Rigorous estimation of the speed of convergence to equilibrium.

S. Galatolo

Dip. Mat, Univ. Pisa

Computation in Dynamics, ICERM, 2016

Overview

- Many questions on the statistical behavior of a dynamical system are related to the speed of convergence to equilibrium: a measure of the speed of convergence to the limit

$$L^n m \rightarrow \mu.$$

- We will see a tool for the **rigorous** computer aided **explicit** estimation of this rate of convergence and one example of application to the computation of the diffusion coefficient;
- Topics mainly from joint works with: W. Bahsoun, M. Monge, I. Nisoli, X. Niu, B. Saussol.

Let us consider a metric space X with a dynamics defined by $T : X \to X$.
Let us also consider the space $PM(X)$ of probability measures on X.
Define the function

$$L : PM(X) \to PM(X)$$

in the following way: if $\mu \in PM(X)$ then:

$$L\mu(A) = \mu(T^{-1}(A))$$

- Considering measures with sign ($SM(X)$) or complex valued measures we have a vector space and L is linear.

Dynamics and the evolution of a measure

The transfer operator

Let us consider a metric space X with a dynamics defined by $T : X \to X$.
Let us also consider the space $PM(X)$ of probability measures on X.
Define the function

$$L : PM(X) \to PM(X)$$

in the following way: if $\mu \in PM(X)$ then:

$$L\mu(A) = \mu(T^{-1}(A))$$

- Considering measures with sign ($SM(X)$) or complex valued measures we have a vector space and L is linear.
- Invariant measures are fixed points of the transfer operator L.

Dynamics and the evolution of a measure

The transfer operator

Let us consider a metric space X with a dynamics defined by $T : X \to X$. Let us also consider the space $PM(X)$ of probability measures on X. Define the function

$$L : PM(X) \to PM(X)$$

in the following way: if $\mu \in PM(X)$ then:

$$L\mu(A) = \mu(T^{-1}(A))$$

- Considering measures with sign $(SM(X))$ or complex valued measures we have a vector space and L is linear.
- Invariant measures are fixed points of the transfer operator L.
- Many results come from the understanding of the properties of the action of this operator on spaces of suitably regular measures.

Convergence to equilibrium

- Consider two spaces of measures with sign $B_s \subseteq B_w$, with norms $|| \ ||_s \geq || \ ||_w$ and the set of zero average measures

$$V = \{\nu \in \mathcal{B}_s | \nu(X) = 0\}$$

Convergence to equilibrium

- Consider two spaces of measures with sign $B_s \subseteq B_w$, with norms $|| \, ||_s \geq || \, ||_w$ and the set of zero average measures

$$V = \{\nu \in \mathcal{B}_s | \nu(X) = 0\}$$

- We say that L has convergence to equilibrium with speed Φ if for each $g \in V$

$$||L^n g||_w \leq \Phi_1(n) \; ||g||_s$$

Convergence to equilibrium

- Consider two spaces of measures with sign $B_s \subseteq B_w$, with norms $|| \ ||_s \geq || \ ||_w$ and the set of zero average measures

$$V = \{\nu \in \mathcal{B}_s | \nu(X) = 0\}$$

- We say that L has convergence to equilibrium with speed Φ if for each $g \in V$

$$||L^n g||_w \leq \Phi_1(n) \ ||g||_s$$

- e.g. $||L^n g||_w \leq C\lambda^{-n}||g||_s$

Convergence to equilibrium

- Consider two spaces of measures with sign $B_s \subseteq B_w$, with norms $|| \ ||_s \geq || \ ||_w$ and the set of zero average measures

$$V = \{\nu \in \mathcal{B}_s | \nu(X) = 0\}$$

- We say that L has convergence to equilibrium with speed Φ if for each $g \in V$

$$||L^n g||_w \leq \Phi_1(n) \ ||g||_s$$

- e.g. $||L^n g||_w \leq C\lambda^{-n}||g||_s$
- Let us consider a starting probability measure $\nu \in B_s$ and μ invariant, since $(L^n \nu - \mu) \in V$, this estimates the speed

$$||L^n \nu - \mu||_w \to 0$$

Speed of contraction of zero average measures

- Consider $B_s$and the set of zero average measures

$$V = \{\nu \in \mathcal{B}_s | \nu(X) = 0\}$$

Speed of contraction of zero average measures

- Consider B_s and the set of zero average measures

$$V = \{\nu \in \mathcal{B}_s | \nu(X) = 0\}$$

- We say that V has speed of contraction Φ if for each $g \in V$

$$||L^n g||_s \leq \Phi_2(n) \; ||g||_s$$

Speed of contraction of zero average measures

- Consider $B_s$and the set of zero average measures

$$V = \{\nu \in \mathcal{B}_s | \nu(X) = 0\}$$

- We say that V has speed of contraction Φ if for each $g \in V$

$$||L^n g||_s \leq \Phi_2(n) \; ||g||_s$$

- e.g. $||L^n g||_s \leq C\lambda^{-n} ||g||_s$

An approach to the problem

Speed of convergence to equilibrium

$$||L^n g||_w \leq \Phi_1(n)\ ||g||_s$$

$$||L^n g||_s \leq \Phi_2(n)\ ||g||_s$$

- We will see that:

  a low resolution information coming from a computer estimation
  $+$ the knowledge of the fine scale behavior of the transfer operator, due to its regularizing action on a suitable space
  $=$ information on the rate of convergence.

Assumptions: Regularizing action of the operator

- Suppose L satisfies a Lasota Yorke (Doeblin Fortet) inequality: there is $\lambda_1 < 1$ s.t.

$$||L^n \nu||_s \leq A\lambda_1^n ||\nu||_s + B||\nu||_w .$$

Assumptions: Regularizing action of the operator

- Suppose L satisfies a Lasota Yorke (Doeblin Fortet) inequality: there is $\lambda_1 < 1$ s.t.
$$||L^n v||_s \leq A\lambda_1^n ||v||_s + B||v||_w.$$
- This is satisfied for example with $|| \ ||_s = || \ ||_{BV}$ and $|| \ ||_w = || \ ||_1$ for piecewise expanding maps

Assumptions: Regularizing action of the operator

- Suppose L satisfies a Lasota Yorke (Doeblin Fortet) inequality: there is $\lambda_1 < 1$ s.t.
$$||L^n \nu||_s \leq A\lambda_1^n ||\nu||_s + B||\nu||_w.$$
- This is satisfied for example with $|| \, ||_s = || \, ||_{BV}$ and $|| \, ||_w = || \, ||_1$ for piecewise expanding maps
- and with suitable anisotropic norms for (piecewise) hyperbolic systems.

Assumptions: The construction of a low resolution approximation

- Consider a projection π_δ on a finite dimensional space. Suppose

$$||\pi_\delta f - f||_w \leq C\delta||f||_s.$$

Assumptions: The construction of a low resolution approximation

- Consider a projection π_δ on a finite dimensional space. Suppose

$$||\pi_\delta f - f||_w \leq C\delta||f||_s.$$

- Supppose π_δ and L are weak contractions for the norm $||\ ||_w$

Assumptions: The construction of a low resolution approximation

- Consider a projection π_δ on a finite dimensional space. Suppose

$$||\pi_\delta f - f||_w \leq C\delta ||f||_s.$$

- Supppose π_δ and L are weak contractions for the norm $|| \ ||_w$
- Define the finite rank approximation L_δ of L as

$$L_\delta = \pi_\delta L \pi_\delta.$$

Assumptions: The construction of a low resolution approximation

- Consider a projection π_δ on a finite dimensional space. Suppose

$$||\pi_\delta f - f||_w \leq C\delta ||f||_s.$$

- Supppose π_δ and L are weak contractions for the norm $|| \ ||_w$
- Define the finite rank approximation L_δ of L as

$$L_\delta = \pi_\delta L \pi_\delta.$$

- Suppose that there is n_1 such that

$$\forall v \in V, \ ||L_\delta^{n_1}(v)||_w \leq \lambda_2 ||v||_w \tag{1}$$

with $\lambda_2 < 1$

Ulam Discretization

- Space X discretized by a partition I_δ

Ulam Discretization

- Space X discretized by a partition I_δ
- Corresponding L_δ can be seen as: let F_δ be the $\sigma-$algebra associated to I_δ, then:

$$\pi_\delta : SM(X) \to L^1(X) \quad (2)$$

$$\pi_\delta(g) = \mathbf{E}(g|F_\delta) \quad (3)$$

$$L_\delta(f) = \pi_\delta L \pi_\delta \quad (4)$$

Ulam Discretization

- Space X discretized by a partition I_δ
- Corresponding L_δ can be seen as: let F_δ be the $\sigma-$algebra associated to I_δ, then:

$$\pi_\delta : SM(X) \to L^1(X) \tag{2}$$

$$\pi_\delta(g) = \mathbf{E}(g|F_\delta) \tag{3}$$

$$L_\delta(f) = \pi_\delta L \pi_\delta \tag{4}$$

- System approximated by a Markov Chain with probabilities

$$P_{ij} = m(T^{-1}(I_j) \cap I_i)/m(I_i)$$

Ulam Discretization

- Space X discretized by a partition I_δ
- Corresponding L_δ can be seen as: let F_δ be the $\sigma-$algebra associated to I_δ, then:

$$\pi_\delta : SM(X) \to L^1(X) \quad (2)$$
$$\pi_\delta(g) = \mathbf{E}(g|F_\delta) \quad (3)$$
$$L_\delta(f) = \pi_\delta L \pi_\delta \quad (4)$$

- System approximated by a Markov Chain with probabilities

$$P_{ij} = m(T^{-1}(I_j) \cap I_i)/m(I_i)$$

- Much literature on the (more or less rigorous) approximation of invariant measures and other by this method (e.g. Bose, Bahsoun, Ding, Froyland, Keane, Li, Murray,Young, Zhou...)

Lemma

Under the previous assumption L_δ, L satisfy an approximation inequality: $\exists\ C, D$ such that $\forall \nu, \forall n \geq 0$:

$$||(L_\delta^n - L^n)\nu||_w \leq \delta(C||\nu||_s + nD||\nu||_w). \quad (5)$$

Putting together in a system Lasota-Yorke and the previous lemma: starting measure $g_0 \in V$, let us denote $g_{i+1} = L^{n_1} g_i$.

$$\begin{cases} ||\mathcal{L}^{n_1} g_i||_s \leq A\lambda_1^{n_1} ||g_i||_s + B||g_i||_w \\ ||\mathcal{L}^{n_1} g_i||_w \leq ||\mathcal{L}_\delta^{n_1} g_i||_w + \delta(C||g_i||_s + n_1 D||g_i||_w) \end{cases}, \tag{6}$$

$$\begin{cases} ||\mathcal{L}^{n_1} g_i||_s \leq A\lambda_1^{n_1} ||g_i||_s + B||g_i||_w \\ ||\mathcal{L}^{n_1} g_i||_w \leq \lambda_2 ||g_i||_w + \delta(C||g_i||_s + n_1 D||g_i||_w) \end{cases}.$$

Convergence to equilibrium

Compacting it in a vector notation,

$$\begin{pmatrix} ||g_{i+1}||_s \\ ||g_{i+1}||_w \end{pmatrix} \preceq \begin{pmatrix} A\lambda_1^{n_1} & B \\ \delta C & \delta n_1 D + \lambda_2 \end{pmatrix} \begin{pmatrix} ||g_i||_s \\ ||g_i||_w \end{pmatrix} \quad (7)$$

here $\preceq$ indicates the component-wise $\leq$ relation (both coordinates are less or equal).

Convergence to equilibrium

Let $\mathcal{M} = \begin{pmatrix} A\lambda_1^{n_1} & B \\ \delta C & \delta n_1 D + \lambda_2 \end{pmatrix}$. What said above allows to bound $||g_i||_s$ and $||g_i||_w$ by a sequence

$$\begin{pmatrix} ||g_i||_s \\ ||g_i||_w \end{pmatrix} \preceq \mathcal{M}^i \begin{pmatrix} ||g_0||_s \\ ||g_0||_w \end{pmatrix}$$

which can be computed explicitly. This gives a way have an explicit estimate on the speed of convergence for the norms $|| \ ||_s$ and $|| \ ||_w$ at a given time.

$$||\mathcal{L}^{kn_1} g||_s \leq C_1 \rho^k ||g||_s.$$

- A similar approach allows the estimation of escape rates

Example

1−d Lorenz map

$$T(x) = \begin{cases} \theta \cdot |x - 1/2|^{\alpha} & 0 \leq x < 1/2 \\ 1 - \theta \cdot |x - 1/2|^{\alpha} & 1/2 < x \leq 1 \end{cases}$$

with $\alpha = 57/64$ and $\theta = 109/64$. L is the transfer operator associated to $F = T^4$
The matrix that corresponds to our data is such that

$$M \preceq \begin{bmatrix} 0.2915 & 4049 \\ 7.75 \cdot 10^{-8} & 0.058 \end{bmatrix}.$$

We have the following estimates:

$$\|L^k g\|_{BV} \leq (16356)\cdot(0.387)^{\lfloor \frac{k}{10} \rfloor}||g||_{BV}.$$
$$\|L^k g\|_{L^1} \leq (4050)\cdot(0.387)^{\lfloor \frac{k}{10} \rfloor}||g||_{BV}.$$

We can also use the coefficients of the powers of the matrix (computed using interval arithmetics) to obtain upper bounds as in the following table:

iterations	bound for $\|\|L^h g\|\|_1$
$h = 20$	$3 \cdot 10^{-6}\|\|g\|\|_{BV} + 3.5 \cdot 10^{-2}\|\|g\|\|_1$
$h = 40$	$5 \cdot 10^{-7}\|\|g\|\|_{BV} + 5.1 \cdot 10^{-3}\|\|g\|\|_1$
$h = 60$	$7 \cdot 10^{-8}\|\|g\|\|_{BV} + 7.6 \cdot 10^{-4}\|\|g\|\|_1$
$h = 80$	$10^{-8}\|\|g\|\|_{BV} + 1.2 \cdot 10^{-4}\|\|g\|\|_1$

Application: estimating the diffusion coefficient

Theorem (Diffusion coefficient for the Lanford map)

Let

$$T(x) = 2x + \frac{1}{2}x(1-x) \quad (mod\ 1). \tag{8}$$

(a) *T admits a unique absolutely continuous invariant measure ν and if ψ is a function of bounded variation the Central Limit Theorem holds:*

$$\frac{1}{\sqrt{n}}\left(\sum_{i=0}^{n-1}\psi(T^i x) - n\int_I \psi d\nu\right) \xrightarrow{law} \mathcal{N}(0,\sigma^2).$$

(b) *For $\psi = x^2$ the diffusion coefficient $\sigma^2 \in [0.3458, 0.4152]$.*

How can it be estimated rigorously

:

- σ^2 is known to have the following expression

$$\sigma^2 := \int_I \hat{\psi}^2 h dm + 2 \sum_{i=1}^{\infty} \int_I L^i(\hat{\psi} h)\hat{\psi} dm, \quad (9)$$

  where

$$\hat{\psi} := \psi - avg \text{ and } avg := \int_I \psi h dm.$$

- Applying the above technique to the Lanford map one obtains:

$$\|L^{28k}(\hat{\psi}h)\|_{L^1} \leq (1.007) \times 0.05^k \|\hat{\psi}h\|_{BV}$$

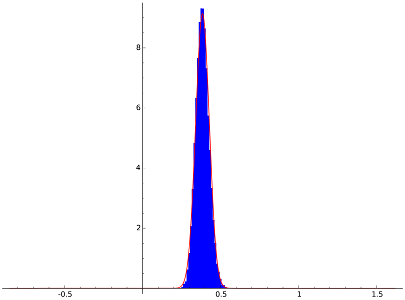
- Thus we can find l such that

$$\sum_{i=l}^{\infty} \int_I L^i(\hat{\psi}h)\hat{\psi} dm$$

  is as small as wanted. Reducing the estimation to a finite sum.

An informal verification

Taking 200 iterates and 20000 starting points



compare the distribution of deviations from averages with the estimated normal distibution.

Essential referencess

S. Galatolo, I. Nisoli, B. Saussol *An elementary way to rigorously estimate convergence to equilibrium and escape rates* J. Comp. Dyn. (2015)

W. Bahsoun, S. Galatolo, I. Nisoli, X. Niu *Rigorous approximation of diffusion coefficients for expanding maps.* J. Stat Phys. (2016)