refined partitions: $\pi_{\alpha_0\cdots\alpha_{n-1}} = \pi_{\alpha_{n-1}} \circ \Phi^{n-1} \times \cdots \pi_{\alpha_1} \circ \Phi^1 \times \pi_{\alpha_0}$.
- $H_{KS}(\mu) = \lim_{n\to\infty} \frac{1}{n} H_n(\mu)$, where $H_n(\mu) = \sum_{|\boldsymbol{\alpha}|=n} -\mu(\pi_{\boldsymbol{\alpha}}) \log \mu(\pi_{\boldsymbol{\alpha}})$. Indicator of localization: μ very localized (e.g. $\mu = \delta_\gamma$) $\Longrightarrow H(\mu)$ small.
- If $\mu(\pi_{\boldsymbol{\alpha}}) \leq Ce^{-\beta|\boldsymbol{\alpha}|}$ when $|\boldsymbol{\alpha}| \to \infty$, then $H(\mu) \geq \beta$.

$\Longrightarrow$ can we show that $\mu_{sc}(\pi_{\boldsymbol{\alpha}}) \leq Ce^{-\beta|\boldsymbol{\alpha}|}$?

Quantizing the partition. Hyperbolic dispersion estimate

Smoothen and quantize π_j into $\Pi_j = \pi_j(x, hD)$, to form a quantum partition of unity: $Id = \sum_{j=1}^{J} \Pi_j$.

Π_j = microlocal quasiprojector on the phase space region V_j.

Refine the quantum partition using Schrödinger evolution $U^t = e^{-itP_h/h}$:

$$\Pi_{\boldsymbol{\alpha}} \stackrel{\text{def}}{=} U^{-n+1}\Pi_{\alpha_{n-1}} \cdots U^1 \Pi_{\alpha_1} U^1 \Pi_{\alpha_0}$$

- evolution of observables: $U^{-t} a(x, hD) U^t = a \circ \Phi^t(x, hD) + \mathcal{O}_t(h)$ (Egorov theorem)
- product of observables: $a(x, hD) b(x, hD) = (ab)(x, hD) + \mathcal{O}(h)$

$\Longrightarrow \quad \Pi_{\boldsymbol{\alpha}} = \pi_{\boldsymbol{\alpha}}(x, hD) + \mathcal{O}_n(h)$.

$\ominus$ correspondence breaks down when $V_{\boldsymbol{\alpha}}$ becomes "quantum", that is for $n > T_E = \frac{\log 1/h}{\lambda_{\max}}$ the Ehrenfest time.

$\oplus$ beyond T_E, exponential decay, governed by the unstable Jacobian along $\boldsymbol{\alpha}$-trajectories:

$$\|\Pi_{\boldsymbol{\alpha}}\|_{L^2 \to L^2} \leq \min\left(1, C h^{-(d-1)/2} J^u(\boldsymbol{\alpha})^{-1/2}\right)$$

Hyperbolic dispersion estimate.

Lower bounds on the entropy

Formally, the weight of ψ_h inside $V_{\boldsymbol{\alpha}}$ is $\|\Pi_{\boldsymbol{\alpha}}\psi_h\|^2$, which decays exponentially when $n > T_E$:

$$\|\Pi_{\boldsymbol{\alpha}}\psi_h\|^2 \leq h^{-(d-1)}\, e^{-n\Lambda_{\min}}$$

$\oplus$ lower bound on quantum entropy $H_n(\psi_h) \geq n\Lambda_{\min} - (d-1)|\log h^{-1}|$.

$\ominus$ for times $n \gg T_E$, impossible to relate $H_n(\mu_{sc})$ with $H_n(\psi_h)$.

We obtain a nontrivial bound by taking $n = 2T_E$:

Theorem ([Anantharaman'06,Anantharaman-N'07])

If Φ^t is Anosov, any semiclassical measure μ_{sc} satisfies

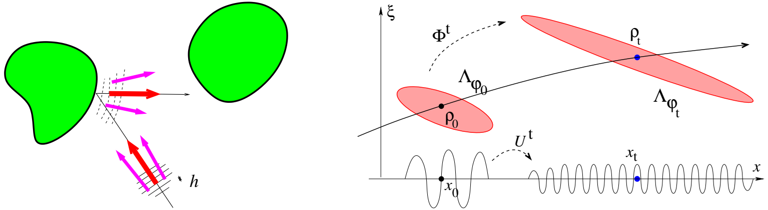
$$H(\mu_{sc}) \geq \int_{S^*X} \log J^u(\rho)\, d\mu_{sc}(\rho) - \frac{(d-1)\lambda_{\max}}{2}.$$

If X is 2-dim. with nonpositive curv., $H(\mu_{sc}) \geq \frac{1}{2} \int_{S^*X} \log J^u(\rho)\, d\mu_{sc}(\rho)$ [Rivière'10]

- (Ruelle: $H(\mu) \leq \int_{S^*X} \log J^u(\rho)\, d\mu(\rho)$, with equality iff $\mu = \mu_{Liouv}$).

- $\exists$ toy Anosov models (quantum maps) for which this lower bound is reached, $\mu_{sc} = \frac{1}{2}\delta_\gamma + \frac{1}{2}\mu_{Liouv}$ [Faure-N-DeBièvre'03].

Semiclassical propagation of Lagrangian states



A **Lagrangian state** $\psi_h(x) = a(x)e^{i\frac{\varphi(x)}{h}}$ is microlocalized on the Lagrangian leaf $\Lambda_\varphi = \{(x, d\varphi(x)),\ x \in \operatorname{supp} a\} \subset T^*X$.
Ex: local plane wave $a(x)e^{i\frac{\eta\cdot x}{h}}$ microlocalized on $\Lambda_\eta = \{(x,\eta),\ x \in \operatorname{supp} a\}$.

Lagrangian states enjoy a simple semiclassical evolution:

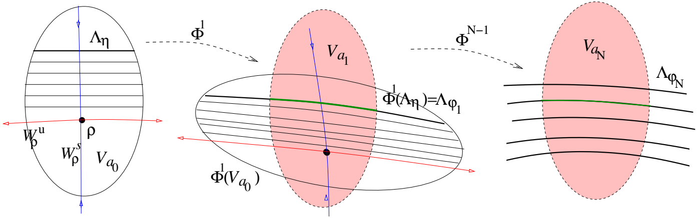
- $U^t(a\, e^{i\varphi/h}) = a_t e^{i\varphi_t/h} + \mathcal{O}(h)$, with $\Lambda_{\varphi_t} = \Phi^t(\Lambda_\varphi)$.
- the amplitude a_t is transported like a half-density:

$$a_t(x_t) = a(x_0)|\det(\partial x_t/\partial x_0)|^{-1/2}, \quad \text{where} \quad (x_t, d\varphi_t(x_t)) = \Phi^t(x_0, d\varphi(x_0))$$

- applying a pseudodiff $F(x, hD)$ only modifies the symbol:

$$[F(x,hD)\, ae^{i\varphi/h}](x) = F(x, d\varphi(x))\, a(x)e^{i\varphi(x)/h} + \mathcal{O}(h)$$

Proof of Hyperbolic dispersive estimate



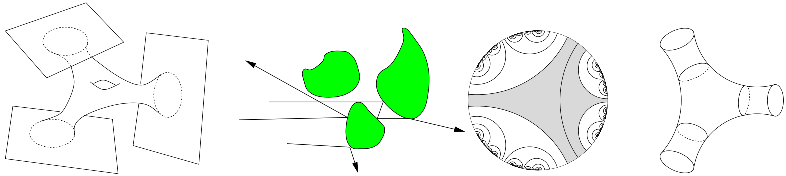
We want to show: $\|\Pi_{\alpha_{n-1}} \cdots U^1 \Pi_{\alpha_1} U^1 \Pi_{\alpha_0} \psi\|_{L^2} \lesssim h^{-\frac{d-1}{2}} J^u(\boldsymbol{\alpha}) \|\psi\|_{L^2}$

- Any state $\Pi_{\alpha_0}\psi$ can be "Fourier" expanded into $\Pi_{\alpha_0}\psi(x) = h^{-\frac{d-1}{2}} \int_I d\eta \, a(x) e^{i\frac{\eta\cdot x}{h}} \tilde{\psi}(\eta)$
- propagate individual Lagrangian states: $U^1(a\, e^{i\eta\cdot x/h}) = a_1 e^{i\varphi_1/h}$, with $\Lambda_{\varphi_1} = \Phi^1(\Lambda_\eta)$.
- the quasiprojector Π_1 cuts off the amplitude (norm reduction)
- propagate $a_1 e^{i\varphi_1/h}$ into $a_2 e^{i\varphi_2/h}$, then truncate, etc.
- Hyperbolicity $\Longrightarrow \Lambda_{\varphi_N}$ aligns along W^u, and $a_N \sim a_1 J^u(\alpha_1 \cdots \alpha_N)^{-1/2}$.
- linearity $\Longrightarrow \|\Pi_{\boldsymbol{\alpha}}\psi\| \lesssim h^{\frac{d-1}{2}} J^u(\alpha_1 \cdots \alpha_N)^{-1/2} \|\psi\|$. □

Open quantum chaos:

Chaotic scattering systems

Classical & Quantum scattering



Assume now that (X, g) is of infinite volume (and "nice" near infinity).

- (X, g) smooth, Euclidean near infinity.
- $X = \mathbb{R}^d \setminus$ smooth compact obstacles.
- $X = \Gamma\backslash\mathbb{H}^2$ with $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ *convex co-compact*.

• Geodesic flow $\Phi^t : S^*X \to S^*X$ may be complicated in the "interaction region".

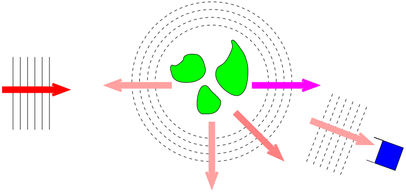
• Quantum particle still described by the Schrödinger equation

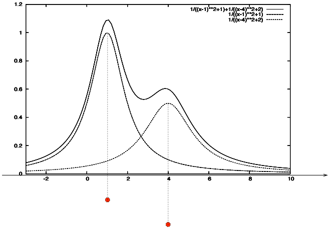
$$\psi(t) = U^t\psi(0), \qquad U^t = e^{-itP_h/h}, \qquad P_h = -h^2\Delta_X.$$

Quantum scattering $\rightsquigarrow$ resonances replace eigenvalues

Given $\psi_0 \in L^2_{comp}(X)$, we want to understand the long time evolution of $\psi(t) = U^t \psi_0$ (dispersion of the waves towards infinity).

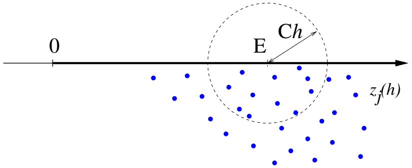
X of infinite volume $\Rightarrow \operatorname{Spec} P_h$ **absolutely continuous** on $[c_0 h^2, \infty)$.
Is that all?





Experimental spectra often feature **peaks**, called resonances.
Mathematically: discrete, complex, generalized eigenvalues of P_h.

Resonances in quantum scattering



P_h selfadjoint $\Longrightarrow (P_h - z)^{-1} : L^2 \to L^2$ bounded for $\{\operatorname{Im} z > 0\}$ ("physical sheet"), becomes unbounded as $\operatorname{Im} z \searrow 0$.

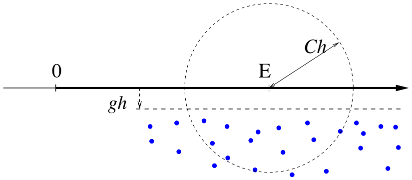
However, for any cutoff $\chi \in C_c^\infty(X)$, the truncated resolvent $\chi(P_h - z)^{-1}\chi$ can be meromorphically continued from $\{\operatorname{Im} z > 0\}$ to $\{\operatorname{Im} z < 0\}$.
Poles of finite multiplicities $\{z_j(h)\}$: resonances of P_h.

Each $z_j(h) \leftrightarrow$ **metastable state** $\psi_j(x)$ ($\notin L^2$), with **lifetime** $\tau_j = \frac{h}{2|\operatorname{Im} z_j|}$.
$\rightsquigarrow$ long-living resonance if $\operatorname{Im} z_j(h) = \mathcal{O}(h)$ (physically meaningful).

Can we give a sense to an expansion like:

$$\psi(t) = \sum_{z_j} c_j \, e^{-itz_j/h} \, \psi_j + rem. \ ? \qquad (\psi_j \text{ not in } L^2!)$$

Distribution of long living resonances



Resonances replace eigenvalues $\rightsquigarrow$ spectral questions:

- fixing $E > 0$, what do we know about the long-living resonances near E?
  How close are they from the real axis?
  How many are they?
- Applications to time evolution: correlation functions

$$\langle \varphi, e^{-itP(h)/h}\psi_0\rangle_{L^2} = \sum_{z_j} \langle \varphi, \psi_j\rangle \, e^{-itz_j/h} + rem., \qquad \varphi, \psi_0 \in C_c^\infty.$$

Semiclassical regime $\rightarrow$ how does the classical dynamics influence this distribution?

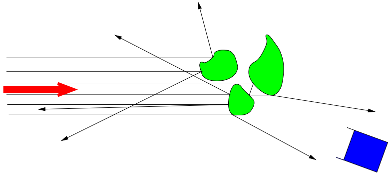
Distribution of resonances – Trapped set

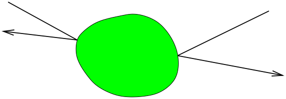
- most trajectories are transient, spend a finite time in the interaction region.
- there may exist trapped trajectories.

trapped set $\Gamma_E = \Gamma_E^+ \cap \Gamma_E^-$, $\Gamma_E^\pm = \{\rho \in p^{-1}(E),\ \Phi^t(\rho) \not\to \infty,\ t \to \mp\infty\}$.

Γ_E compact, flow-invariant.

Intuition: the distribution of the $\{z_j(h)\}$ near E depends on $\Phi^t \restriction_{\Gamma_E}$.





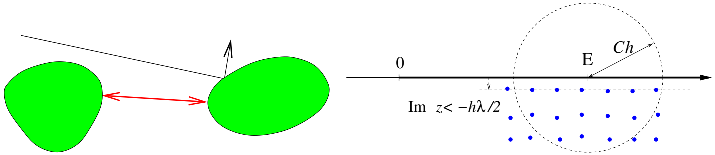
Ex. 1: $\Gamma_E = \emptyset$. $\Longrightarrow$ fast dispersion, NO long-living resonance

Γ_E a single hyperbolic orbit

The distribution of $\{z_j(h)\}$ near E depends on the classical trapped set Γ_E.

Ex. 2: $d = 2$, $\Gamma_E =$ 1 hyperbolic periodic orbit γ_E.

Can use a **Quantum Birkhoff Normal Form** for P_h near γ_E.



$\Longrightarrow$ resonances on a deformed half-lattice
[IKAWA'85,GÉRARD-SJÖSTRAND'87...]

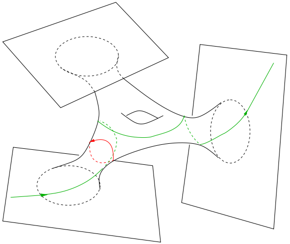
The **resonance gap** is determined by λ_E, the Lyapunov exponent of γ_E.

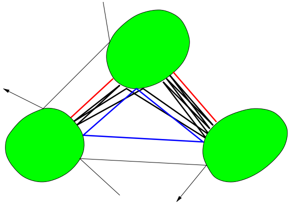
Γ_E a chaotic fractal set

Ex. 3: Γ_E a **fractal hyperbolic repeller**, with $\Phi^t_{\restriction \Gamma_E}$ Axiom A flow (unif. hyperb.)

Examples:

- (X, g) of negative curvature near Γ_E
- $N \geq 3$ convex obstacles in $\mathbb{R}^d$ with nonshadowing property
- $X = \Gamma \setminus \mathbb{H}^2$, with Γ convex co-compact.



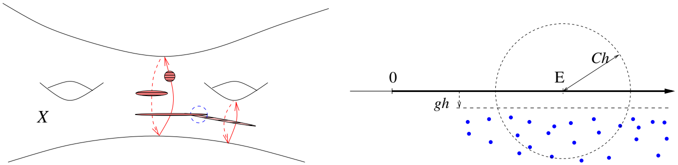


Γ_E “thin” enough: fast dispersion and resonance gap

Theorem ([Ikawa’88, Gaspard-Rice’89, N-Zworski’09])

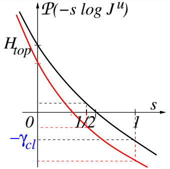
Assume Γ_E is hyperbolic, and thin enough so that $\mathcal{P}(-1/2\log J^u; \Gamma_E) < 0$. Then, in the limit $h \to 0$, all resonances in $D(E, Ch)$ satisfy

$$\frac{\operatorname{Im} z_j(h)}{h} \leq \mathcal{P}(-1/2\log J^u) + o(1)_{h\to 0}$$ *"resonance gap"*



$\oplus$ hyperbolic dispersion $\Longrightarrow$ wavepackets “leak away” from Γ_E.
$\ominus$ interferences between wavepackets on different trajectories may reduce the global leakage from Γ_E.
$\oplus$ if Γ_E is thin, interferences cannot completely suppress the leakage $\Longrightarrow$ lifetimes $\tau_j(h)$ are uniformly bounded.

Topological pressure at $1/2$



$\mathcal{P}(-\log J^u) = -\gamma_{cl} < 0$,
but $\mathcal{P}(-1/2\log J^u)$ can take both signs.

If $\dim X = 2$:
$\mathcal{P}(-1/2\log J^u) < 0 \iff \dim_H \Gamma_E < 2$

Proof of thm (sketch): Want to control the decay of $\Pi_\Gamma U^n\psi$ as $n\to\infty$.
Quantum partition of unity near Γ_E: $\Pi_\Gamma = \sum_j \Pi_j$.

- Decompose $\Pi_{\alpha_0}\psi(x) = h^{-\frac{d-1}{2}} \int_I d\eta\, a(x) e^{i\frac{\eta\cdot x}{h}}\, \tilde\psi(\eta)$

$$\Pi_\Gamma U^n(a\,e^{i\frac{\eta\cdot x}{h}}) \approx \sum_{|\boldsymbol{\alpha}|=n} U_{\boldsymbol{\alpha}}(a\,e^{i\frac{\eta\cdot x}{h}}) = \sum_{|\boldsymbol{\alpha}|=n} a_{\boldsymbol{\alpha}}\, e^{i\frac{\varphi_{\boldsymbol{\alpha}}}{h}}, \quad U_{\boldsymbol{\alpha}} = U^1\Pi_{\alpha_{n-1}}\cdots U^1 .$$

- Apply the *triangle inequality* (allows interferences):

$$\|\Pi_\Gamma U^n(a\,e^{i\frac{\eta\cdot x}{h}})\| \lesssim \sum_{|\boldsymbol{\alpha}|=n} \|U_{\boldsymbol{\alpha}}(a\,e^{i\frac{\eta\cdot x}{h}})\| \approx \sum_{|\boldsymbol{\alpha}|=n} J^u(\boldsymbol{\alpha})^{-1/2} \lesssim e^{n\mathcal{P}(-1/2\log J^u)}$$

- Sum over $\psi_\eta \rightsquigarrow$ extra factor $h^{-\frac{d-1}{2}} \leq e^{n\epsilon}$ if we take $n \gg \log h^{-1}$. $\square$

How sharp is the bound $\mathcal{P}(-1/2\log J^u)$? (cf. next 2 talks)

Are there partial cancellations in $\sum_{\boldsymbol{\alpha}\sim\Gamma_E} a_{\boldsymbol{\alpha}}\, e^{i\frac{\varphi_{\boldsymbol{\alpha}}}{h}}$?

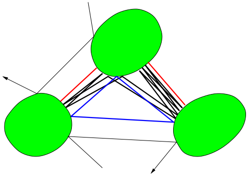
Need to control:
• the relative positions of the nearby leaves $\Lambda_{\varphi_{\boldsymbol{\alpha}}}$
• the relative phases between the $\varphi_{\boldsymbol{\alpha}}$.

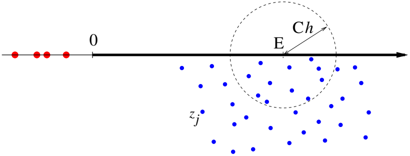
Most precise results obtained for $X=\Gamma\backslash\mathbb{H}^2$:
• the laminations are smooth.
• resonances of Δ_X correspond to zeros of the Selberg zeta function

- [NAUD'05] adapts Dolgopyat's method $\rightsquigarrow$ resonance gap increased by ϵ_1.
- Conjecture [JAKOBSON-NAUD'11]: at high frequency, $\frac{\operatorname{Im} z_j}{h}\leq-\frac{\gamma_{cl}}{2}+o(1)$.
- [DYATLOV-ZAHL'15, FAURE-WEICH'15, TSUJII'16]: quantitative predictions for ϵ_1, using better informations on the structure of Γ_E.
- [FAURE-WEICH'15, TSUJII'16]: improvement of gap for *classical* (R-P) resonances in partially expanding maps / semiflows.
- [PETKOV-STOYANOV'10] adapt Dolgopyat's method to study the N-obstacles system on $\mathbb{R}^2$.

Thank you for your attention

Counting resonances: fractal Weyl law





Theorem ([SJÖSTRAND'90, SJÖSTRAND-ZWORSKI'07, N-SJ-ZW'11])

Assume Γ_E *is a hyperbolic repeller. Then,*

$$\forall C > 0, \qquad \#\{\mathrm{Res}(P_h) \cap D(E, C\,h)\} = \mathcal{O}(h^{-\mu_E}),$$

where $\mu_E = \frac{\dim(\Gamma_E) - 1}{2}$ *(Minkowski dimension).*

Intuition:

1. the metastable states are microlocalized in a $\sqrt{h}$-nbhd of K_E (uncertainty principle)
2. Each "quantum box" (phase space volume $\sim h^d$ can accomodate at most one quantum state.
3. $\leadsto$ count the number of "quantum boxes" in this nbhd.

Conjecture: for C large enough this upper bound is *sharp* [LIN-ZWORSKI].

Fractal Weyl law?

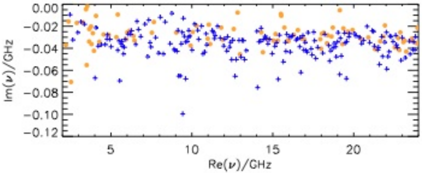
Conjecture: $\#\{\mathrm{Res}(P(h)) \cap D(E, C\,h)\} \asymp h^{-\mu_E}$

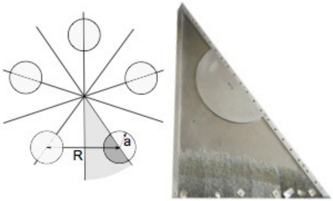
- $X = \Gamma \setminus \mathbb{H}^{n+1}$: Selberg trace formula $\rightarrow$ non-optimal lower bound

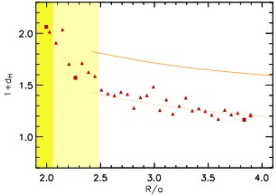
$$\#\{\mathrm{Res}(P(h)) \cap D(E, C\,h)\} \gtrsim 1$$ [GUILLOPÉ-ZWORSKI'99, PERRY'03]

- numerics for various systems seem to confirm this fractal Weyl law [LIN'01, LU-SRIDHAR-ZWORSKI'03, GUILLOPÉ-LIN-ZWORSKI'04].

Quasi-2D "open" microwave table, desymmetrized version of the 5-disk scatterer.

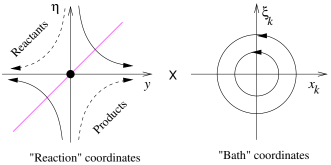






Experimental studies for the 5-disk scatterer [KUHL *et al.*'12].

Two examples of normal hyperbolicity



• Chemical reaction dynamics [Goussev *et al.*'10]. K = Normally Hyperbolic Invariant Manifold. Near a saddle-center-center fixed point the flow on K is approximately *integrable* ⇒ Quantum Normal Form:

$$P(h) = E_0 + \frac{\lambda}{2}(y\,\frac{h}{i}\partial_y + \frac{h}{i}\partial_y\, y) + \sum_{k=2}^{d} \frac{\omega_k}{2}((\frac{h}{i}\partial_{x_k})^2 + x_k^2) + \textit{smaller}$$

$\rightsquigarrow$ resonances $z_{\ell,n_k} \approx E_0 - ih\lambda(\ell + 1/2) + \sum_{k=2}^{d} h\omega_k(n_k + 1/2)$

• General relativity: wave propagation on Kerr-de Sitter metric (rotating black hole, positive cosmological constant).
The system is also separable ⇒ explicit resonances (called **quasi-normal modes** in this setting) [Dyatlov'10].