Nonlinear Modulational Instability of Dispersive PDE Models

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Dispersive long waves models

Consider the KDV type equations,

$$\partial_t u + \partial_x(\mathcal{M}u + f(u)) = 0, \tag{1}$$

where $\mathcal{M}$ is a Fourier multiplier operator satisfying $\widehat{\mathcal{M}u}(\xi) = \alpha(\xi)\widehat{u}(\xi)$.
Assume $f(s) \in C^1(\mathbf{R}, \mathbf{R})$ and
(A1) $\mathcal{M}$ is a self-adjoint operator, and the symbol $\alpha : \mathbf{R} \mapsto \mathbf{R}^+$ is even and regular near 0.
(A2) There exist constants $m, c_1, c_2 > 0$, such that

$$\text{(Differential case)} \quad c_1 |\xi|^m \leq \alpha(\xi) \leq c_2 |\xi|^m, \text{ for large } \xi, \tag{2}$$

or

$$\text{(Smoothing case)} \quad c_1 |\xi|^{-m} \leq \alpha(\xi) \leq c_2 |\xi|^{-m}, \text{ for large } \xi. \tag{3}$$

For the classical KDV equation, $\mathcal{M} = -\partial_x^2$. For Benjamin-Ono, Whitham and intermediate long-wave equations, $\alpha(\xi) = |\xi|$, $\sqrt{\frac{\tanh \xi}{\xi}}$ and $\xi \coth(\xi H) - H^{-1}$ respectively.

Periodic Traveling waves

A periodic traveling wave (TW) solution is of the form $u(x,t) = u_c(x - ct)$, where $c \in \mathbf{R}$ is the traveling speed and u_c satisfies the equation

$$\mathcal{M}u_c - cu_c + f(u_c) = a,$$

for a constant a. In general, the periodic TWs are a three-parameter family of solutions depending on period T, travel speed c and the constant a. The stability of periodic TWs to perturbations of the same period had been studied a lot in the literature. Take minimal period $T = 2\pi$. We assume that $u_c(x - ct)$ is orbitally stable in the energy norm $\inf_{y\in\mathbb{T}} \|u - u_c(x+y)\|_{H^{\frac{m}{2}}(\mathbb{T}_{2\pi})}$ for the differential case $\|\mathcal{M}(\cdot)\|_{L^2} \sim \|\cdot\|_{H^m}$, and $\inf_{y\in\mathbb{T}} \|u - u_c(x+y)\|_{L^2(\mathbb{T}_{2\pi})}$ for the smoothing case $\|\mathcal{M}(\cdot)\|_{H^m} \sim \|\cdot\|_{L^2}$.

Modulational instability

The modulational instability (also called Benjamin-Feir, side-band instability) is to consider perturbations of different period and even localized perturbations. Consider the linearized equation $\partial_t u = JLu$, where $J = \partial_x$ and $L := \mathcal{M} - c + f'(u_c)$. By the standard Floquet-Bloch theory, any bounded eigenfunction $\phi(x)$ of the linearized operator JL takes the form $\phi(x) = e^{ikx} v_k(x)$, where $k \in [0,1]$ is a parameter and $v_k \in L^2(\mathbb{T}_{2\pi})$. Then $JLe^{ikx} v_k(x) = \lambda(k) e^{ikx} v_k(x)$ is equivalent to $J_k L_k v_k = \lambda(k) v_k$, where

$$J_k = \partial_x + ik,\ \ L_k = \mathcal{M}_k - c + f'(u_c).$$

Here, $\mathcal{M}_k$ is the Fourier multiplier operator with the symbol $m(\xi + k)$. We say that u_c is linearly modulationally unstable if there exists $k \in (0,1)$ such that the operator $J_k L_k$ has an unstable eigenvalue $\lambda(k)$ with $\operatorname{Re} \lambda(k) > 0$ in the space $L^2(\mathbb{T}_{2\pi})$.

Linear Modulational instability

First, it can be shown that (Lin & Zeng, 2016):
If u_c is orbitally stable under perturbations of the same period, then for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that $|k| < \varepsilon_0$ implies $\sigma(J_k L_k) \cap \{|z| \geq \delta\} \subset i\mathbf{R}$.
Thus when k is small enough, the unstable eigenvalues of $J_k L_k$ can only bifurcate from the zero eigenvalue of JL. Since $\dim \ker(JL) = 3$, the perturbation of zero eigenvalue of JL for $J_k L_k$ $(0 < k \ll 1)$ can be reduced to the eigenvalue perturbation of a 3 by 3 matrix. This had been studied extensively in the literature and instability conditions were obtained for various dispersive models, by Bronski, Johnson, Hur, Kapitula, Hărăguș, Deconinck, ...

Nonlinear instability-smooth case

Theorem 1 (L, Shasha Liao, Jiayin Jin, arxiv: 1704.08618) Assume $f \in C^\infty(\mathbf{R})$, $\mathcal{M}$ satisfies (A1)-(A2) and u_c is linearly modulationally unstable. When $\mathcal{M}$ is smoothing, assume in addition that

$$c - \|f'(u_c)\|_{L^\infty(\mathbb{T}_{2\pi})} \geqslant \delta_0 > 0.$$

Then u_c is nonlinearly unstable in the following sense:
i) (Multiple periodic perturbations) $\exists\ q \in \mathbb{N},\ \theta_0 > 0$, such that for any $s \in \mathbb{N}$ and arbitrary $\delta > 0$, there exists a solution $u_\delta(t,x)$ to the nonlinear equation satisfying $\|u_\delta(0,x) - u_c(x)\|_{H^s(\mathbb{T}_{2\pi q})} <$ δ and $\inf_{y \in q\mathbb{T}} \|u_\delta(T^\delta, x) - u_c(x+y)\|_{L^2(\mathbb{T}_{2\pi q})} \geqslant \theta_0$, where $T^\delta \sim |\ln \delta|$.
ii) (Localized perturbations) $\exists\ \theta_0 > 0$, such that for any $s \in \mathbb{N}$ and arbitrary small $\delta > 0$, there exists $T^\delta \sim |\ln \delta|$ and a solution $u_\delta(t,x)$ to the nonlinear equation in the traveling frame

$$\partial_t U - c\partial_x U + \partial_x(\mathcal{M}U + f(U)) = 0,$$

satisfying $\|u_\delta(0,x) - u_c(x)\|_{H^s(\mathbb{R})} < \delta$ and $\|u_\delta(T^\delta, x) - u_c(x)\|_{L^2(\mathbb{R})} \geqslant \theta_0$.

Remark

1. When $\mathcal{M}$ is smoothing (e.g. Whitham equation), the additional assumption

$$c - \|f'(u_c)\|_{L^\infty(\mathbb{T}_{2\pi})} \geqslant \delta_0 > 0,$$

is used to show the regularity of TWs and the unstable eigenfunctions. For Whitham equation with $f = u^2$, this assumption is verified for small amplitude waves and numerically confirmed for large amplitude waves.
2. The nonlinear instability for multi-periodic perturbations is proved in the orbital distance since the equation is translation invariant. For localized perturbations, we study the equation in the space $u_c + H^s(\mathbf{R})$ which is not translation invariant. Therefore, we do no use the orbital distance for nonlinear instability under localized perturbations.

Theorem 2 (L, Shasha Liao, Jiayin Jin) Assume $\mathcal{M}$ is differential with $m \geq 1$, that is,

$$c_1 |\xi|^m \leq \alpha(\xi) \leq c_2 |\xi|^m, \quad m \geq 1, c_1, c_2 > 0, \text{ for large } \xi, \tag{4}$$

and

$$f \in C^{2n+2}(\mathbf{R}), \text{ where } n \geqslant \frac{1}{2}\max\{1+m, 1\} \text{ is an integer}, \tag{5}$$

Suppose u_c is linearly modulationally unstable. Then u_c is nonlinearly unstable for both multi-periodic and localized perturbations in the sense of Theorem 1, with the initial perturbation arbitrarily small in $H^{2n}(\mathbb{T}_{2\pi q})$ or $H^{2n}(\mathbf{R})$.

Remark

In Theorem 2, the assumption $f \in C^{2n+2}(\mathbf{R})$ is only used to prove that the nonlinear equation is locally well-posed in $H^{2n}(\mathbb{T}_{2\pi q})$ and $u_c + H^{2n}(\mathbf{R})$ by Kato's approach of nonlinear semigroup. Assuming the local well-posedness in the energy space $H^{\frac{m}{2}}$, we only need much weaker assumptions on f to prove nonlinear instability:
$f \in C^1(\mathbf{R})$ and there exist $p_1 > 1,\ p_2 > 2$, such that

$$\left| f(u+v) - f(v) - f'(v)u \right| \leq C\left(|u|_\infty, |v|_\infty\right)|u|^{p_1}, \tag{6}$$

$$\left| F(u+v) - F(v) - f(v)u - \frac{1}{2}f'(v)u^2 \right| \leq C\left(|u|_\infty, |v|_\infty\right)|u|^{p_2}, \tag{7}$$

where $F(u) = \int_0^u f(s)\,ds$. When $f \in C^2(\mathbf{R})$, the conditions (6)-(7) are automatically satisfied with $(p_1, p_2) = (2, 3)$.

Ideas of the proof

The proof consists of two steps. First, we use the Hamiltonian structure of the linearized equation to get the semigroup estimates for both periodic and localized perturbations. This is obtained by using the general theory in a recent paper of Lin and C. Zeng.

Second, there is a difficulty of the loss of derivative in the nonlinear term for KDV type equations. For smooth f, this loss of derivative was overcome by using the approach of Grenier by constructing higher order approximation solutions. For non-smooth f and differential $\mathcal{M}$, it can be overcome by a bootstrap argument.

The nonlinear instability for semilinear equations (BBM, Schrödinger, Klein-Gordon etc.) is much easier. For multi-periodic perturbations, one can even construct invariant (stable, unstable and center) manifolds which characterize the complete local dynamics near unstable TWs.

Linear Hamiltonian system

First we consider the following abstract framework of linear Hamiltonian system

$$\partial_t u = JLu, \ \ u \in X,$$

where X is a Hilbert space.

(H1) $J : X^* \to X$ is a skew-adjoint operator.

(H2) $L : X \to X^*$ generates a bounded bilinear symmetric form $\langle L\cdot, \cdot \rangle$ on X. There exists a decomposition $X = X_- \oplus \ker L \oplus X_+$ satisfying that $\langle L\cdot, \cdot \rangle |_{X_-} < 0, \ \dim X_- = n^- (L) < \infty$, and

$$\langle Lu, u \rangle \geq \delta_1 \|u\|_X^2 \ \ , \text{ for some } \delta_1 > 0 \text{ and any } u \in X_+.$$

(H3) The above $X_\pm$ satisfy

$$\ker i^*_{X_+ \oplus X_-} = \{ f \in X^* \mid \langle f, u \rangle = 0, \forall u \in X_- \oplus X_+ \} \subset D(J),$$

where $i^*_{X_+ \oplus X_-} : X^* \to (X_+ \oplus X_-)^*$ is the dual operator of the embedding $i_{X_+ \oplus X_-}$.

The assumption **(H3)** is automatically satisfied when $\dim \ker L < \infty$, as in the current case.

Theorem (L & Zeng, arxiv 1703.04016) Under assumptions (H1)-(H3), we have

$$X = E^u \oplus E^c \oplus E^s,$$

satisfying: i) E^u, E^s and E^c are invariant under e^{tJL}. ii) $\exists\ M > 0,\ \lambda_u > 0$, such that

$$\begin{aligned} \left| e^{tJL}|_{E^s} \right|_X &\leq Me^{-\lambda_u t}, \quad \forall\ t \geq 0, \\ |e^{tJL_c}|_{E^u}|_X &\leq Me^{\lambda_u t}, \quad \forall\ t \leq 0. \end{aligned}$$

and

$$|e^{tJL_c}|_{E^c}|_X \leq M(1 + |t|^{k_0}),\ \forall\ t \in \mathbf{R}.$$

where $k_0 \leq 2n^-(L)$.

Exponential trichotomy (continue)

For $k \geq 1$, define the space $X^k \subset X$ to be

$$X^k = \{u \in X \mid (JL)^n u \in X, \ n = 1, \cdots, k.\}$$

and

$$\|u\|_{X^k} = \|u\|_X + \|JLu\|_X + \cdots + \left\|(JL)^k u\right\|_X .$$

Assume $E^{u,s} \subset X^k$, then the exponential trichotomy holds true for X^k with

$$X^k = E^u \oplus E^c_k \oplus E^s, \ E^c_k = E^c \cap X^k$$

Semigroup estimates (multiple periodic)

First, from the definition of linear MI, the unstable frequencies are in open intervals $I \subset (0,1)$. Pick a rational number $k_0 = \frac{p}{q} \in I$ with $p, q \in \mathbb{N}$. Then $e^{ik_0 x} v_{k_0}(x)$ is of period $2\pi q$ and JL has an unstable eigenvalue in $L^2(\mathbb{T}_{2\pi q})$. It leads us to consider the nonlinear instability of u_c in $L^2(\mathbb{T}_{2\pi q})$. By the above general theorem on linear Hamiltonian PDEs, we have

Lemma

Consider the semigroup e^{tJL} associated with the linearized equation near TW $u_c(x-ct)$ in the traveling frame $(x-ct, t)$, then the exponential trichotomy holds true in the spaces $H^s(\mathbb{T}_{2\pi q})$ $\left(s \geq \frac{m}{2}, q \in \mathbb{N}\right)$ when $\mathcal{M}$ is differential and in $H^s(\mathbb{T}_{2\pi q})$ $(s \geq 0, q \in \mathbb{N})$ when $\mathcal{M}$ is smoothing.

Continue

As an immediate corollary of the above lemma, we get the following upper bound on the growth of the semigroup e^{tJL} which is used in the proof of nonlinear instability.

Corollary

Let λ_0 be the growth rate of the most unstable eigenvalue of JL in $L^2(\mathbb{T}_{2\pi q})$. Then for any $\varepsilon > 0$, there exists constant C_ε such that

$$\left\| e^{tJL} \right\|_{H^s(\mathbb{T}_{2\pi q})} \leq C_\varepsilon e^{(\lambda_0+\varepsilon)t}, \text{ for any } t > 0,$$

where $q \in \mathbb{N}$, $s \geq \frac{m}{2}$ when $\mathcal{M}$ is differential and $s \geq 0$ when $\mathcal{M}$ is smoothing.

Semigroup estimates for localized perturbations

For localized perturbation in $H^s(\mathbb{R})$, we cannot use the general theorem directly since L has bands of negative continuous spectra. But notice that: If $u \in H^s(\mathbb{R})$, by using Fourier transform, we can write $u(x) = \int_0^1 e^{i\xi x} u_\xi(x)\, d\xi$, where $u_\xi \in H_x^s(\mathbb{T})$ and $\|u(x)\|^2_{H^s(\mathbf{R})} \approx \int_0^1 \|u_\xi(x)\|^2_{H^s(\mathbb{T}_{2\pi})} d\xi$. Since $e^{tJL}u(x) = \int_0^1 e^{i\xi x} e^{tJ_\xi L_\xi} u_\xi(x)\ d\xi$, we have $\left\|e^{tJL}u\right\|^2_{H^s(\mathbb{R})} \approx \int_0^1 \left\|e^{tJ_\xi L_\xi} u_\xi\right\|^2_{H_x^s(\mathbb{T})} d\xi$ and and the estimate of e^{tJL} in $H^s(\mathbb{R})$ is reduced to prove the semigroup estimate of $e^{tJ_\xi L_\xi}$ in $H_x^s(\mathbb{T})$ uniformly for $\xi \in (0,1)$.

Lemma

Let λ_0 be the maximal growth rate of $J_\xi L_\xi$, $\xi \in (0,1)$. For every $s \geq \frac{m}{2}$ and any $\varepsilon > 0$, there exists $C(s,\varepsilon) > 0$ such that

$$\|e^{JLt}U_0(x)\|_{H^s(\mathbb{R})} \leqslant C(s,\varepsilon)e^{(\lambda_0+\varepsilon)t}\|U_0(x)\|_{H^s(\mathbb{R})}.$$

Nonlinear instability-smooth case

We use the idea of Grenier (CPAM, 2000) in the proof of nonlinear instability of shear flows to construct higher order approximation solutions and overcome the loss of derivative by using energy estimates.

Lemma (Energy estimates)

Consider the solution of the following equation

$$\partial_t v - c\partial_x v + \partial_x \mathcal{M} v + \partial_x \left(f\left(u_c + U + v\right) - f\left(u_c + U\right)\right) = R,$$

with $v(0,\cdot) = 0$, *and* $U\left(t,\cdot\right) \in H^4\left(\mathbb{T}\right)$, $R\left(t,\cdot\right) \in H^2\left(\mathbb{T}\right)$ *are given. Assume that*

$$\sup_{0\leq t\leq T} \left\|U\right\|\left(t\right)_{H^4\left(\mathbb{T}\right)} + \left\|v\right\|_{H^2\left(\mathbb{T}\right)}\left(t\right) \leq \beta,$$

then there exists a constant $C\left(\beta\right)$ *such that for* $0 \leq t \leq T$,

$$\partial_t \left\|v\right\|_{H^2} \leq C\left(\beta\right)\left\|v\right\|_{H^2} + \left\|R\right\|_{H^2}.$$

Higher order approximate solution

Choose integer N such that $(N+1)\lambda_0 > C(1)$. We construct an approximate solution U^{app} to the nonlinear problem of the form

$$U^{app}(t,x) = u_c(x) + \sum_{j=1}^{N} \delta^j U_j(t,x),$$

where

$$U_1(t,x) = v_g(x)e^{\lambda t} + \bar{v}_g(x)e^{\bar{\lambda} t},$$

is the most rapidly growing real-valued $2\pi q$-periodic solution of the linearized equation. The construction is by induction and such that

$$\|U_j(t,x)\|_{H^{l+1-j}(\mathbb{T})} \leqslant C(N)e^{j\lambda_0 t}, \text{ for } j = 1, 2, \cdots, N,$$

where $l = 4 + N$.

Error estimate

The construction of U^{app} is to ensure that the error term

$$R_{app} = \partial_t U^{app} - c\partial_x U^{app} + \partial_x(\mathcal{M} U^{app} + f(U^{app})),$$

satisfies

$$\|R_{app}\|_{H^2} \leq C(N)\,\delta^{N+1} e^{(N+1)\lambda_0 t}, \quad \text{for } 0 \leq t \leq T^\delta,$$

where $\delta e^{\lambda_0 T^\delta} = \theta$ for some $\theta < 1$ small. Let $U_\delta(t,x)$ be the solution to the nonlinear equation with initial value $u_c(x) + \delta U_1(0,x)$, and let $v = U_\delta - U^{app}$.

continue

The error term v satisfies

$$\begin{cases} \partial_t v - c\partial_x v + \partial_x \mathcal{M} v + \partial_x \left(f\left(U^{app} + v\right) - f\left(U^{app}\right)\right) = -R_{app} \\ v(0, \cdot) = 0. \end{cases}$$

By using the lemma on energy estimates, when θ is small we can show that

$$\partial_t \left\| v \right\|_{H^2} \leq C\left(1\right) \left\| v \right\|_{H^2} + \left\| R_{app} \right\|_{H^2}, \text{ for } 0 \leq t \leq T^{\delta}.$$

Thus by Gronwall, $\left\| v \right\|_{H^2}(t) \leq C\left(N\right) \delta^{N+1} e^{(N+1)\lambda_0 t}$.

Continue

The nonlinear instability follows since at the time T_δ with $\delta e^{\lambda_0 T^\delta} = \theta$,

$$\begin{aligned}
& \left\| U_\delta(T^\delta, x) - u_c \right\|_{L^2(\mathbb{T})} \\
\geqslant \quad & \left\| U^{app}(T^\delta, x) - u_c(x) \right\|_{L^2(\mathbb{T})} - \| v(T^\delta, x) \|_{H^2(\mathbb{T})} \\
\geqslant \quad & C_1 \delta e^{\lambda_0 T^\delta} - C_2 \left(\delta e^{\lambda_0 T^\delta} \right)^2 \\
= \quad & C_1 \theta - C_2 \theta^2 \geq \frac{1}{2} C_1 \theta,
\end{aligned}$$

when θ is chosen to be small. The orbital instability can be shown with some extra estimates.

Localized case

There is no genuine unstable eigenfunction of JL in $L^2(\mathbb{R})$. Choose $u_1(0) = \int_I e^{i\xi x} v_\xi(x)\, d\xi$, where I is a small interval centered at the most unstable frequency ξ_0 with maximal growth rate λ_0 and $v_\xi(x)$ is the most unstable eigenfunction of $J_\xi L_\xi$ $(\xi \in I)$ with unstable eigenvalue $\lambda(\xi)$. Then

$$U_1(t,x) = e^{tJL} u_1(0) = \int_I v_\xi(x)\, e^{\lambda(\xi)t} e^{i\xi x}\, d\xi.$$

By using stationary phase type arguments, it can be shown that

$$\|U_1(t,x)\|_{L^2(\mathbb{R})} \approx \frac{Ce^{\lambda_0 t}}{(1+t)^{\frac{1}{2l}}},$$

where l is a positive integer and λ_0 is the maximal growth rate. By using the semigroup estimates in $H^s(\mathbb{R})$, the rest of the proof is similar to the periodic case.

Non-smooth case

In Theorem 1, the assumption $f(u) \in C^\infty$ is required in order to construct approximation solutions to sufficiently high order to close the energy estimates. When $f(u)$ is only C^1 and $\|\mathcal{M}(\cdot)\|_{L^2} \sim \|\cdot\|_{H^m}$ $(m \geq 1)$, the nonlinear instability can be proved by bootstrap arguments. This is done in three steps. First, by using the energy conservation

$$H(u) = \frac{1}{2}\langle Lu, u\rangle - \int_{\mathbf{R}} \left(F(u+u_c) - F(u_c) - f(u_c)u - \frac{1}{2} f'(u_c)u^2 \right) dx,$$

we can show

$$\|u(t)\|_{L^2} \sim \frac{\delta e^{\lambda_0 s}}{(1+s)^{\frac{1}{l}}} \Longrightarrow \|u(t)\|_{H^{m/2}} \sim \frac{\delta e^{\lambda_0 s}}{(1+s)^{\frac{1}{l}}}.$$

Second, we bootstrap $\|u(t)\|_{H^{-1}}$ from $\|u(t)\|_{H^{m/2}}$. The nonlinear solution $u_\delta(t)$ for the unstable perturbation can be written as

$$\begin{aligned} u_\delta(t) &= e^{tJL}u_\delta(0) - \int_0^t e^{(t-s)JL}\partial_x \left(f(u_\delta(s)+u_c) - f(u_c) - f'(u_c)u_\delta(s) \right) \\ &= u_l(t) + u_n(t). \end{aligned}$$

Continue

We have the semigroup estimate in $H^{-1}(\mathbf{R})$: for any $\varepsilon > 0$ there exist $C(\varepsilon) > 0$ such that

$$\|e^{tJL}u(x)\|_{H^{-1}(\mathbf{R})} \leqslant C(\varepsilon)e^{(\lambda_0+\varepsilon)t}\|u(x)\|_{H^{-1}(\mathbf{R})}, \ \forall t > 0.$$

Thus

$$\begin{aligned}
\|u_n(t)\|_{H^{-1}} &\lesssim \int_0^t \left\|e^{(t-s)JL}\right\|_{H^{-1}} \left\|f(u_\delta(s)+u_c) - f(u_c) - f'(u_c)\,u_\delta(s)\right\|_{L^2} \\
&\lesssim \int_0^t e^{(\lambda_0+\varepsilon)(t-s)} \|u_\delta(s)\|^{p_1}_{H^{\frac{m}{2}}}\, ds, \qquad (p_1 > 1) \\
&\lesssim \int_0^t e^{(\lambda_0+\varepsilon)(t-s)} \left(\frac{\delta e^{\lambda_0 s}}{(1+s)^{\frac{1}{l}}}\right)^{p_1} ds \\
&\lesssim \left(\frac{\delta e^{\lambda_0 t}}{(1+t)^{\frac{1}{l}}}\right)^{p_1},
\end{aligned}$$

by choosing $\varepsilon < (p_1 - 1)\lambda_0$.

Interpolation

By interpolation of L^2 by H^{-1} and $H^{\frac{m}{2}}$,

$$\begin{aligned}\|u_n(t)\|_{L^2} &\leq \|u_n(t)\|_{H^{-1}}^{\alpha_1} \|u_n(t)\|_{H^{\frac{m}{2}}}^{1-\alpha_1} \quad \left(\alpha_1 = \frac{m}{m+2}\right) \\ &\lesssim \left(\frac{\delta e^{\lambda_0 t}}{(1+t)^{\frac{1}{l}}}\right)^{\alpha p_1 + 1 - \alpha_1},\end{aligned}$$

where $p_3 = \alpha p_1 + 1 - \alpha_1 > 1$. At $t = T_\delta$ with $\frac{\delta e^{\lambda_0 T_\delta}}{(1+T_\delta)^{\frac{1}{l}}} = \theta$,

$$\begin{aligned}\|u_\delta(T_\delta)\|_{L^2} &\geq \|u_l(T_\delta)\|_{L^2} - \|u_n(T_\delta)\|_{L^2} \\ &\geq C_0 \frac{\delta e^{\lambda_0 T_\delta}}{(1+T_\delta)^{\frac{1}{l}}} - C' \left(\frac{\delta e^{\lambda_0 T_\delta}}{(1+T_\delta)^{\frac{1}{l}}}\right)^{p_3} \\ &= C_0\theta - C'\theta^{p_3} \geq \frac{1}{2} C_0 \theta,\end{aligned}$$

when θ is small.

Remark

1. In the above proof, the semigroup estimates of e^{tJL} in H^{-1} is used to overcome the loss of derivative of the nonlinear term $\partial_x f(u)$, since

$$\|\partial_x f(u)\|_{H^{-1}} \approx \|f(u)\|_{L^2},$$

which is controllable in $H^{\frac{m}{2}}$. To estimate $e^{tJL}|_{H^{-1}}$, by duality it suffices to estimate $e^{tLJ}|_{H^1}$, which is reduced to estimate $e^{tL_\xi J_\xi}|_{H^1(\mathbb{T}_{2\pi})}$ uniformly for $\xi \in [0,1]$. The estimate of $e^{tL_\xi J_\xi}|_{H^1(\mathbb{T}_{2\pi})}$ is obtained by a decomposition of the spectral projections of L_ξ near 0 and away from 0, and then conjugate $e^{tL_\xi J_\xi}$ to $e^{tJ_\xi L_\xi}$ by L_ξ^{-1} on the part away from 0.

2. The idea of overcoming the loss of derivative by bootstrapping the growth of higher order norms from a lower order one was originated in the work of (Guo, Strauss 95) for the Vlasov-Poisson system. This approach was later extended to other problems including 2D Euler equation (Bardos, Guo, Strauss 02) (Lin, 04) and Vlasov-Maxwell systems (Lin, Strauss 07). Here, our approach of bootstrapping the lower order norm (H^{-1}) from a higher order norm $(H^{\frac{m}{2}})$ and then closing by interpolation seems to be new.

Examples-Fractional KDV

Consider the Fractional KDV equation

$$\partial_t u + \partial_x (\Lambda^m u - u^p) = 0,$$

where $\Lambda = \sqrt{-\partial_x^2}$, $m > \frac{1}{2}$ and either $p \in \mathbb{N}$ or $p = \frac{q}{n}$ with q and n being even and odd natural numbers, respectively. It is proved by (Johnson, 2013) that TWs of small amplitude are linearly MI if $m \in (\frac{1}{2}, 1)$ or if $m > 1$ and $p > p^*(m)$, where

$$p^*(m) := \frac{2^m(3+m) - 4 - 2m}{2 + 2^m(m-1)}.$$

Examples-Whitham equation

Whitham equation

$$\partial_t u + \mathcal{M}\partial_x u + \partial_x (u^2) = 0,$$

where

$$\widehat{\mathcal{M}f}(\xi) = \sqrt{\frac{\tanh \xi}{\xi}}\widehat{f}(\xi).$$

It is clear that $\|\mathcal{M}(\cdot)\|_{H^{1/2}} \sim \|\cdot\|_{L^2}$. When $f(u) = u^2$, for small amplitude TWs, the condition

$$c - 2\|u_c\|_{L^\infty(\mathbb{T}_{2\pi})} \geqslant \epsilon > 0,$$

is satisfied and it is also true for large amplitude waves by numerics. It was shown by Hur & Johnson (2015) that the small TWs of small period are linearly modulationally unstable.

Summary

1. For other dispersive models with energy-momentum functional bounded from below $(n^{-}(L) < \infty)$, we could use the similar approach to prove that linear MI implies nonlinear MI for both periodic and localized perturbations.
2. The remaining problem is to prove nonlinear MI when the energy-momentum functional is indefinite. An important example is 2D water waves for which the linear MI for small amplitude Stokes waves was first found by (Benjamin & Feir 1967) and later proved by (Bridges and Mielke, 1995) for the finite depth case.
3. For multi-periodic perturbations, it is possible to construct invariant manifolds (stable, unstable and center) which give the complete dynamics near the orbit of unstable waves. For localized perturbations, it is not clear how to describe the local dynamics for general initial data. The long time dynamics for unstable perturbations is more challenging.