Conformal maps – Computability and Complexity

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Conformal maps: the objects

Inside the domain: computability and complexity

Boundary behaviour: harmonic measure

Boundary behaviour: Caratheodory extension

Examples

The starting point: what are we computing?

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3. The harmonic measure on $\partial\Omega$ at w: first boundary hitting distribution of Brownian motion started at w (or one of a score of other definitions).

Computing the Riemann map

Constructive Riemann Mapping Theorem.(Hertling, 1997) The following are equivalent:

(i) Ω is a lower-computable open set, $\partial\Omega$ is a lower-computable closed set, and $w_0 \in \Omega$ is a computable point;

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Idea of the proof The lower-computability of Ω implies that one can compute a sequence of rational polygonal domains Ω_n such that $\Omega = \cup\Omega_n$. The maps $f_n : \mathbb{D} \mapsto \Omega_n$ are explicitly computable (by Schwarz-Christoffel, for example) and converge to f. To check that $f_n(z)$ approximates $f(z)$ well enough, we just need to approximate the boundary from below by centers of rational balls intersecting it.

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KNOWN: P $\neq$ EXP.
CONJECTURED: P $\subsetneq$ NP $\subsetneq$ #P $\subsetneq$ PSPACE $\subsetneq$ EXP.

A lower bound on computational complexity

Theorem (B-Braverman-Yampolsky). Suppose there is an algorithm A that given a simply-connected domain Ω with a linear-time computable boundary, a point $w_0 \in \Omega$ with $\mathrm{dist}(w_0, \partial\Omega) > \frac{1}{2}$ and a number n, computes $20n$ digits of the conformal radius $f'(0))$, then we can use one call to A to solve any instance of a $\#SAT(n)$ with a linear time overhead. In other words, $\#P$ is poly-time reducible to computing the conformal radius of a set.

Any algorithm computing values of the uniformization map will also compute the conformal radius with the same precision, by Distortion Theorem.

An upper bound on computational complexity

Theorem (B-Braverman-Yampolsky). There is an algorithm A that computes the uniformizing map in the following sense:
Let Ω be a bounded simply-connected domain, and $w_0 \in \Omega$. Assume that the boundary of a simply connected domain Ω, $\partial\Omega$, $w_0 \in \Omega$, and $w \in \Omega$ are provided to A by an oracle. Then A computes $g(w)$ with precision n with complexity $PSPACE(n)$.

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Later improved by Rettinger to $\#P$.

The proof of lower bound

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$$\mathbb{D} \setminus \cup_{l \in L} \{|z - \exp(2\pi i l 2^{-n})| \leq 2^{-10n}\},$$

the unit disk with k very small and spaced out half balls removed.

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The key estimate:
if $f : (\mathbb{D}, 0) \to (\Omega_L, 0)$ is conformal, $f'(0) > 0$ and n is large enough, then

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The boundary of Ω_L is computable in linear time, given the access to Φ. The estimate implies that using the algorithm A we can evaluate $|L| = k$, and solve the $\#SAT$ problem on Φ.

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A compact set $K \subset \mathbb{C}$ which contains at least two points is **uniformly perfect** if there exists some $C > 0$ such that for any $x \in K$ and $r > 0$, we have

$$(B(x, Cr) \setminus B(x, r)) \cap K = \emptyset \implies K \subset B(x, r).$$

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We do not assume that Ω is simply-connected, but we need the uniform perfectness of the complement: there exists a computable regular domain for which the harmonic measure is not computable.

Approximating harmonic measure: capacity density condition.

Theorem (Pommerenke, 1979): For a domain with uniformly perfect boundary there exists a constant $\nu = \nu(C) < 1$ such that for any $y \in \Omega$

$$\mathbf{P}[|B_T^y - y| \geq 2\,\mathrm{dist}(y, \partial\Omega)] < \nu.$$

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Take any computable ϕ. We need to compute $\mathbf{E}(\phi(B_T))$. Compute the interior polygonal δ-approximation Ω' to Ω for small enough δ. Then it is easy to see that $\mathbf{E}(\phi(B_T) - \phi(B_{T'}))$ is small, since with high probability B_T is close to $B_{T'}$.

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Closer related to the Modulus of local connectivity $m'(\delta)$ of $\mathbb{C} \setminus \Omega$:
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Theorem(B-Rojas-Yampolsky) The Carathéodory extension of $f : \mathbb{D} \to \Omega$ is computable iff f is computable and there exists a computable Carathéodory modulus of Ω.
Furthermore, there exists a domain Ω with computable Carathéodory modulus but no computable modulus of local connectivity.

General simply-connected domains: Carathéodory metric.

Carthéodory metric on (Ω, w):

$$\operatorname{dist}_C(z_1, z_2) = \inf \operatorname{diam}(\gamma),$$

where γ is a closed curve or crosscut in Ω separating $\{z_1, z_2\}$ from w_0.
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Computable Carathéodory Theorem (B-Rojas-Yampolsky): In the presence of oracles for w_0 and for $\partial\Omega$, both $\hat{f}$ and $\hat{g} = \hat{f}^{-1}$ are computable.

Warshawski's theorems

Oscillation of f near boundary:

$$\omega(r) := \sup_{|z_0|=1, |z_1|<1,\ |z_2|<1, |z_1-z_0|<r, |z_2-z_0|<r} |f(z_1) - f(z_2)|.$$

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Warshawski's Theorem (1950): $\omega(r) \leq \eta\left(\left(\frac{2\pi A}{\log 1/r}\right)^{1/2}\right)$, for all $r \in (0,1)$.
Here A is the area of Ω, and $\eta(\delta)$ is Carathéodory modulus.

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The estimate $|f(z) - f((1-r)z)| \leq \omega(r)$ for $|z| = 1$ allows one to compute $f(z)$ using $f(rz)$ for r close to 1.

Other direction: Lavrentieff-type estimate

A refinement of Lavrentieff estimate(1936) (Also proven by Ferrand(1942) and Beurling in the 50ties). Let $M = \mathrm{dist}(\partial\Omega, w_0)$, γ be a crosscut with $\mathrm{dist}(\partial\Omega, w_0) \geq M/2$, $\epsilon^2 < M/4$. Then

$$\mathrm{diam}(\gamma) < \epsilon^2 \implies \mathrm{diam}(f^{-1}(N_\gamma)) \leq \frac{30\epsilon}{\sqrt{M}}.$$

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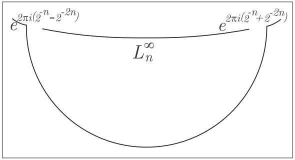
The estimate implies that

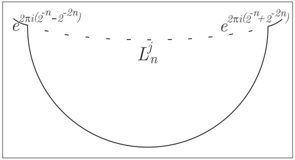
$$\mathrm{diam}(N_\gamma) \leq 2\omega(\mathrm{diam}(f^{-1}(N_\gamma))) \leq 2\omega\left(\frac{30\epsilon}{\sqrt{M}}\right).$$

Thus, if f is computable up to the boundary, $2\omega\left(\frac{30\epsilon}{\sqrt{M}}\right)$ is a computable Carathéodory modulus.

A domain with computable boundary and noncomputable harmonic measure.

Let $B \subset \mathbb{N}$ be a lower-computable, non-computable set. We modify the unit circle by inserting the following "gates" at $\exp 2\pi i\,(2^{-n})$:





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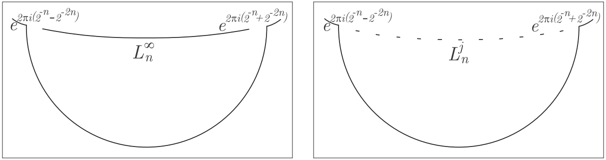

Specifically, if $n \in B$ is enumerated at stage j we take the interval $\left[\exp 2\pi i \left(2^{-n} - 2^{-2n}\right), \exp 2\pi i \left(2^{-n} + 2^{-2n}\right)\right]$ and insert j equally spaced small arcs such that the harmonic measure of the "outer part of the gate" is at least $1/2 \times 2^{-2n}$, producing a j-**gate.**

A domain with computable boundary and noncomputable harmonic measure.

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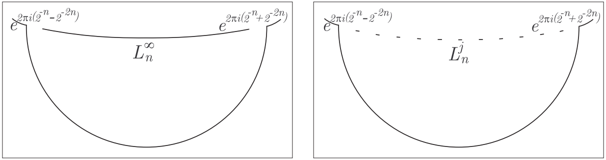

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Otherwise, if $n \notin B$, we almost cover the gate with one interval so that the harmonic measure on the the "outer part of the gate" is at most 2^{-100n}, making an ∞-**gate.**

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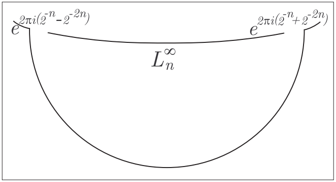

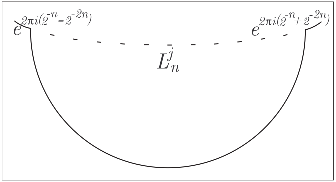

The resulting domain Ω is regular.

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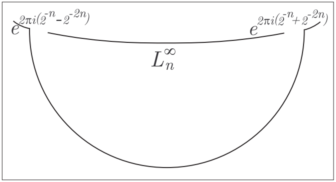

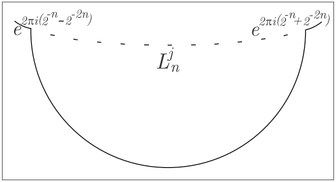

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To compute its boundary with precision $1/j$, run an algorithm enumerating B for j steps. Insert j-gate for all n which are not yet enumerated.

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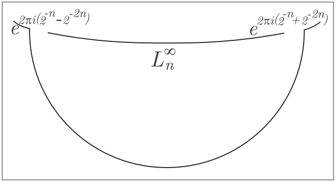

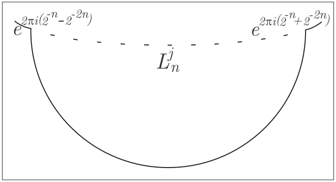

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But if the harmonic measure of Ω would be computable, we would just have to compute it with precision 2^{-10n} to decide if $n \in B$. This contradicts non-computability of B!

A domain with computable Carathéodory extension and no computable modulus of local connectivity: construction

Let again $B \subset \mathbb{N}$ be a lower-computable, non-computable set. Set $x_i = 1 - 1/2i$.

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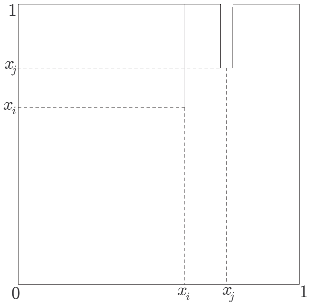

If $i \notin B$, then we add a straight line (*i-line*) to I going from $(x_i, 1)$ to (x_i, x_i).

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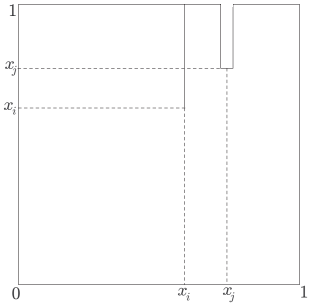

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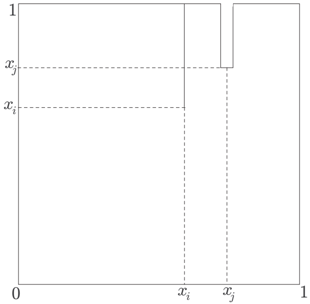
If $i \in B$ and it is enumerated in stage s, we remove *i-fjord*, i.e. the rectangle

$$[(x_i - s_i, (x_i + s_i] \times [x_i, 1]$$

where $s_i = \min\{2^{-s}, 1/(3i^2)\}$.

The example: $\partial\Omega$ and Carathéodory modulus are computable.



Computing a 2^{-s} Hausdorff approximation of $\partial\Omega$. Run an algorithm enumerating B for $s+1$ steps. For all those i's that have been enumerated so far, draw the corresponding i-fjords. For all the other i's, draw a i-line.

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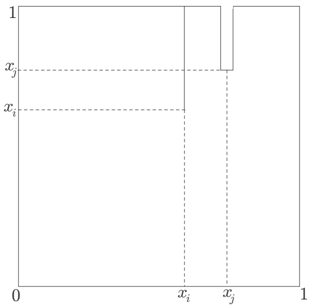

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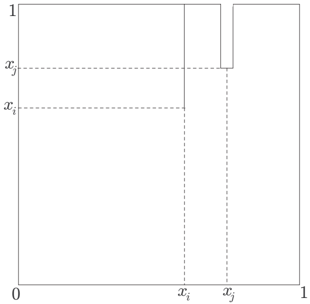
Carathéodory modulus: $2\sqrt{r}$.

The example: Modulus of local connectivity $m(r)$ is not computable



Compute B using $m(r)$.

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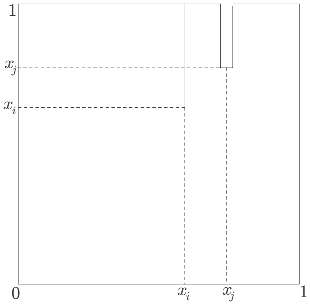

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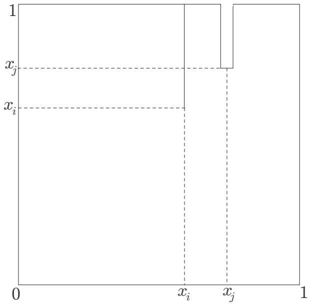

Compute B using $m(r)$. First, for $i \in \mathbb{N}$, compute $r_i \in \mathbb{Q}$ such that

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If $i \in B$ then i is enumerated in fewer than r_i steps. Our algorithm to compute B will emulate the algorithm for enumerating B for r_i steps.