Conformal maps – Computability and Complexity

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Conformal maps: the objects

Inside the domain: computability and complexity

Boundary behaviour: harmonic measure

Boundary behaviour: Caratheodory extension

Examples
The starting point: what are we computing?

1. The Riemann map: "given" a simply connected domain $\Omega$ and a point $w \in \Omega$, "compute" the conformal map $f : (\mathbb{D}, 0) \mapsto (\Omega, w)$
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2. Carathéodory extension of $f$.

Given by Carathéodory Theorem: Let $\Omega \subset \mathbb{C}$ be a simply-connected domain. A conformal map $f : (D, 0) \mapsto (\Omega, w)$ extends to a continuous map $D \mapsto \Omega$ iff $\partial \Omega$ is locally connected.

A set $K \subset \mathbb{C}$ is called locally connected if there exists a modulus of local connectivity $m(\delta)$: a non-decreasing function decaying to 0 as $\delta \to 0$ and such that for any $x, y \in K$ with $|x - y| < \delta$ one can find a connected $C \subset K$ containing $x$ and $y$ with $\text{diam} C < m(\delta)$.

$f$ extends to a homeomorphism $D \mapsto \Omega$ iff $\partial \Omega$ is a Jordan curve.

3. The harmonic measure on $\partial \Omega$ at $w$: first boundary hitting distribution of Brownian motion started at $w$ (or one of a score of other definitions).
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2. *Carathéodory extension of $f$*. Given by

**Carthéodory Theorem:** Let $\Omega \subset \mathbb{C}$ be a simply-connected domain. A conformal map $f : (\mathbb{D}, 0) \mapsto (\Omega, w)$ extends to a continuous map $\overline{\mathbb{D}} \mapsto \overline{\Omega}$ iff $\partial \Omega$ is locally connected.
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A set \( K \subset \mathbb{C} \) is called **locally connected** if there exists **modulus of local connectivity** \( m(\delta) \): a non-decreasing function decaying to 0 as \( \delta \to 0 \) and such that for any \( x, y \in K \) with \( |x - y| < \delta \) one can find a connected \( C \subset K \) containing \( x \) and \( y \) with \( \text{diam} \ C < m(\delta) \).
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Computing the Riemann map

**Constructive Riemann Mapping Theorem. (Hertling, 1997)** The following are equivalent:

(i) $\Omega$ is a lower-computable open set, $\partial \Omega$ is a lower-computable closed set, and $w_0 \in \Omega$ is a computable point;

(ii) The maps $g$ and $f$ are both computable conformal bijections.
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**Idea of the proof** The lower-computability of $\Omega$ implies that one can compute a sequence of rational polygonal domains $\Omega_n$ such that $\Omega = \bigcup \Omega_n$. The maps $f_n : \mathbb{D} \mapsto \Omega_n$ are explicitly computable (by Schwarz-Christoffel, for example) and converge to $f$. To check that $f_n(z)$ approximates $f(z)$ well enough, we just need to approximate the boundary from below by centers of rational balls intersecting it.
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$NP$ – solution can be checked in polynomial time.
$\#P$ – can be reduced to counting the number of satisfying assignments for a given propositional formula ($\#SAT$).
$PSPACE$ – solvable in space polynomial in the input size.
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KNOWN: $P \neq EXP$.

CONJECTURED: $P \subset NP \subset \#P \subset PSPACE \subset EXP$. 
Theorem (B-Braverman-Yampolsky). Suppose there is an algorithm $A$ that given a simply-connected domain $\Omega$ with a linear-time computable boundary, a point $w_0 \in \Omega$ with $\text{dist}(w_0, \partial \Omega) > \frac{1}{2}$ and a number $n$, computes $20n$ digits of the conformal radius $f'(0))$, then we can use one call to $A$ to solve any instance of a $\#\text{SAT}(n)$ with a linear time overhead. In other words, $\#P$ is poly-time reducible to computing the conformal radius of a set. Any algorithm computing values of the uniformization map will also compute the conformal radius with the same precision, by Distortion Theorem.
Theorem (B-Braverman-Yampolsky). There is an algorithm $A$ that computes the uniformizing map in the following sense: Let $\Omega$ be a bounded simply-connected domain, and $w_0 \in \Omega$. Assume that the boundary of a simply connected domain $\Omega$, $\partial \Omega$, $w_0 \in \Omega$, and $w \in \Omega$ are provided to $A$ by an oracle. Then $A$ computes $g(w)$ with precision $n$ with complexity $PSPACE(n)$. The algorithm uses solution of Dirichlet problem with random walk and de-randomization. Later improved by Rettinger to $\#P$. 

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The proof of lower bound

For a propositional formula $\Phi$ with $n$ variables, let $L \subset \{0, 1, \ldots, 2^n - 1\}$ be the set of numbers corresponding to its satisfying instances. Let $k$ be the number of elements of $L$. 
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Let $\Omega_L$ be defined as

$$\mathbb{D} \setminus \bigcup_{l \in L} \{ |z - \exp(2\pi il2^{-n})| \leq 2^{-10n} \},$$

the unit disk with $k$ very small and spaced out half balls removed.
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**The key estimate:**

If $f : (\mathbb{D}, 0) \rightarrow (\Omega_L, 0)$ is conformal, $f'(0) > 0$ and $n$ is large enough, then

$$|f'(0) - 1 + k 2^{-20n-1}| < \frac{1}{100} 2^{-20n}.$$
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The boundary of $\Omega_L$ is computable in linear time, given the access to $\Phi$. The estimate implies that using the algorithm A we can evaluate $|L| = k$, and solve the $\#SAT$ problem on $\Phi$. 
Computability of harmonic measure

A measure $\mu$ on a metric space $X$ is called *computable* if for any computable function $\phi$, the integral $\int_X \phi \, d\mu$ is computable.

Theorem (B-Braverman-Rojas-Yampolsky). If a closed set $K \subset \mathbb{C}$ is computable, uniformly perfect, and has a connected complement, then in the presence of oracle for $w \not\in K$, the harmonic measure of $\Omega = \mathbb{C} \setminus K$ at $w_0$ is computable.

A compact set $K \subset \mathbb{C}$ which contains at least two points is uniformly perfect if there exists some $C > 0$ such that for any $x \in K$ and $r > 0$, we have $(B(x,Cr) \setminus B(x,r)) \cap K = \emptyset = \Rightarrow K \subset B(x,r)$.

In particular, every connected set is uniformly perfect. We do not assume that $\Omega$ is simply-connected, but we need the uniform perfectness of the complement: there exists a computable regular domain for which the harmonic measure is not computable.
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Approximating harmonic measure: capacity density condition.

**Theorem (Pommerenke, 1979):** For a domain with uniformly perfect boundary there exists a constant $\nu = \nu(C) < 1$ such that for any $y \in \Omega$

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P[|B_T^y - y| \geq 2 \text{dist}(y, \partial \Omega)] < \nu.
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Here $B_T^y$ is the first hitting of the boundary by Brownian motion started at $y$. 
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Take any computable \( \phi \). We need to compute \( \mathbb{E}(\phi(B_T)) \). Compute the interior polygonal \( \delta \)-approximation \( \Omega' \) to \( \Omega \) for small enough \( \delta \). Then it is easy to see that \( \mathbb{E}(\phi(B_T) - \phi(B_{T'})) \) is small, since with high probability \( B_T \) is close to \( B_{T'} \).
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**Carathéodory modulus.** A non-decreasing function $\eta(\delta)$ is called the Carathéodory modulus of $\Omega$ if $\eta(\delta) \to 0$ as $\delta \to 0$ and if for every crosscut $\gamma$ with $\text{diam}(\gamma) < \delta$ we have $\text{diam} N_\gamma < \eta(\delta)$. Here $N_\gamma$ is the component of $\Omega \setminus \gamma$ not containing $w_0$. 
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Closer related to the Modulus of local connectivity $m'(\delta)$ of $\mathbb{C} \setminus \Omega$: $m'(\delta) \leq 2\eta(\delta) + \delta$. 

Theorem (B-Rojas-Yampolsky)

The Carathéodory extension of $f : D \to \Omega$ is computable iff $f$ is computable and there exists a computable Carathéodory modulus of $\Omega$. Furthermore, there exists a domain $\Omega$ with computable Carathéodory modulus but no computable modulus of local connectivity.
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Furthermore, there exists a domain $\Omega$ with computable Carathéodory modulus but no computable modulus of local connectivity.
General simply-connected domains: Carathéodory metric.

Carathéodory metric on \((\Omega, w)\):

\[
\text{dist}_C(z_1, z_2) = \inf \text{diam}(\gamma),
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where \(\gamma\) is a closed curve or crosscut in \(\Omega\) separating \(\{z_1, z_2\}\) from \(w_0\). (Defined as continuous extension when one of the points is equal to \(w_0\).)
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The closure of \(\Omega\) in Carathéodory metric is called the Carathéodory compactification, \(\hat{\Omega}\). It is obtained from \(\Omega\) by adding the prime ends.
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**Computable Carathéodory Theorem (B-Rojas-Yampolsky):** In the presence of oracles for $w_0$ and for $\partial \Omega$, both $\hat{f}$ and $\hat{g} = \hat{f}^{-1}$ are computable.
Warshawski’s theorems

Oscillation of $f$ near boundary:

$$\omega(r) := \sup_{|z_0|=1, |z_1|<1, |z_2|<1, |z_1-z_0|<r, |z_2-z_0|<r} |f(z_1) - f(z_2)|.$$
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**Warshawski’s Theorem (1950):** $\omega(r) \leq \eta \left( \left( \frac{2\pi A}{\log 1/r} \right)^{1/2} \right)$, for all $r \in (0, 1)$.

Here $A$ is the area of $\Omega$, and $\eta(\delta)$ is Carathéodory modulus.
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The estimate $|f(z) - f((1 - r)z)| \leq \omega(r)$ for $|z| = 1$ allows one to compute $f(z)$ using $f(rz)$ for $r$ close to 1.
Other direction: Lavrentieff-type estimate

A refinement of Lavrentieff estimate (1936) (Also proven by Ferrand (1942) and Beurling in the 50ties). Let $M = \text{dist}(\partial \Omega, w_0)$, $\gamma$ be a crosscut with $\text{dist}(\partial \Omega, w_0) \geq M/2$, $\epsilon^2 < M/4$. Then

$$\text{diam}(\gamma) < \epsilon^2 \implies \text{diam}(f^{-1}(N_\gamma)) \leq \frac{30\epsilon}{\sqrt{M}}.$$
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The estimate implies that

$$\text{diam}(N_\gamma) \leq 2\omega(\text{diam}(f^{-1}(N_\gamma))) \leq 2\omega\left(\frac{30\epsilon}{\sqrt{M}}\right).$$

Thus, if $f$ is computable up to the boundary, $2\omega\left(\frac{30\epsilon}{\sqrt{M}}\right)$ is a computable Carathéodory modulus.
A domain with computable boundary and noncomputable harmonic measure.

Let $B \subset \mathbb{N}$ be a lower-computable, non-computable set. We modify the unit circle by inserting the following "gates" at $\exp 2\pi i \left(2^{-n}\right)$:
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Specifically, if $n \in B$ is enumerated at stage $j$ we take the interval $\left[\exp 2\pi i \left(2^{-n} - 2^{-2n}\right), \exp 2\pi i \left(2^{-n} + 2^{-2n}\right)\right]$ and insert $j$ equally spaced small arcs such that the harmonic measure of the ”outer part of the gate” is at least $1/2 \times 2^{-2n}$, producing a $j$-gate.
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Otherwise, if $n \notin B$, we almost cover the gate with one interval so that the harmonic measure on the "outer part of the gate" is at most $2^{-100n}$, making an $\infty$-gate.
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The resulting domain $\Omega$ is regular.
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To compute its boundary with precision $1/j$, run an algorithm enumerating $B$ for $j$ steps. Insert $j$-gate for all $n$ which are not yet enumerated.
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But if the harmonic measure of $\Omega$ would be computable, we would just have to compute it with precision $2^{-10n}$ to decide if $n \in B$. This contradicts non-computability of $B$!
A domain with computable Carathéodory extension and no computable modulus of local connectivity: construction

Let again $B \subset \mathbb{N}$ be a lower-computable, non-computable set. Set $x_i = 1 - 1/2i$. 
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Let again $B \subset \mathbb{N}$ be a lower-computable, non-computable set. Set $x_i = 1 - 1/2^i$.

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If $i \not\in B$, then we add a straight line ($i$-line) to I going from $(x_i, 1)$ to $(x_i, x_i)$.

If $i \in B$ and it is enumerated in stage $s$, we remove $i$-fjord, i.e. the rectangle

$$[(x_i - s_i, x_i + s_i) \times [x_i, 1]$$

where $s_i = \min\{2^{-s}, 1/(3i^2)\}$. 
The example: $\partial \Omega$ and Carathéodory modulus are computable.

Computing a $2^{-s}$ Hausdorff approximation of $\partial \Omega$. Run an algorithm enumerating $B$ for $s + 1$ steps. For all those $i$’s that have been enumerated so far, draw the corresponding $i$-fjords. For all the other $i$’s, draw a $i$-line.
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Carathéodory modulus: $2\sqrt{r}$.
The example: Modulus of local connectivity \( m(r) \) is not computable

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m(2 \cdot 2^{-r_i}) < \frac{x_i}{2}.
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The example: Modulus of local connectivity $m(r)$ is not computable

**Compute $B$ using $m(r)$.** First, for $i \in \mathbb{N}$, compute $r_i \in \mathbb{Q}$ such that

$$m(2 \cdot 2^{-r_i}) < \frac{x_i}{2}.$$ 

If $i \in B$ then $i$ is enumerated in fewer than $r_i$ steps. Our algorithm to compute $B$ will emulate the algorithm for enumerating $B$ for $r_i$ steps.