Some practical and theorical issues in computational ergodic theory

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Introducing the problem

This talk is about a series of works joint with W. Bahsoun, S. Galatolo, M. Monge, X.Niu, F. Poloni and B. Saussol. The aim of our research is to rigorously estimate some quantitites associated to a given dynamical system, effectively proving theorems regarding the behaviour of these systems.

About our results

We obtained results in the following areas:

- approximate the invariant measure
- give a rigorous upper bound for speed of decay of correlations/convergence to equilibrium
- approximate the diffusion coefficient
- approximate the linear response.

In this talk I will explain how we tackle the task of approximating the linear response under a random perturbation for some one dynamical systems, in the same setting as W. Bahsoun talk.

An example of approximation of the density in L^1

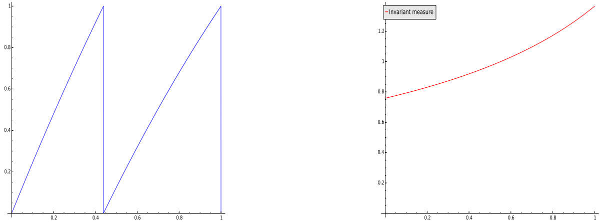


Figure: The approximated invariant density of the map $2x + \frac{1}{2}x(1-x)$ mod 1

$$||f - f_\delta||_{L^1} \leq 1.7 \cdot 10^{-5}, \quad \delta = 2^{-25}$$

Theorem: The Lyapunov exponent for the Lanford map lies in $[1.312, 1.318]$ (old estimate) and the diffusion coefficient for the observable x^2 is contained in $[0.3458, 0.4152]$.

A geometric lorenz map

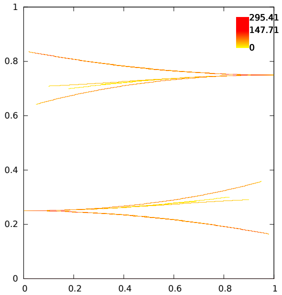
$$T(x) = \begin{cases} \theta|x-1/2|^{\alpha} & 0 \leq x < 1/2 \\ 1-\theta|x-1/2|^{\alpha} & \frac{1}{2} < x \leq 1 \end{cases} \quad (1)$$

with constants $\alpha = 51/64$, $\theta = 109/64$, and

$$G(x,y) = \begin{cases} (y-1/2)|x-1/2|^{\beta} + 1/4 & 0 \leq x < 1/2 \\ (y-1/2)|x-1/2|^{\beta} + 3/4 & \frac{1}{2} < x \leq 1 \end{cases}$$

with $\beta = 396/256$.

The computed approximation

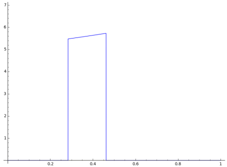


Theorem: the dimension of the attractor of this Lorenz map lies in the interval $[1.24063, 1.24219]$ (partition size 16384×1024).

The transfer operator

The main object of our investigation is the transfer operator:

$$L\mu(A) = \mu(T^{-1}A)$$



In the figure an approximation of the action of the transfer operator for the Lanford map.

The dynamical system

The strategy that we have developed works for dynamical systems $T : S^1 \to S^1$ such that $|T'(x)| > 1$ and $T \in C^3(S^1)$.
For those systems the following two Lasota-Yorke inequalities hold:

$$||\mathcal{L}^n f||_{\tilde{C}^1} \leq M \cdot \lambda^n ||f||_{\tilde{C}^1} + C||f||_\infty$$
$$||\mathcal{L}^n f||_{\tilde{C}^2} \leq M(\lambda^2)^n ||f||_{\tilde{C}^2} + D||f||_{\tilde{C}^1}.$$

This implies that T has an a.c.i.m. f which lies in $C^2(S^1)$, and that, since f is continuous on the torus, f' lies in the space of average zero densities

The strategy for computing the linear response 1

As we saw in W. Bahsoun's talk, we want to approximate the linear response when we add a stochastic perturbation.
Let $\varepsilon \in (0,1)$. For $f \in L^\infty(\mathbb{T})$ let K_ε denote the operator defined as:

$$K_\varepsilon f(x) = \int_{\mathbb{T}} \varepsilon^{-1} j(\varepsilon^{-1}(x-y)) f(y) dy,$$

where $j \in C^\infty(\mathbb{R}, \mathbb{R}^+)$, supp$(j) \subset [-1/2, 1/2]$ and $\int_{\mathbb{R}} j(y)dy = 1$.
Denote $L_\epsilon = K_\epsilon L$ and by f_ϵ its fixed point. We want to compute $\tilde{h}$ such that

$$\lim_{\epsilon \to 0} || \frac{f_\epsilon - f}{\epsilon} - \tilde{h} ||_{C^0} = 0.$$

The strategy for computing the linear response 2

We want to approximate:

$$(\mathrm{Id} - L)^{-1}\hat{L}f = \sum_{i=0}^{+\infty} L^i \hat{L}f,$$

where $Lf = f$ and $\hat{L}f = \gamma f'$ and this equality is true since $\hat{L}f$ is in the space of average 0 measures, where $\gamma := \int j(\xi)|\xi|d\xi$ and j is the convolution kernel that gives the random perturbation.

The strategy for computing the linear response 3

The main idea behind the strategy is the following:

- find a rigorous approximation g of $\hat{L}f$ (we need a discretization of L acting on C^2)
- estimate the tail of the sum $\sum_{i=0}^{+\infty} L^i g$
- approximate $L^i g$ by iterating a discretized operator (we need a discretization of L acting on C^1).

General ideas about discretizations

Knowing that an operator satisfies a L-Y inequality with respect to a strong norm and a weak norm

$$||L^n f||_s \leq A\lambda^n ||f||_s + B||f||_w$$

The idea about discretization Π_η is that they have to be well behaved with respect to the same couple of norms

$$||\Pi_\eta f||_s \leq P_s ||f||_s, \quad ||\Pi_\eta f||_w \leq P_w ||f||_w + \eta Q_w ||f||_s$$

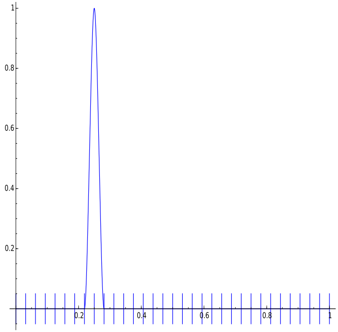
This permits us to prove a L-Y for a $\Pi_\eta L^k \Pi_\eta$, ensuring spectral stability.
Many people have been working in the topic of approximation of invariant measures using Ulam type approximations
[Bahsoun,Bose,Dellnitz,Ding,Froyland,Ippei,Junge,Li,Liverani,Murray,Padberg-Gehle,Tehresu...]

A C^1 partition of unity

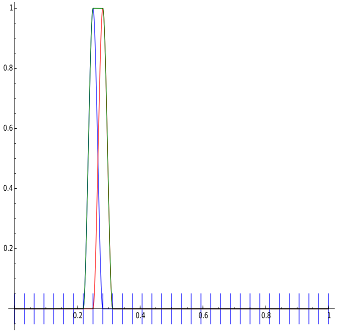
Let

$$\phi(x) = \begin{cases} 1 - 3x^2 - 2x^3 & x \in [-1, 0] \\ 1 - 3x^2 + 2x^3 & x \in [0, 1] \end{cases} . \quad (2)$$

Fix m, we define a partition of unity by $\phi_{i,m}(x) = \phi(m \cdot x - i)$, $i = 0, \ldots, m$.



: $\phi_{8,32}(x)$



: $\phi_{8,32}(x) + \phi_{9,32}(x)$

A C^1 discretization

Denote by $\eta = 1/m$ the C^1 discretization we use is the following:

$$\Pi_\eta(f)(x) := \sum_i f(a_i) \cdot \phi_i(x) + \left(\int_0^1 f dm - \sum_{i=0}^m f(a_i) \int_0^1 \phi_i dm \right) \kappa(x),$$

where $\kappa(x) = 2\sum_{j=0}^m \phi_{2j+1,2m}(x)$, and it is used to ensure that $\int \Pi_\eta f dx = \int f dx$.
We have that:

1. $||\Pi_\eta f||_\infty \leq 5||f||_\infty$;
2. $||\Pi_\eta f||_\infty \leq ||f||_\infty + 2\frac{||f'||_\infty}{m}$;
3. $||\Pi_\eta f||_{\tilde{C}^1} \leq \frac{11}{2}||f||_{\tilde{C}^1}$;
4. $||\Pi_\eta f - f||_\infty \leq 3\frac{||f'||_\infty}{m}$;

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A C^2 discretization

Define:

$$\tilde{g}(x) := \sum_i f'(a_i) \cdot \phi_i(x).$$

We define the operator

$$\tilde{\Pi}_\eta(f)(x) := \left(\int f dx - I(f) \right) + \int_0^x \tilde{g}(\xi) d\xi,$$

This projection also satisfies similar inequalities.

Approximating fixed points

Suppose f and f_δ are fixed points of L and L_δ and that

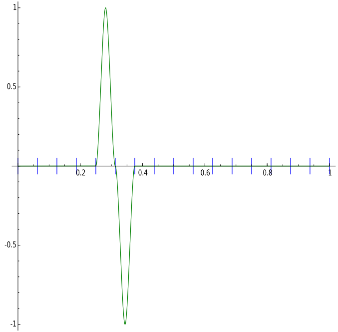
- $||Lf - L_\delta f||_w \leq K\delta$,
- there exists N such that $||L_\delta^N g|| \leq \alpha ||g||_w$ for all g with zero average
- $||L_\delta^i||_w \leq C_i$ for all $i = 0, \dots, N-1$

then

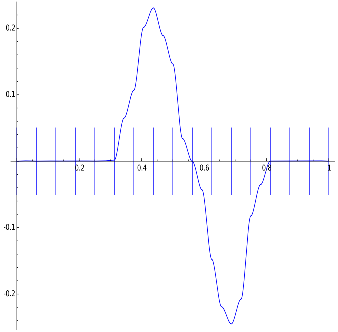
$$||f - f_\delta|| \leq \frac{1}{1-\alpha} K\delta \sum_{i=0}^{N-1} C_i$$

The HMCHUCAC principle, i.e., some ideas about resolution

How Mathematics Can Help Us in the Contemporary Age of Computation?



: In the kernel



: After 2 iterations, out of the kernel

Acting a C^0 approximation for the transfer operator of the Lanford map.

Coarse-Fine inequalities

Given two discretizations L_δ and $L_{\tilde{\delta}}$ and knowing that $||L_\delta|_V||_w \leq \alpha$ we want to find $\tilde{N}$,$\tilde{\alpha}$ such that $||L_{\tilde{\delta}}^{\tilde{N}}|_V|| \leq \tilde{\alpha}$.
If the projections is well behaved:

$$||L_\delta^k f - L_{\tilde{\delta}}^k f||_w \leq \delta C||f||_s + \delta k D||f||_w.$$

Therefore:

$$\begin{aligned} ||L_{\tilde{\delta}}^N f||_w &\leq ||L_\delta^N f||_w + ||L_\delta^N f - L_{\tilde{\delta}}^N f||_w \\ &\leq \alpha||f||_w + \delta C||f||_s + \delta k D||f||_w. \end{aligned}$$

To get a usable inequality we look at:

$$||L_{\tilde{\delta}}^{N+K} f||_w \leq M\alpha||f||_w + \delta C\lambda^k||f||_s + \delta CB||f||_w + \delta k DM||f||_w,$$

since $L_{\delta'}$ acts on a finite dimensional space $||f||_s \leq \Gamma||f||_w$.

Estimating the convergence to equilibrium and the tails

In the talk by Stefano Galatolo, was explained how, by knowing that $||L_\delta^N|_V||_w \leq \alpha$, it is possible to estimate a, b and ρ such that

$$||L^{kN}g||_w \leq \frac{1}{a}\rho^k||g||_s$$
$$||L^{kN}g||_s \leq \frac{1}{b}\rho^k||g||_s$$

Bounding the iteration error

Suppose there are two norms $|| \, ||_s \geq || \, ||_w$, such that $\forall f \in \mathcal{B}, \forall n \geq 1$

$$||\mathcal{L}^n f||_s \leq A\lambda_1^n ||f||_s + B||f||_w. \tag{3}$$

Let π_δ be a finite rank operator satisfying:

- $\mathcal{L}_\delta = \pi_\delta \mathcal{L} \pi_\delta$ with $||\pi_\delta v - v||_w \leq K\delta ||v||_s$;
- π_δ, $\mathcal{L}^i$ and $\mathcal{L}_\delta^i$ are bounded for the norm $|| \, ||_w$: $||\pi_\delta||_w \leq P$ and $\forall i > 0$, $||\mathcal{L}^i||_w \leq M$, $||\mathcal{L}_\delta^i||_w \leq M_\delta$.

Then

$$\begin{aligned} ||(\mathcal{L} - \mathcal{L}_\delta) f||_w &\leq K\delta(A\lambda_1 + P)||f||_s + K\delta B||f||_w \\ ||\mathcal{L}^n f - \mathcal{L}_\delta^n f||_w &\leq \delta K M_\delta \left(\frac{(A\lambda_1 + P)A}{1 - \lambda_1} ||f|| + nB(A\lambda_1 + P + M)||f||_w \right) \end{aligned}$$

One experiment

Let

$$T_0(x) = S(x) + P(x) + 0.005 \cdot \sin(64\pi \cdot x),$$

where

$$S(x) = \frac{31x}{1-x}.$$

Let

$$T_1(x) = 32x - 1 + 0.005 \cdot \sin(64\pi \cdot x).$$

Define

$$T(x) : \begin{cases} T_0(x - 2k/32), & x \in [2k/32, (2k+1)/32] \\ T_1(x - (2k+1)/32), & x \in [(2k+1)/32, (2k+2)/32], \end{cases} \tag{4}$$

where $k = 0, 1, \ldots, 15$.

Plot of the dynamic

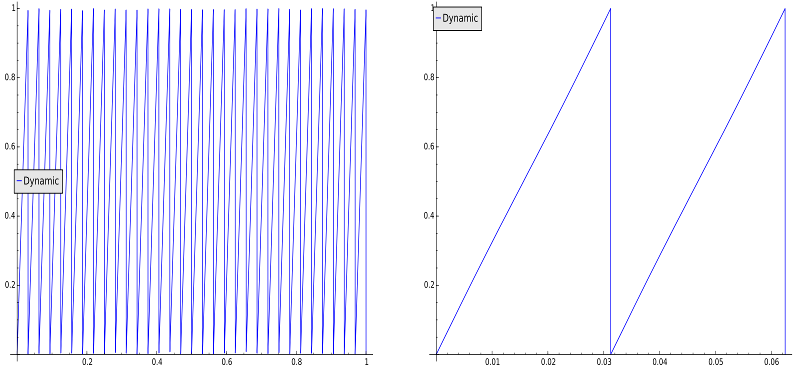


Figure: The graph of the map T as defined in (4)

The approximation of the linear response

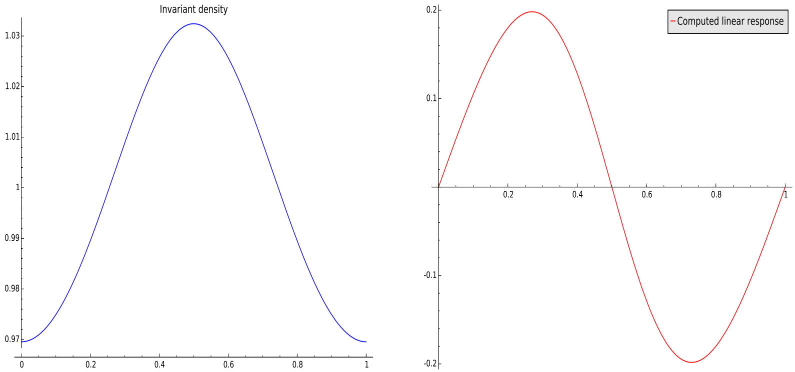


Figure: The density and its linear response

Some numbers about the experiment

Partition size for the estimate of the convergence to equilibrium in C^0: $1/65536$.
Partition size of the C^1 partition: $1/4194304$.
Partition size of the C^2 partition: $1/4194304$.
Rigorous error:

$$||\hat{h} - \gamma \sum_{k=0}^{14} \mathcal{L}_{\eta}^{k} \tilde{f}_{\eta}||_{\infty} \leq 0.044\gamma.$$