On the Muskat problem

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Outline of the Talk

Part I: Introduction to the Muskat problem

Part II: Global in time existence and uniqueness results in 2D and 3D for the Muskat problem

- On the global existence for the Muskat problem
- On the Muskat problem: global in time results in 2D and 3D

Part III: Large time Decay for the Muskat problem

- Large time Decay Estimates for the Muskat equation

Part IV: Absence of singularity formation for the Muskat problem

- Absence of splash singularities for SQG sharp fronts and the Muskat problem

Introduction to the Muskat problem

Consider the general transport equation

$$\rho_t + u \cdot \nabla \rho = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

Here ρ is an "active scalar" which is driven by the incompressible velocity u:

$$\nabla \cdot u = 0.$$

This type of system comes up in many contexts in fluid dynamics and beyond by taking a suitable choice of u.

- Vortex Patch Problems
- Surface Quasi-geostrophic equation (SQG):

$$u \overset{\text{def}}{=} R^{\perp}\rho = (-R_2\rho, R_1\rho), \quad \widehat{R_j} = i\frac{\xi_j}{|\xi|}$$

- Muskat Problem (using Darcy's law.)

Vortex Patch problems

Contour equation:

$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0, \\ \\ u = \nabla^{\perp} \Delta^{-1} \omega, \end{cases}$$

where the vorticity is given by

$$\omega(x_1, x_2, t) = \begin{cases} \omega_0, & \Omega(t) \\ 0, & \mathbb{R}^2 \setminus \Omega(t). \end{cases}$$

- Chemin (1993)
- Bertozzi & Constantin (1993)

Fluids in porous media and Hele-Shaw cells

The Muskat problem assumes u is given by Darcy's law:

Darcy's law:

$$\frac{\mu}{\kappa} u = -\nabla p - \mathrm{g}\,\rho\, e_n,$$

u velocity, p pressure, μ viscosity, κ permeability, ρ density, g acceleration due to gravity and e_n is the last canonical basis vector with $n = 2, 3$.

Widely noted similarity to Hele-Shaw (Saffman & Taylor (1958)):

Hele-Shaw:

$$\frac{12\mu}{b^2} u = -\nabla p - (0, \mathrm{g}\,\rho),$$

b distance between the plates.

Below we normalize physical constants to one WLOG

Patch problem for IPM: Muskat (1934)

$$\rho_t + u \cdot \nabla \rho = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

where ρ is the scalar density which is driven by

the incompressible velocity u: $\quad \nabla \cdot u = 0.$

For the Muskat problem, the velocity satisfies Darcy's law:

$$u = -\nabla p - (0, \rho).$$

We consider "sharp fronts" (where ρ^1 and ρ^2 are constants):

$$\rho = \begin{cases} \rho^1, & x \in \Omega(t) \\ \rho^2, & x \in \mathbb{R}^2 \setminus \Omega(t), \end{cases}$$

For the transport equation, initial data of this form propagate this structure forward in time, where $\Omega(t)$ is a moving domain.

Contour equation

In this situation, the interphase $\partial\Omega(t)$ is a free boundary:

$$\partial\Omega(t) = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)), \alpha \in \mathbb{R}\}\,.$$

For the Muskat problem we obtain the Contour equation:

$$z_t(\alpha) = \frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{(z_1(\alpha)-z_1(\beta))}{|z(\alpha)-z(\beta)|^2}\,(\partial_\alpha z_2(\alpha)-\partial_\alpha z_2(\beta))\,d\beta.$$

We characterize the free boundary as a graph $(\alpha, f(\alpha,t))$:

$$\Omega(t) = \left\{(x_1,x_2)\in\mathbb{R}^2 : x_2 > f(x_1,t)\right\}.$$

This structure is preserved and $f(\alpha,t)$ satisfies the equation

$$f_t(\alpha,t) = \frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} d\beta \frac{(\partial_\alpha f(\alpha,t)-\partial_\alpha f(\alpha-\beta,t))\,\alpha}{\alpha^2+(f(\alpha,t)-f(\alpha-\beta,t))^2}.$$

Some Fundamental Questions for Muskat

- Existence of front type solutions? Siegel M, Caflisch R, Howison S (2004); Escher J, Matioc BV (2011); Constantin P, Córdoba D, Gancedo F, S. (2013); Beck T, Sosoe P, Wong P (2014); Constantin, Córdoba, Gancedo, S, Rodríguez-Piazza (2015); Constantin, Gancedo, Shvydkoy, Vicol (Preprint 2016), Deng, Lei, Lin (Preprint 2016)...
- Possible singularity formation for scenarios with large initial data?
  Castro A, Córdoba D, Fefferman C, Gancedo F, Lopez-Fernandez M (2012); Castro A, Córdoba D, Fefferman C, Gancedo F (2013), Coutand-Shkoller (2015)...

Additional (incomplete collection of) References

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The linearized equation

This equation for f can be linearized around the flat solution:

$$f^L_t(\alpha,t) = -\frac{\rho^2-\rho^1}{2}\Lambda(f^L)(\alpha,t), \qquad \Lambda = (-\Delta)^{1/2}.$$

The linearized equation can be solved by Fourier transform:

$$\hat{f}^L(\xi) = \hat{f}_0(\xi)\exp\Big(-\frac{\rho^2-\rho^1}{2}|\xi|t\Big).$$

- $\rho^2 > \rho^1$ stable case, we have well-posedness.
- $\rho^2 < \rho^1$ unstable case, we have ill-posedness.
  See Ambrose (2004), Córdoba & Gancedo (2007), ...
- Also we have the L^2 evolution for the linear equation:

$$\frac{d}{dt}\|f^L\|^2_{L^2}(t) = -\frac{\rho^2-\rho^1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\frac{f^L(\alpha,t)-f^L(\beta,t)}{\alpha-\beta}\Big)^2 d\alpha d\beta dt.$$

This is a smoothing estimate. Similar in 3D.

Smoothing for the non-linear equation?

$$f_t(\alpha,t) = \frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} d\beta \frac{(\partial_\alpha f(\alpha,t) - \partial_\alpha f(\alpha-\beta,t))\,\beta}{\beta^2 + (f(\alpha,t) - f(\alpha-\beta,t))^2}.$$

Satisfies L^2 maximum principle:

$$\frac{d}{dt}\|f\|_{L^2}^2(t) = -\frac{\rho^2-\rho^1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\left(1+\left(\frac{f(\alpha,t)-f(\beta,t)}{\alpha-\beta}\right)^2\right) d\alpha d\beta$$

For which it is possible to bound as follows:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\left(1+\left(\frac{f(\alpha,t)-f(\beta,t)}{\alpha-\beta}\right)^2\right) d\alpha d\beta \le 4\pi\sqrt{2}\|f\|_{L^1}(t).$$

Don't see a non-linear smoothing effect at the level of f in L^2.
See P. Constantin, D. Córdoba, F. Gancedo - S. (2013).
Also a similar "no-smoothing" statement also in 3D.

Global-existence results for the stable case

In 2D:

$$f_t(\alpha,t) = \frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{\beta(\partial_\alpha f(\alpha,t) - \partial_\alpha f(\alpha-\beta,t))}{\beta^2 + (f(\alpha,t) - f(\alpha-\beta,t))^2} d\beta,$$
$$f(\alpha,0) = f_0(\alpha), \quad \alpha \in \mathbb{R}.$$

In 3D:

$$f_t(x,t) = \frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}^2} \frac{(\nabla f(x,t) - \nabla f(x-y,t)) \cdot y}{[|y|^2 + (f(x,t) - f(x-y,t))^2]^{3/2}} dy,$$
$$f(x,0) = f_0(x), \quad x \in \mathbb{R}^2.$$

We suppose that $\rho^2 > \rho^1$

Crucial norm: $\quad \|f\|_s = \int |\xi|^s |\widehat{f}(\xi)| d\xi, \quad s \geq 0.$

Let f be a solution to the Muskat problem in 3D ($d = 2$), or in 2D ($d = 1$) with initial data $f_0 \in H^l(\mathbb{R}^d)$ some $l \geq 1 + d$.

Theorem (Constantin-Córdoba-Gancedo- Rodríguez-Piazza- S)

In 2D ($d = 1$) we suppose for some $0 < \delta < 1$ that

$$\|f_0\|_1 \leq c_0, \quad 2\sum_{n\geq 1}(2n+1)^{1+\delta} c_0^{2n} \leq 1, \quad c_0 \geq \frac{1}{3}$$

In 3D ($d = 2$) we suppose for some $0 < \delta < 1$ that

$$\|f_0\|_1 \leq k_0, \quad \pi\sum_{n\geq 1}(2n+1)^{1+\delta} \frac{(2n+1)!}{(2^n n!)^2} k_0^{2n} \leq 1, \quad k_0 \geq \frac{1}{5}.$$

Then there is a unique Muskat solution with initial data f_0 that satisfies $f \in C([0,T]; H^l(\mathbb{R}^d))$ for any $T > 0$.

A few recent papers

- Constantin, Gancedo, Shvydkoy, Vicol (Preprint 2016): Local well posedness for initial data with finite slope. Global well posedness for initial data with very small slope:

$$f_0 \in L^2(\mathbb{R}),\ f_0'' \in L^p(\mathbb{R}), 1 < p \leq \infty, \quad \|f_0'\|_{L^\infty} \ll 1$$

- Matioc (Preprint 2016): Well posedness 2D ($d = 1$) for initial data $f_0 \in H^l(\mathbb{R})$ for $l \in (3/2, 2)$. (with surface tension for $l \in (2, 3)$.)
  (One may combine this with all the previously mentioned results to get a slightly lower regularity initial data.)

- Tofts (Preprint 2016): Well posedness in 2D ($d = 1$) with surface tension, including global unique solutions for small data. Building on previous local well posedness work of Ambrose 2014

Ideas from the proof ...

We set $\rho^2 - \rho^1 = 2$ WLOG and we only discuss the 2D case.

- One can show the following differential inequality:

$$\frac{d}{dt}\|f\|_{H^l}^2 \le CP(\|\nabla f\|_{L^\infty})|\nabla^2 f|_{C^\delta}\|f\|_{H^l}^2.$$

  Our goal will be to uniformly in time bound $\|f(t)\|_{H^l}$
- We can further expand out the non-linear problem as

$$f_t = -\Lambda(f) - N(f),$$

  where

$$N(f) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\partial_\alpha f(\alpha) - \partial_\alpha f(\alpha-\beta)}{\beta} \frac{\left(\frac{f(\alpha)-f(\alpha-\beta)}{\beta}\right)^2}{1+\left(\frac{f(\alpha)-f(\alpha-\beta)}{\beta}\right)^2} d\beta.$$

- Then by a Taylor expansion we have

$$N(f) = \frac{1}{\pi}\sum_{n\ge 1}(-1)^n \int_{\mathbb{R}} \frac{\partial_\alpha f(\alpha) - \partial_\alpha f(\alpha-\beta)}{\beta}\left(\frac{f(\alpha)-f(\alpha-\beta)}{\beta}\right)^{2n} d\beta.$$

... Ideas from the proof ...

- We have the following differential inequality

$$\frac{d}{dt}\|f\|_1(t) \leq -\|f\|_2(t) + \int d\xi\ |\xi||\mathcal{F}(N)(\xi)|,$$

  And our goal is to understand the non-linear term.
- Using the Taylor expansion we can prove the bound

$$\int |\xi||\mathcal{F}(N)(\xi)|d\xi \leq 2\|f\|_2(t)\sum_{n\geq 1}(2n+1)\|f\|_1^{2n}(t),$$

- Then for $\|f_0\|_1$ sufficiently small we get the uniform estimate

$$\|f\|_1(t) \leq \|f_0\|_1.$$

- Similarly for $0 < \delta < 1$ we can show that

$$\int |\xi|^{1+\delta}|\mathcal{F}(N)(\xi)|d\xi \leq 2\|f\|_{2+\delta}(t)\sum_{n\geq 1}(2n+1)^{1+\delta}\|f\|_1^{2n}(t).$$

... Ideas from the proof.

- We use the inquality for some $0 < \mu < 1$

  $$1 > 2\sum_{n\geq 1}(2n+1)^{1+\delta}\|f_0\|_1^{2n} = 1-\mu \geq 2\sum_{n\geq 1}(2n+1)^{1+\delta}\|f\|_1^{2n}(t),$$

  To establish that

  $$\int |\xi|^{1+\delta}|\mathcal{F}(N)(\xi)|d\xi \leq (1-\mu)\|f\|_{2+\delta}(t),$$

- This proves the following differential inequality

  $$\frac{d}{dt}\|f\|_{1+\delta}(t) \leq -\mu\|f\|_{2+\delta}(t),$$

- Or alternatively

  $$\|f\|_{1+\delta}(t) + \mu\int_0^t ds\,\|f\|_{2+\delta}(s) \leq \|f_0\|_{1+\delta},$$

- Then we finally obtain our desired uniform in time bound:

  $$\|f\|_{H^l}(t) \leq \|f_0\|_{H^l}\exp(CP(c_0)\int_0^t \|f\|_{2+\delta}(s)ds).$$

3. Large time Decay for the Muskat problem

To study the large time decay, the choice of good Functional spaces is essential.

- We use the *s-norm* for $s > -d$ ($d = 1$ or $d = 2$)

$$\|f\|_s \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^s |\hat{f}(\xi)| \, d\xi$$

- For $s = -d$ (and $s \geq -d$) define the *Besov-type s-norm*:

$$\|f\|_{s,\infty} \stackrel{\text{def}}{=} \left\| \int_{C_j} |\xi|^s |\hat{f}(\xi)| \, d\xi \right\|_{l_j^\infty} = \sup_{j \in \mathbb{Z}} \int_{C_j} |\xi|^s |\hat{f}(\xi)| \, d\xi,$$

  where $C_j = \{\xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| < 2^j\}$.
- Note that we have the inequality

$$\|f\|_{s,\infty} \leq \int_{\mathbb{R}^d} |\xi|^s |\hat{f}(\xi)| \, d\xi = \|f\|_s.$$

  Also have that $\|f\|_{-d/p,\infty} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ for $p \in [1, 2]$

Optimal Linear Decay Rate

$$f^L_t(\alpha,t) = -\Lambda(f^L)(\alpha,t), \quad \Lambda = (-\Delta)^{1/2}, \quad f^L(\alpha,t) = e^{t\Lambda} f_0.$$

- Can be solved by Fourier transform:

$$\hat{f}^L(\xi,t) = \hat{f}_0(\xi)\exp\Big(-\frac{\rho^2-\rho^1}{2}|\xi|t\Big).$$

- If $f_0(x)$ a tempered distribution vanishing at infinity and satisfying $\|f_0\|_{\nu,\infty} < \infty$, then can be shown that

$$\|f_0\|_{\nu,\infty} \approx \left\| t^{s-\nu} \left\| e^{t\Lambda} f_0 \right\|_s \right\|_{L^\infty_t((0,\infty))}, \quad \text{for any } s \geq \nu.$$

- Equivalence above implies the optimal time decay rate

$$\left\| e^{t\Lambda} f_0 \right\|_s \approx t^{-s+\nu} C(\|f_0\|_{\nu,\infty}), \quad \text{for any } s > \nu.$$

Theorem (Patel-S 2016)

Let f be a solution to the non-linear Muskat problem in 3D ($d = 2$), or in 2D ($d = 1$), given by the previous theorems. The initial data satisfies $f_0 \in H^l(\mathbb{R}^d)$ some $l \geq 1 + d$.

- *For $-d < s < l - 1$, we have the uniform in time estimate*

$$\|f\|_s(t) \lesssim 1. \tag{1}$$

- *For $0 \leq s < l - 1$ have the uniform in time decay estimate*

$$\|f\|_s(t) \leq C(\|f_0\|_{\nu,\infty})(1 + t)^{-s+\nu}, \tag{2}$$

  where we allow ν to satisfy $-d \leq \nu < s$.

Corollary (Patel-S 2016)

For $0 \leq s < l - 1$ we have the uniform time decay estimate

$$\|f\|_{\dot{W}^{s,\infty}}(t) \lesssim C(\|f_0\|_{\nu,\infty})(1 + t)^{-s+\nu}, \quad (-d \leq \nu < s)$$

A few previous results on bounds and large time decay

- Córdoba-Gancedo (2009):
  - Maximum principle: $\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}$.
  - Also optimal time decay rate:

$$\|f\|_{L^\infty(\mathbb{R}^d)}(t) \leq \frac{\|f_0\|_{L^\infty(\mathbb{R}^d)}}{\left(1 + C(\|f_0\|_{L^\infty(\mathbb{R}^d)}, \|f_0\|_{L^1(\mathbb{R}^d)})t\right)^d}$$

- Constantin, Gancedo, Shvydkoy, Vicol (Preprint 2016): Time decay rate in 2D:

$$\|f''\|_{L^\infty(\mathbb{R})}(t) \leq \frac{\|f_0''\|_{L^\infty(\mathbb{R})}}{1 + C(\|f_0''\|_{L^\infty(\mathbb{R})}, \|f_0'\|_{L^\infty(\mathbb{R})})t}$$

- Constantin, Córdoba, Gancedo, Rodriguez-Piazza, S (2015):
  - $\|\nabla f_0\|_{L^\infty(\mathbb{R}^2)} < 1/3$ then the solution with initial data f_0 satisfies the uniform in time bound $\|\nabla f\|_{L^\infty(\mathbb{R}^2)}(t) < 1/3$.
- Constantin, Córdoba-Gancedo, S (2013):
  - $\|\nabla f_0\|_{L^\infty(\mathbb{R})} < 1$ then $\|\nabla f\|_{L^\infty(\mathbb{R})}(t) < 1$.

Some useful Functional Inequalities

- For $s > -\frac{d}{p}$ and $r > s + d/q$ and $p, q \in [1,2]$ we have

$$\|f\|_s \lesssim \|f\|_{L^p(\mathbb{R}^d)}^{1-\theta} \|f\|_{\dot{W}^{r,q}(\mathbb{R}^d)}^{\theta}, \quad \theta = \frac{s + d/p}{r + d\left(\frac{1}{p} - \frac{1}{q}\right)} \in (0,1)$$

- For $s = -\frac{d}{p}$ and $p \in [1,2]$ we further

$$\|f\|_{s,\infty} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \quad \text{(includes } s = -d \text{ and } p = 1\text{)}$$

- For $s > -\frac{d}{2}$ these imply

$$\|f\|_s \lesssim \|f\|_{H^r(\mathbb{R}^d)} \quad (r > s + d/2).$$

- For $1 \leq p \leq 2$, $r > s + \frac{d}{p}$ and $s > -\frac{d}{p}$, we also conclude

$$\|f\|_s \lesssim \|f\|_{W^{r,p}(\mathbb{R}^d)}.$$

Idea's of the Proof

Two main steps:

Lemma (Step 1: General Decay Lemma)

For some $\mu \in \mathbb{R}$, $\|g_0\|_\mu < \infty$ *and* $\|g(t)\|_{\nu,\infty} \leq C_0$ *for some* $\nu \geq -d$ *satisfying* $\nu < \mu$. *Differential inequality holds for* $C > 0$*:*

$$\frac{d}{dt}\|g\|_\mu \leq -C\|g\|_{\mu+1}.$$

Then we have the uniform in time estimate

$$\|g\|_\mu(t) \lesssim (1+t)^{-\mu+\nu}.$$

Lemma (Step 2: Prove uniform in time bounds using Step 1)

$$\|f\|_s \lesssim 1, \quad (-d < s < 2)$$

and prove $\|f\|_{s,\infty} \lesssim 1$ *for* $-d \leq s < 2$ *including* $s = -d$.

Overview of the proof of Step 2

- We have a unform bound on H^3 from:

$$\|f\|_{H^3(\mathbb{R}^2)}(t) \leq \|f_0\|_{H^3(\mathbb{R}^2)} \exp(CP(k_0)\|f_0\|_{1+\delta}/\mu). \tag{3}$$

- Embeddings grant uniform bound on $\|f\|_s(t)$:

$$\|f\|_s(t) \lesssim \|f\|_{H^3}(t) \lesssim 1, \quad (-1 < s < 2).$$

- Previous bound plus the decay lemma gives us time decay:

$$\|f\|_s \lesssim (1+t)^{-s+\nu}, \quad -1 < \nu < s, \quad 0 \leq s \leq 1.$$

- Prove a weaker inequality to obtain stronger bounds

$$\frac{d}{dt}\|f\|_s(t) \lesssim \|f\|_1, \quad -2 < s < -1.$$

- Use the time decay of $\|f\|_1(t) \lesssim (1+t)^{-1-\epsilon}$ to prove

$$\|f\|_s(t) \lesssim 1, \quad -2 < s \leq -1.$$

4. The Multi-Phase Muskat Problem

$$\rho(x,t) = \begin{cases} \rho^1, & x \in \{x_2 > f(x_1,t)\}, \\ \rho^2, & x \in \{f(x_1,t) > x_2 > g(x_1,t)\}, \\ \rho^3, & x \in \{g(x_1,t) > x_2\}, \end{cases}$$

Stable Situation: $\rho^1 < \rho^2 < \rho^3$.The equations of motion are

$$\begin{aligned} f_t(\alpha,t) =& \rho^{32} \int_{\mathbb{R}} \frac{\beta(\partial_\alpha f(\alpha) - \partial_\alpha f(\alpha-\beta))}{\beta^2 + (f(\alpha) - f(\alpha-\beta))^2} d\beta \\ &+ \rho^{21} \int_{\mathbb{R}} \frac{\beta(\partial_\alpha f(\alpha) - \partial_\alpha g(\alpha-\beta))}{\beta^2 + (f(\alpha) - g(\alpha-\beta))^2} d\beta, \end{aligned}$$

$$\begin{aligned} g_t(\alpha,t) =& \rho^{21} \int_{\mathbb{R}} \frac{\beta(\partial_\alpha g(\alpha) - \partial_\alpha g(\alpha-\beta))}{\beta^2 + (g(\alpha) - g(\alpha-\beta))^2} d\beta \\ &+ \rho^{32} \int_{\mathbb{R}} \frac{\beta(\partial_\alpha g(\alpha) - \partial_\alpha f(\alpha-\beta))}{\beta^2 + (g(\alpha) - f(\alpha-\beta))^2} d\beta. \end{aligned}$$

where $\rho^{ji} = \frac{\rho^j - \rho^i}{2\pi}$ for $i,j = 1,2,3$. These are derived similarly.

4. Absence of singularities for Multi-Phase Muskat

- splat or squirt singularity: the free boundary intersects on a surface. Then a positive volume of the fluid between the interphases would be ejected in finite time.

  Ruled out in Cordoba-Gancedo (2010).

  They prove that $\frac{d}{dt}\mathrm{Vol}\Omega(t) \geq 0$ where $\Omega(t)$ is roughly the region between the interfaces.
- splash singularity: the free boundary intersects at a single point.

  Ruled out in Gancedo-S (2014) stated below.

  (See also recent related work on free boundary Euler by Fefferman-Ionescu-Lie (2015) and Coutand-Shkoller (2015).)

Suppose: $\lim_{\alpha\to\infty} f(\alpha,t) = f_\infty > g_\infty = \lim_{\alpha\to\infty} g(\alpha,t)$.

Theorem (Gancedo-S. (2014))

Suppose the free boundaries $f(\alpha,t)$ and $g(\alpha,t)$ are smooth for $\alpha \in \mathbb{R}$ and $t \in [0,T)$ with $T > 0$ arbitrary. Define the distance:

$$0 < S(t) = \min_{\alpha\in\mathbb{R}}(f(\alpha,t) - g(\alpha,t)) \ll \min\{f_\infty - g_\infty, 1\}. \tag{4}$$

Then the following uniform lower bound for $t \in [0,T)$ holds:

$$S(t) \geq \exp\left(\ln(S(0))\exp\left(\int_0^t C(f,g)(s)ds\right)\right). \tag{5}$$

Here $C(f,g)$ is a smooth function of $\|f''\|_{L^\infty} + \|g''\|_{L^\infty}$ and $\|f\|_{L^\infty} + \|g\|_{L^\infty}$. And of course $\ln(S(0)) < 0$.

More generally we have a unified method to establish the absence of splash singularities for these types of systems in different scenarios. In particular, an analogous theorem also holds for SQG sharp fronts.

Verification of some Numerical Evidence

- Córdoba D, Fontelos MA, Mancho AM, Rodrigo JL (2005) observed that computer solutions of the SQG sharp front system exhibit pointwise collapse and the curvature blows-up at the same finite time.
- We prove that in order to have a pointwise collapse, the second derivative, and therefore the curvature, has to blow-up.

Idea's of the Proof

- We observe that the minimum is attained a.e.

$$S(t) = \min_\alpha \left(f(\alpha, t) - g(\alpha, t)\right) = f(\alpha_t, t) - g(\alpha_t, t),$$

- Crucial identity for smooth solutions:

$$\partial_\alpha f(\alpha_t, t) = \partial_\alpha g(\alpha_t, t).$$

- We plug this identity into the equation

$$\begin{aligned} S_t(t) &= \int_{|\beta|<S(t)} d\beta + \int_{S(t)<|\beta|<1} d\beta + \int_{|\beta|>1} d\beta \\ &= I + II + III. \end{aligned}$$

- Naturally: $I + III \leq CS(t)$.

Idea's of the Proof Cont...

Recall $S_t(t) = I + II + III$ where $I + III \leq CS(t)$.

- We further split $II = \rho^{21} II_1 + \rho^{32} II_2$ where for instance

$$II_1 = \int_{S(t)<|\beta|<1} d\beta \frac{\beta \delta_\beta f'(\alpha_t)[(\delta_\beta(g,f)(\alpha_t))^2 - (\delta_\beta f(\alpha_t))^2]}{D(g,f,\beta)},$$

$\delta_\beta(f,g)(\alpha) = f(\alpha) - g(\alpha - \beta)$ and $\delta_\beta f(\alpha) = \delta_\beta(f,f)(\alpha)$

$$D(g,f,\beta) \stackrel{\text{def}}{=} [\beta^2 + (\delta_\beta f(\alpha_t))^2][\beta^2 + (\delta_\beta(g,f)(\alpha_t))^2].$$

- Using the previous identities after a lengthy calculation we find subtle hidden non-intuitive cancellation:

$$II_1 = -\int_{S(t)<|\beta|<1} \frac{\beta \delta_\beta f'(\alpha_t) S(t) \delta_\beta(g,f)(\alpha_t)}{D(g,f,\beta)} d\beta - \int_{S(t)<|\beta|<1} \frac{\beta \delta_\beta f'(\alpha_t) S(t) \delta_\beta f(\alpha_t)}{D(g,f,\beta)} d\beta,$$

- Thus $II \leq -CS(t) \ln S(t)$.
- Therefore: $S_t(t) \geq -C(f,g) S(t) \ln S(t)$. **Q.E.D.**

THANK YOU!

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