Global well-posedness of the primitive equations of oceanic and atmospheric dynamics

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Outline

1. Primitive equations (PEs)
2. Full viscosity case
3. Horizontal viscosity case

Primitive equations (PEs)

Hydrostatic approximation

In the context of the horizontal large-scale ocean and atmosphere, an important feature is

$$\begin{aligned}\text{Aspect ratio } &= \frac{\text{the depth}}{\text{the width}} \\ &\simeq \frac{\text{several kilometers}}{\text{several thousands kilometers}} \\ &\ll 1.\end{aligned}$$

Small aspect ratio is the main factor to imply

Hydrostatic Approximation

Formal small aspect ratio limit

Consider the anisotropic Navier-Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu_1 \Delta_H u - \nu_2 \partial_z^2 u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{cases} \quad \text{in } M \times (0, \varepsilon),$$

where $u = (v, w)$, with $v = (v^1, v^2)$, and M is a domain in $\mathbb{R}^2$.

Suppose that $\nu_1 = O(1)$ and $\nu_2 = O(\varepsilon^2)$. Changing of variables:

$$\begin{cases} v_\varepsilon(x, y, z, t) = v(x, y, \varepsilon z, t), \\ w_\varepsilon(x, y, z, t) = \frac{1}{\varepsilon} w(x, y, \varepsilon z, t), \\ p_\varepsilon(x, y, z, t) = p(x, y, \varepsilon z, t), \end{cases}$$

for $(x, y, z) \in M \times (0, 1)$.

Formal small aspect ratio limit (continue)

Then u_ε and p_ε satisfy the scaled Navier-Stokes equations

$$(SNS)\begin{cases} \partial_t v_\varepsilon + (u_\varepsilon \cdot \nabla) v_\varepsilon - \Delta v_\varepsilon + \nabla_H p_\varepsilon = 0, \\ \nabla_H \cdot v_\varepsilon + \partial_z w_\varepsilon = 0, \\ \varepsilon^2(\partial_t w_\varepsilon + u_\varepsilon \cdot \nabla w_\varepsilon - \Delta w_\varepsilon) + \partial_z p_\varepsilon = 0, \end{cases} \quad \text{in } M \times (0,1).$$

Formally, if $(v_\varepsilon, w_\varepsilon, p_\varepsilon) \to (V, W, P)$, then $\varepsilon \to 0$ yields

$$(PEs)\begin{cases} \partial_t V + (U \cdot \nabla) V - \Delta V + \nabla_H P = 0, \\ \nabla_H \cdot V + \partial_z W = 0, \\ \boxed{\partial_z P = 0}, \quad (\text{Hydrostatic Approximation}), \end{cases} \quad \text{in } M \times (0,1).$$

where $U = (V, W)$.

The above formal limit can be rigorously justified:

- weak convergence (L^2 initial data, weak solution of SNS $\rightharpoonup$ weak solution of PEs, no convergence rate), Azérad–Guillén (SIAM J. Math. Anal. 2001)
- strong convergence & convergence rate (H^m initial data, $m \geq 1$, strong solution of SNS $\rightarrow$ strong solution of PEs, with convergence rate $O(\varepsilon)$), JL–Titi

The primitive equations (PEs)

Equations:

$$\begin{cases} \partial_t v + (v \cdot \nabla_H) v + w \partial_z v - \nu_1 \Delta_H v - \nu_2 \partial_z^2 v \\ \qquad + \nabla_H p + f_0 k \times v = 0, \\ \boxed{\partial_z p + T = 0}, \qquad \text{(hydrostatic approximation)} \\ \nabla_H \cdot v + \partial_z w = 0, \\ \partial_t T + v \cdot \nabla_H T + w \partial_z T - \mu_1 \Delta_H T - \mu_2 \partial_z^2 T = 0. \end{cases}$$

Unknowns:

- velocity (v, w), with $v = (v^1, v^2)$, pressure p, temperature T

Constants:

- viscosities ν_i, diffusivity μ_i, $i = 1, 2$, Coriolis parameter f_0

Remark: some properties of the PEs

- The vertical momentum equation reduces to the hydrostatic approximation;
- There is no dynamical information for the vertical velocity, and it can be recovered only by the incompressiblity condition;
- The strongest nonlinear term $w\partial_z v = -\partial_z^{-1}\nabla_H \cdot v\partial_z v \approx (\nabla v)^2$.

Remark: on the coefficients

- The viscosities ν_1 and ν_2 may have different values
- The diffusivity coefficients μ_1 and μ_2 may have different values
- In case of $\nu_1 = 0$, the primitive equations look like the Prandtl equations (without the term $f_0 k \times v$)
- Due to the **strong horizontal turbulent mixing**, which creates the horizontal eddy viscosity, $\nu_1 > 0$.

PEs with full dissipation: weak solutions

Global existence:

- Lions–Temam–Wang (Nonlinearity 1992A, 1992B, J. Math. Pures Appl. 1995)

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Conditional uniqueness:

- z-weak solutions ($v_0 \in X := \{f | f, \partial_z f \in L^2\}$): Bresch et al. (Differential Integral Equations 2003),
- continuous initial data: Kukavica et al. (Nonlinearity 2014),
- certain discontinuous initial data (v_0 is small L^∞ perturbation of some $f \in X$): JL–Titi (SIAM J. Math. Anal. 2017)

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Remark

Unlike the Navier-Stokes equations, the above uniqueness conditions for the PEs are imposed on the initial data of the solutions, rather than on the solutions themselves.

PEs with full dissipation: strong solutions

- Local strong: Guillén-González et al. (Differential Integral Equations 2001);
- Global strong (2D): Bresch–Kazhikhov–Lemoine (SIAM J. Math. Anal. 2004);

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- **Global strong (3D):** Cao–Titi (arXiv 2005/Ann. Math. 2007), Kobelkov (C. R. Math. Acad. Sci. Paris 2006), Kukavica–Ziane (C. R. Math. Acad. Sci. Paris 2007, Nonlinearity 2007), Hieber–Kashiwabara (Arch. Rational Mech. Anal. 2016)

Remark: PEs$\leftrightsquigarrow$NS

One of the key observations of Cao–Titi 2007:

(i) $v = \bar{v} + \tilde{v}, v = \frac{1}{2h}\int_{-h}^{h} v dz$;

(ii) p appears only in the equations for $\bar{v}$ $(2D)$, but not in those for $\tilde{v}$.

$\Longrightarrow$ $L_t^\infty(L_{\mathbf{x}}^6)$ of v (~~Navier-Stokes equations~~).

Primitive equations without any dissipation

The **inviscid** primitive equations may **develop finite-time singularities**

- Cao – Ibrahim – Nakanishi – Titi (Comm. Math. Phys. 2015)
- Wong (Proc. Amer. Math. Soc. 2015)

Our goals

Question: How about the case in between (PEs with partial viscosity or diffusivity)? Blow-up in finite time or global existence?

We will focus on the structure of the system itself instead of the effects caused by the boundary: **always suppose the periodic boundary conditions, and $\Omega = \mathbb{T}^2 \times (-h, h)$.**

Full viscosity case

Theorem (Cao–Titi, Comm. Math. Phys. 2012)

$$\left.\begin{array}{r}\text{Full Viscosities}\\ \text{\&Vertical Diffusivity}\\ (v_0, T_0) \in H^4 \times H^2\\ \text{Local well-posedness}\end{array}\right\} \Longrightarrow \text{Global well-posedness}$$

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Theorem (Cao–JL–Titi, J. Differential Equations 2014)

$$\left.\begin{array}{r}\text{Full Viscosities}\\ \text{\&Horizontal Diffusivity}\\ (v_0, T_0) \in H^2 \times H^2\end{array}\right\} \Longrightarrow \text{Global well-posedness}$$

Ideas I (to overcome the strongest nonlinearity)

- The hard part of the pressure depends only on two spatial variables x, y

$$\partial_z p + T = 0 \quad \Rightarrow \quad p = p_s(x, y, t) - \int_{-h}^{z} T dz';$$

- Use anisotropic treatments on different derivatives of the velocity ($\partial_z >> \nabla_H$):

$$\partial_z(w\partial_z v) = \partial_z w \partial_z v + \cdots = -\boxed{\nabla_H \cdot v \partial_z v} + \cdots ,$$

$$\partial_h(w\partial_z v) = \partial_h w \partial_z v + \cdots = -\boxed{\int_{-h}^{z} \partial_h \nabla_H \cdot v d\xi \partial_z v} + \cdots ;$$

- The Ladyzhenskaya type inequalities can be applied to

$$\int_M \left(\int_{-h}^{h} |f| dz \right) \left(\int_{-h}^{h} |g||h| dz \right) dxdy.$$

Horizontal viscosity case

Horizontal viscosity + horizontal diffusivity

PEs with horizontal viscosity + horizontal diffusivity:

$$
\begin{cases}
\partial_t v + (v \cdot \nabla_H) v + w \partial_z v - \nu_1 \Delta_H v \\
\qquad + \nabla_H p + f_0 k \times v = 0, \\
\partial_z p + T = 0, \qquad \text{(hydrostatic approximation)} \\
\nabla_H \cdot v + \partial_z w = 0, \\
\partial_t T + v \cdot \nabla_H T + w \partial_z T - \mu_1 \Delta_H T = 0.
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\left.\begin{array}{r}
\text{Horizontal Viscosity} \\
\text{\&Horizontal Diffusivity} \\
(v_0, T_0) \in H^2 \times H^2
\end{array}\right\} \Longrightarrow \text{Global well-posedness}
$$

Some improvement of the above result:

Theorem (Cao–JL–Titi, J. Funct. Anal. 2017)

$$\left.\begin{array}{r}\text{Horizontal Viscosity}\\ \&\text{Horizontal Diffusivity}\\ (v_0, T_0) \in H^1\end{array}\right\} \Longrightarrow \text{Local well-posedness}$$

Some improvement of the above result:

Theorem (Cao–JL–Titi, J. Funct. Anal. 2017)

$$\left.\begin{array}{r}\text{Horizontal Viscosity}\\ \text{\&Horizontal Diffusivity}\\ (v_0, T_0) \in H^1\end{array}\right\} \Longrightarrow \text{Local well-posedness}$$

$$\left.\begin{array}{r}\text{Horizontal Viscosity}\\ \text{\&Horizontal Diffusivity}\\ (v_0, T_0) \in H^1 \cap L^\infty,\\ \partial_z v_0 \in L^q, \text{for some } q \in (2, \infty)\end{array}\right\} \Longrightarrow \text{Global well-posedness}$$

Remark

Local-in-space estimates are used for local well-posedness, as

(i) Nonlinearity of $w\partial_z v = -\partial_z^{-1}\nabla_H \cdot v\partial_z v$ is critical.

(ii) Some smallness on initial data is required if using the global-in-space type energy estimates.

Main Difficulties

- Absence of the dynamical information on w
  $\Longrightarrow$ Strongest nonlinear term $w\partial_z v \sim (\nabla v)^2$;
- Absence of the vertical viscosity
  $\Longrightarrow$ Need to estimate somewhat a priori $\int_0^{\mathcal{T}} \|v\|_\infty^2 dt$.

Energy inequality for ω

All high order estimates depend on

$$L^\infty(L^2) \cap L^2(0, T; H^1) \text{ estimates on } \omega := \partial_z v.$$

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$$L^\infty(L^2) \cap L^2(0, T; H^1) \text{ estimates on } \omega := \partial_z v.$$

Note that ω satisfies

$$\partial_t \omega + (v \cdot \nabla_H)\omega + w\partial_z \omega - \Delta_H \omega + (\omega \cdot \nabla_H) v - (\nabla_H \cdot v)\omega = 0.$$

Multiplying the above equation by ω, one will encounter

$$\int (\omega \cdot \nabla_H) v \cdot \omega = -\int v \nabla_H \cdot (\omega \otimes \omega) \leq \frac{1}{2} \int |\nabla_H \omega|^2 + C \int |v|^2 |\omega|^2$$

$\Longrightarrow$

$$\frac{d}{dt}\|\omega\|_2^2 + \|\nabla_H \omega\|_2^2 \leq C \int_\Omega |v|^2 |\omega|^2 dx.$$

Absence of vertical viscosity asks for $\|v\|_{L^2_t(L^\infty_x)}$

If we have full viscosities, then

$$\begin{aligned}\int |v|^2|\omega|^2 \leq& \|v\|_4^2\|\omega\|_3\|\omega\|_6 \leq \|v\|_4^2\|\omega\|_2^{\frac{1}{2}}\|\omega\|_6^{\frac{3}{2}} \\ \leq& C\|v\|_4^2\|\omega\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{3}{2}} \leq \frac{1}{2}\|\nabla\omega\|_2^2 + C\|v\|_4^8\|\omega\|_2^2.\end{aligned}$$

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$$\leq C\|v\|_4^2\|\omega\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{3}{2}} \leq \frac{1}{2}\|\nabla\omega\|_2^2 + C\|v\|_4^8\|\omega\|_2^2.$$

Since we only have $\|\nabla_H\omega\|_2^2$, we have to

$$\int |v|^2|\omega|^2 \leq \|v\|_\infty^2\|\omega\|_2^2.$$

The absence of the vertical viscosity **forces** us to do somewhat

a priori $\int_0^{\mathcal{T}} \|v\|_\infty^2 dt$ estimates !!

Try some ways

We may try:

- ~~Maximal principle~~: p is nonlocal;
- ~~Uniform L^q estimates and let $q \to \infty$~~: p is nonlocal;
- ~~Interpolation inequalities~~ ($\|v\|_\infty \leq C\|v\|_{\text{low}}^{\theta}\|v\|_{\text{high}}^{1-\theta}$): only leads to the local-in-time estimate.

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Our idea:

- Though we are not able to get the uniform L^q estimates on v, we may be able to get the **precise growth of $\|v\|_q$** w.r.t q;
- Such growth information may control the main part of $\|v\|_\infty$, while the remaining part depends only on the logarithm of the higher order norms, i.e.

$$\|v\|_\infty \le \underline{\text{“growth information of } \|v\|_q\text{”}} \log \|v\|_{\text{high order}}$$

Ideas II (to overcome the absence of vertical viscosity)

- **Precise L^q estimates** of v:

$$\boxed{\|v\|_q \le C\sqrt{q}}, \quad C \text{ is independent of } q;$$

Remark: The above estimates is independent of μ_1, μ_2.

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Remark: The above estimates is independent of μ_1, μ_2.

- A **logarithmic Sobolev embedding** inequality:

$$\|v\|_{L^\infty} \le C \max\left\{1, \sup_{q\ge 2} \frac{\|v\|_{L^q}}{\sqrt{q}}\right\} \log^{\frac{1}{2}}(\mathcal{N}_{\mathbf{p}}(v) + e),$$

where $\mathcal{N}_{\mathbf{p}}(v) = \sum_{i=1}^{3}(\|v\|_{p_i} + \|\partial_i v\|_{p_i})$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$.

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- A **logarithmic Gronwall inequality** (and its variations):

$$\frac{d}{dt}A + B \lesssim A \log B \quad \Longrightarrow \quad A(t) + \int_0^t B(s)ds < \infty.$$

Why Δ_H is enough?

- The pressure satisfies (ignoring the temperature):

$$\frac{1}{2h}\int_{-h}^{h}\nabla_H\cdot\left\{\partial_t v+\nabla_H\cdot(v\otimes v)+\partial_z(wv)-\Delta_H v+\nabla_H p(x^H,t)=0\right\}dz$$

$$\Longrightarrow \quad \boxed{-\Delta_H p(x^H,t)=\frac{1}{2h}\int_{-h}^{h}\nabla_H\cdot\nabla_H\cdot(v\otimes v)dz}$$

- Only the horizontal derivatives are involved in the following

$$\int_M\left(\int_{-h}^{h}|f|dz\right)\left(\int_{-h}^{h}|g\phi|dz\right)dx^H$$
$$\leq C\|f\|_2\|g\|_2^{\frac{1}{2}}\|\nabla_H g\|_2^{\frac{1}{2}}\|\phi\|_2^{\frac{1}{2}}\|\nabla_H\phi\|_2^{\frac{1}{2}}$$

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Theorem (Cao-JL-Titi)

$$\left.\begin{array}{r} \text{Horizontal Viscosity} \\ \text{\&Vertical Diffusivity} \\ v_0 \in H^2,\ T_0 \in H^1 \\ \nabla_H T_0 \in L^q \text{ for some } q \in (2, \infty) \end{array}\right\} \Rightarrow \text{Global well-posedness}$$

$$\omega := \partial_z v, \quad \theta := \nabla_H^\perp \cdot v,$$

$$\eta := \nabla_H \cdot v + \boxed{\int_{-h}^{z} T d\xi - \frac{1}{2h}\int_{-h}^{h}\int_{-h}^{z} T d\xi dz},$$

Remark

- We need more smoothness of v_0 than that of T_0;
- The velocity v has the nonstandard regularities:

$$\nabla_H \partial_z v \in L_t^2(H_{\mathbf{x}}^1), \quad \boxed{(\eta, \theta) \in L_t^2(H_{\mathbf{x}}^2)}$$

$$\not\Longrightarrow \nabla_H v \in L_t^2(H_{\mathbf{x}}^2)$$

- However, if in addition that $T_0 \in H^2$, then v has the standard regularities:

$$\boxed{\nabla_H v \in L_t^2(H_{\mathbf{x}}^2)}.$$

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  $\Longrightarrow$ Need to estimate somewhat a priori $\int_0^{\mathcal{T}} \|v\|_\infty^2 dt$;

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- Absence of the vertical viscosity
  $\Longrightarrow$ Need to estimate somewhat a priori $\int_0^{\mathcal{T}} \|v\|_\infty^2 dt$;
- Absence of the horizontal diffusivity
  $\Longrightarrow$ Need to estimate somewhat a priori $\int_0^{\mathcal{T}} \|\nabla_H v\|_\infty dt$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla_H T\|_2^2 + \|\nabla_H \partial_z T\|_2^2 \\ &= -\int_\Omega \nabla_H T \cdot \nabla_H v \cdot \nabla_H T + \cdots \\ &\leq \int_\Omega |\nabla_H v| |\nabla_H T|^2 + \cdots ;\end{aligned}$$

- Mismatching of regularities between v and T: $\nabla_H \int_{-h}^z T d\xi$ is involved in the momentum equation, but temperature has only smoothing effect in vertical direction.

Ideas III (to overcome mismatching of the regularities)

To overcome the difficulties caused by the mismatching of the regularities between v and T, we introduce:

$$\eta := \nabla_H \cdot v + \int_{-h}^{z} T d\xi - \frac{1}{2h} \int_{-h}^{h} \int_{-h}^{z} T d\xi dz, \quad \theta := \nabla_H^{\perp} \cdot v,$$

when working on $\|v\|_{L_t^\infty(H_x^1)}$, and

$$\varphi := \nabla_H \cdot \partial_z v + T, \qquad \psi := \nabla_H^{\perp} \cdot \partial_z v,$$

when working on $\|v\|_{L_t^\infty(H_x^2)}$.

Equations for (η, θ)

Then, (η, θ) satisfies

$$\begin{aligned} \partial_t \theta - \Delta_H \theta &= -\nabla_H^{\perp} \cdot [(v \cdot \nabla_H) v + w \partial_z v + f_0 k \times v], \\ \int_{-h}^{h} \eta dz &= 0, \\ \partial_t \eta - \Delta_H \eta &= -\nabla_H \cdot [(v \cdot \nabla_H) v + w \partial_z v + f_0 k \times v] - wT \\ &\quad + \partial_z T - \int_{-h}^{z} \nabla_H \cdot (vT) d\xi + \mathcal{H}(x, y, t), \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}(x, y, t) = & \ \tfrac{1}{2h} \int_{-h}^{h} \nabla_H \cdot \left(\nabla_H \cdot (v \otimes v) + f_0 k \times v \right) dz \\ & + \tfrac{1}{2h} \int_{-h}^{h} \left(\int_{-h}^{z} \nabla_H \cdot (vT) d\xi + wT \right) dz. \end{aligned}$$

Advantages of η and θ

Some advantages of η and θ:

- η and θ have more regularities than $\nabla_H v$ (η and θ have standard regularities, but $\nabla_H v$ does not);
- Only ∇T, instead of $\nabla_H^2 T$ (which appears in the equations for $\nabla_H v$), is involved in the equations of η and θ;
- For the aim of getting $L_t^\infty(L_{\mathbf{x}}^2)$ estimates on η and θ, one does not need appeal to ∇T.

$$\Downarrow$$

One can achieve $\|v\|_{L_t^\infty(H_{\mathbf{x}}^1)}$ by performing the $\|(\omega, \eta, \theta)\|_{L_t^\infty(L_{\mathbf{x}}^2)}$ (precise L^q estimates, logarithmic Sobolev, logarithmic Gronwall).

Remark

The $\|v\|_{L_t^\infty(H_{\mathbf{x}}^1)}$ **estimate** is **uniform in** the vertical diffusivity μ_2.

Ideas IV (to overcome absence of horizontal diffusivity)

The absence of horizontal diffusivity requires somewhat

$$\text{a priori} \int_0^{\mathcal{T}} \|\nabla_H v\|_\infty dt,$$

We decompose v as

$$\begin{aligned} v =& \text{"temperature-independent part"} (\leftrightsquigarrow \|(\eta, \theta)\|_{L_t^2(H_x^1)}) \\ &+ \text{"temperature-dependent part"} (\text{boundedness of } T) \\ =& \zeta + \varpi, \end{aligned}$$

where

$$\begin{cases} \nabla_H \cdot \varpi = \frac{1}{|M|} \int_M \Phi dxdy - \Phi, & \text{in } \Omega, \\ \nabla_H^\perp \cdot \varpi = 0, \quad \text{in } \Omega, & \int_M \varpi dxdy = 0, \end{cases}$$

where $\Phi = \int_{-h}^z T d\xi - \frac{1}{2h} \int_{-h}^h \int_{-h}^z T d\xi dz.$

Estimates on ϖ and ζ

- For the temperature-dependent part ϖ: recalling that

$$\nabla_H \cdot \varpi = \frac{1}{|M|}\int_M \Phi dxdy - \Phi, \quad \nabla_H^\perp \cdot \varpi = 0$$

and using the Beale-Kato-Majda type logarithmic Sobolev embedding $\Longrightarrow$

$$\sup_{-h\leq z\leq h} \|\nabla_H \varpi\|_{\infty,M} \leq C\log(e+\|\nabla_H T\|_q).$$

- For the temperature-independent part ζ: Noticing that

$$\nabla_H \cdot \zeta = \eta - \tfrac{1}{|M|}\int_M \Phi dxdy, \quad \nabla_H^\perp \cdot \zeta = \theta$$

and using the Brézis-Gallouet-Wainger type logarithmic Sobolev embedding inequality $\Longrightarrow$

$$\int_{-h}^{h} \|\nabla_H \zeta\|_{\infty,M} dz \leq C\|\nabla_H(\eta,\theta)\|_2 \log^{1/2}(e+\|\Delta_H(\eta,\theta)\|_2).$$

Summary and ongoing works

More related results can be found in a recent survey paper:

- JL–Titi: Recent Advances Concerning Certain Class of Geophysical Flows, (in "Handbook of Mathematical Analysis in Viscous Fluid") arXiv:1604.01695

Summary:

- The **PEs with only horizontal viscosity** admit a unique **global** strong solution, as long as we **still have either horizontal or vertical diffusivity**;
- **Strong horizontal turbulent mixing**, which creates the horizontal eddy viscosity, **is crucial for stabilizing** the oceanic and atmospheric dynamics.

Ongoing works:

- PEs with full or partial viscosity but **without any diffusivity** (need more ideas).
- PEs (with full or partial dissipation) with moisture (different phases).

Thank You!