On long-term existence of water wave models

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The water wave equations

We consider the free boundary incompressible Euler equations

$$v_t + v \cdot \nabla v = -\nabla p - g e_n, \qquad \nabla \cdot v = 0, \qquad x \in \Omega_t,$$

where g is the gravitational constant. The free surface $\Gamma_t = \{z(\alpha, t) : \alpha \in \mathbb{R}\}$ moves with the velocity, according to the kinematic boundary condition

$$(\partial_t z - v)|_{\Gamma_t} \qquad \text{tangent to } \Gamma_t.$$

In the presence of surface tension the pressure on the interface is given by

$$p(x,t) = \sigma \kappa(x,t), \quad x \in \Gamma_t\,,$$

where κ is the mean-curvature of Γ_t and $\sigma > 0$.

Natural questions:

- Local regularity
- Global regularity and asymptotics
- Dynamical formation of singularities

Possible variants: Periodic conditions, finite bottom, two-fluid model.

Local wellposedness: Nalimov (1974), Yosihara (1982), Craig (1985), Wu (1997, 1999), Beyer–Gunther (1998), Christodoulou–Lindblad (2000), Ambrose (2003), Ambrose–Masmoudi (2005), Lannes (2005), Lindblad (2005), Coutand–Shkoller (2007), Cheng–Coutand–Shkoller (2008), Christianson–Hur–Staffilani (2010), Alazard–Burq–Zuily (2011), Shatah–Zeng (2008, 2011).

One has local regularity if $\sigma > 0$ or if the Rayleigh–Taylor condition is satisfied. The time of existence depends on two quantities: the smoothness, say in H^{10}, of the interface and the fluid velocities, and the arc-chord constant of the interface.

Formation of singularities: possible scenarios: (1) loss of regularity, and (2) self-intersection of the interface.

The "splash" singularity of Castro–Cordoba–Fefferman–Gancedo–Gomez-Serrano (new proof of Coutand–Shkoller).

Figure 2. Formation of "splash" singularities.

- The splash singularity cannot form in the two-fluid model (Fefferman–I.–Lie).

Global regularity

Small irrotational global solutions, with either gravity or surface tension (but not both) in 2D or 3D:

- (almost global) 2D gravity waves $g > 0, \sigma = 0$: Wu (2009)
- 3D gravity waves $g > 0, \sigma = 0$: Wu,

Germain–Masmoudi–Shatah;

- 3D capillary waves $g = 0, \sigma > 0$: Germain–Masmoudi–Shatah;
- 2D gravity waves $\sigma = 0, g > 0$: I.–Pusateri, Alazard–Delort

(new proofs in different topologies by Hunter–Ifrim–Tataru (almost global regularity), Ifrim–Tataru (global regularity), Wang (removal of a momentum condition on the velocity field));

- 2D capillary waves $g = 0, \sigma > 0$: I.–Pusateri in the general case,

Ifrim–Tataru assuming one momentum condition on the Hamiltonian variables.

- 3D gravity or capillary waves with finite bottom: Wang.
- 3D gravity waves $g > 0, \sigma > 0$: Deng–I.–Pausader–Pusateri.

- In 2 dimensions (1D interface), there are no resonances if either $g = 0$ or $\sigma = 0$. An important piece of the proof is the *quartic energy inequality* (Wu)

$$\mathcal{E}_N(t) - \mathcal{E}_N(0) \lesssim \Big| \int_0^t \langle \nabla \rangle u \cdot \langle \nabla \rangle u \cdot \langle \nabla \rangle^N u \cdot \langle \nabla \rangle^N u \, dxds \Big|.$$

- Formally, it is similar to Shatah's normal form method. It is important not to lose derivatives in the right-hand side.
- The linearized and nonlinear solution have $t^{-1/2}$ pointwise decay, which leads to almost-global existence. Global existence relies on understanding the scattering theory, i.e. proving *modified scattering* (I.-Pusateri, Alazard-Delort).
- Improvements: paradifferential energy estimates (Alazard-Delort), compatible vector-field structures (I.-Pusateri), modified energy method (Hunter-Ifrim-Tataru).
- The quartic energy inequality was proved in other settings: gravity constant vorticity (Ifrim-Tataru), gravity finite bottom (Harrop-Griffith–Ifrim–Tataru).

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- In 3 dimensions (2D interface), if either $g = 0$ or $\sigma = 0$ then one has and $1/t$ pointwise decay for both the linearized solution and the nonlinear solution. One can close the argument by letting the highest order energy grow slowly.

The "division" problem

Consider a generic evolution problem of the type

$$\partial_t u + i\Lambda u = \mathcal{N}(u, D_x u)$$

where Λ is real and $\mathcal{N}$ is a quadratic nonlinearity. At first iteration

$$u(t) = e^{-it\Lambda}\phi.$$

At second iteration, assuming $\mathcal{N} = \partial_1(u^2)$,

$$\widehat{u}(\xi, t) = e^{-it\Lambda(\xi)}\widehat{\phi}(\xi) + Ce^{-it\Lambda(\xi)} \int \widehat{\phi}(\xi - \eta)\widehat{\phi}(\eta) i\xi_1 \frac{1 - e^{it[\Lambda(\xi) - \Lambda(\eta) - \Lambda(\xi-\eta)]}}{\Lambda(\xi) - \Lambda(\eta) - \Lambda(\xi - \eta)}\, d\eta.$$

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$$\{(\xi, \eta) : \pm\Lambda(\xi) \pm \Lambda(\eta) \pm \Lambda(\xi - \eta) = 0\}.$$

The phases corresponding to bilinear interactions satisfy the following **restricted nondegeneracy condition** of the resonant hypersurfaces: if

$$\Phi(\xi,\eta) := \pm\Lambda(\xi) \pm \Lambda(\eta) \pm \Lambda(\xi-\eta)$$

and

$$\Upsilon(\xi,\eta) := \nabla^2_{\xi,\eta}\Phi(\xi,\eta)\left[\nabla^{\perp}_{\xi}\Phi(\xi,\eta), \nabla^{\perp}_{\eta}\Phi(\xi,\eta)\right],$$

then $\Upsilon(\xi,\eta) \neq 0$ at (almost all) points on the time-resonant set $\Phi(\xi,\eta) = 0$.

In the irrotational case $\mathrm{curl}\, v = 0$, let Φ denote the velocity potential, $v = \nabla\Phi$, and let $\phi(x,t) = \Phi(x, h(x,t), t)$ denote its trace on the interface.

Main Theorem. (Deng, I., Pausader, Pusateri) If $g > 0$, $\sigma > 0$, and

$$\|(h_0, \phi_0)\|_{\text{Suitable norm}} \leq \varepsilon_0 \ll 1$$

then there is a unique smooth global solution of the gravity-capillary water-wave system in 3d, with initial data (h_0, ϕ_0),

$$\begin{cases} \partial_t h = G(h)\phi, \\ \partial_t \phi = -gh + \sigma \mathrm{div}\Big[\dfrac{\nabla h}{(1+|\nabla h|^2)^{1/2}}\Big] - \dfrac{1}{2}|\nabla\phi|^2 + \dfrac{(G(h)\phi + \nabla h \cdot \nabla\phi)^2}{2(1+|\nabla h|^2)}. \end{cases}$$

where $G(h)$ is the (normalized) Dirichlet-Neumann map associated to the domain Ω_t (the Zakharov-Craig-Sulem formulation). The solution $(h,\phi)(t)$ decays in L^∞ at $t^{-5/6+}$ rate as $t \to \infty$.

For sufficiently smooth solutions, this is a Hamiltonian system which admits the conserved energy (Zakharov)

$$\mathcal{H}(h,\phi) := \frac{1}{2}\int_{\mathbf{R}^{n-1}} G(h)\phi\cdot\phi\,dx + \frac{g}{2}\int_{\mathbf{R}^{n-1}} h^2\,dx$$
$$+\sigma\int_{\mathbf{R}^{n-1}} \frac{|\nabla h|^2}{1+\sqrt{1+|\nabla h|^2}}\,dx$$
$$\approx \||\nabla|^{1/2}\phi\|_{L^2}^2 + \|(g-\sigma\Delta)^{1/2}h\|_{L^2}^2.$$

Model equation:

$$(\partial_t + i\Lambda)U = \nabla V\cdot\nabla U + (1/2)\Delta V\cdot U, \qquad U(0)=U_0,$$
$$\Lambda(\xi) := \sqrt{|\xi|+|\xi|^3}, \qquad V := P_{[-10,10]}\Re U.$$

which has the L^2 conservation law

$$\|U(t)\|_{L^2} = \|U_0\|_{L^2}, \quad t\in[0,\infty).$$

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The specific dispersion relation $\Lambda(\xi) = \sqrt{|\xi| + |\xi|^3}$ is important. It is radial and has stationary points when $|\xi| = \gamma_0 := \sqrt{2/\sqrt{3} - 1} \approx 0.393$. As a result, linear solutions $e^{it\Lambda}\phi$ can only have $|t|^{-5/6}$ pointwise decay, even for Schwartz functions.

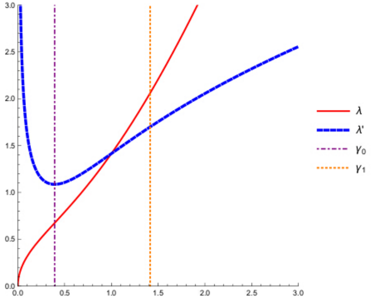

Figure: The dispersion relation $\lambda(r) = \sqrt{r^3 + r}$ and the group velocity λ'. The frequency γ_1 corresponds to the space-time resonant sphere.

Resonant sets:

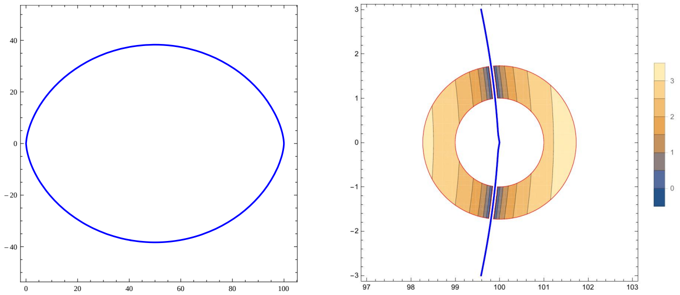

Figure: The first picture illustrates the resonant set $\{\eta : 0 = \Phi(\xi,\eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda(\xi-\eta)\}$ for a fixed large frequency ξ (in the picture $\xi = (100, 0)$). The second picture illustrates the intersection of a neighborhood of this resonant set with the set where $|\xi - \eta|$ is close to γ_0.

Schematically, we prove the following bootstrap proposition:

Proposition: Assume that U is a solution on some time interval $[0, T]$, with initial data U_0. Define, as before, $u(t) = e^{it\Lambda}u(t)$. Assume that

$$\|u_0\|_{H^{N_0} \cap H^{N_1}_\Omega} + \|u_0\|_Z \leq \varepsilon_0 \ll 1$$

and

$$(1+t)^{-p_0}\|u(t)\|_{H^{N_0} \cap H^{N_1}_\Omega} + \|u(t)\|_Z \leq \varepsilon_1 \ll 1$$

for all $t \in [0, T]$. Then, for any $t \in [0, T]$

$$(1+t)^{-p_0}\|u(t)\|_{H^{N_0} \cap H^{N_1}_\Omega} \lesssim \varepsilon_0 \qquad \text{using energy estimates,}$$

$$\|u(t)\|_Z \lesssim \varepsilon_0 \qquad \text{using dispersive analysis.}$$

Energy estimates

- Start with an energy inequality of the form

$$\mathcal{E}_N(t) - \mathcal{E}_N(0) \leq \Big| \int_0^t \int_{\mathbb{R}^2} D^N U \times D^N U \times DU \, dxds \Big|$$

- Transfer to the Fourier space (the I-method of Colliander–Keel–Staffilani–Takaoka–Tao), and let $W = D^N U$

$$\mathcal{E}_N(t) - \mathcal{E}_N(0)$$
$$\leq \Big| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \widehat{\overline{W}}(-\xi) \widehat{W}(\eta) \widehat{U}(\xi - \eta) m(\xi, \eta) \, d\xi d\eta dt \Big|$$

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$$\begin{aligned} &\mathcal{E}_N(t) - \mathcal{E}_N(0) \\ &\leq \Big| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \overline{\widehat{W}}(-\xi) \widehat{W}(\eta) \widehat{U}(\xi - \eta) m(\xi, \eta) \, d\xi d\eta dt \Big| \end{aligned}$$

- Rewrite the energy increment inequality in terms of the profiles $u(t) := e^{it\Lambda}U$, $w(t) := e^{it\Lambda}W$

$$\begin{aligned}&\mathcal{E}_N(t) - \mathcal{E}_N(0)\\ &\leq \Big| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\Phi(\xi,\eta)} \widehat{\overline{w}}(-\xi) \widehat{w}(\eta) \widehat{u}(\xi - \eta) m(\xi, \eta) \, d\xi d\eta ds \Big|\end{aligned}$$

Here

$$\Phi(\xi, \eta) = \pm\Lambda(\xi) \pm \Lambda(\eta) \pm \Lambda(-\xi - \eta).$$

The function Φ (typically) has a codimension 1 vanishing set.

The profiles satisfy equations of the form (with quadratic nonlinearities)

$$\begin{aligned}\partial_t u &= e^{it\Lambda} D(e^{-it\Lambda} u \times e^{-it\Lambda} u);\\ \partial_t w &= e^{it\Lambda} D(e^{-it\Lambda} u \times e^{-it\Lambda} w).\end{aligned}$$

- Decompose the bulk term dyadically over time $\approx 2^m$, frequency $\approx 2^k$, modulation $\approx 2^p$,

$$I_{k,m,p} := \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi_p(\Phi(\xi,\eta)) e^{is\Phi(\xi,\eta)} m(\xi,\eta) \\ \times \widehat{P_k \overline{w}}(-\xi,s)\widehat{P_k w}(\eta,s)\chi_{\gamma_0}(\xi-\eta)\widehat{u}(\xi-\eta,s)\, d\xi d\eta ds.$$

We could estimate this using integration by parts in time (Shatah's normal form method),

$$I_{k,m,p} \approx \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\varphi_p(\Phi(\xi,\eta))}{\Phi(\xi,\eta)} e^{is\Phi(\xi,\eta)} m(\xi,\eta) \\ \times \frac{d}{ds}\left[\widehat{P_k \overline{w}}(-\xi,s)\widehat{P_k w}(\eta,s)\right]\chi_{\gamma_0}(\xi-\eta,s)\widehat{u}(\xi-\eta)\, d\xi d\eta ds.$$

For small p we estimate the integral using an L^2 lemma. This is the critical gain of the argument. It depends on the functions Φ satisfying the "restricted nondegeneracy condition".

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Main L^2 lemma: Assume that $k, m \gg 1$,

$$-m + \delta m \leq p - k/2 \leq -\delta m, \qquad 2^{m-1} \leq |s| \leq 2^{m+1}.$$

Let T_p denote the operator defined by

$$T_p f(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \chi(2^{-p}\Phi(\xi,\eta)) \chi_{\gamma_0}(\xi - \eta) \varphi_k(\eta) a(\xi,\eta) f(\eta) d\eta,$$

where

$$\Phi(\xi,\eta) = \Lambda(\xi) \pm \Lambda(\xi - \eta) - \Lambda(\eta).$$

Then

$$\|T_p\|_{L^2 \to L^2} \lesssim 2^{\delta m}[2^{-m/3+(p-k/2)} + 2^{3(p-k/2)/2}].$$

Depends on the fact that

$$\Upsilon(\xi,\eta) := \nabla^2_{\xi,\eta}\Phi(\xi,\eta)\left[\nabla^{\perp}_{\xi}\Phi(\xi,\eta), \nabla^{\perp}_{\eta}\Phi(\xi,\eta)\right] \neq 0,$$

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This L^2 lemma can be used to control the contribution of small modulation, $p - k/2 \leq -2m/3 - 2\delta m$. For higher modulation we integrate by parts in time. The danger here is a potential loss of derivative, due to the equation

$$\partial_t w = e^{it\Lambda} D(e^{-it\Lambda} u \times e^{-it\Lambda} w).$$

Recall the model

$$(\partial_t + i\Lambda)U = \nabla V \cdot \nabla U + (1/2)\Delta V \cdot U, \qquad U(0) = U_0,$$

$$\Lambda(\xi) := \sqrt{|\xi| + |\xi|^3}, \qquad V := P_{[-10,10]}\Re U.$$

The multiplier in the space-time integrals is

$$m(\xi, \eta) = \frac{(\xi - \eta) \cdot (\xi + \eta)}{2} \frac{(1 + |\eta|^2)^N - (1 + |\xi|^2)^N}{(1 + |\eta|^2)^{N/2}(1 + |\xi|^2)^{N/2}}.$$

which satisfies

$$m(\xi, \eta) \lesssim \mathfrak{d}(\xi, \eta), \quad \text{where} \quad \mathfrak{d}(\xi, \eta) := \frac{[(\xi - \eta) \cdot (\xi + \eta)]^2}{1 + |\xi + \eta|^2}.$$

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The key point is the algebraic correlation

$$\text{if } |\Phi(\xi,\eta)| \lesssim 1 \text{ then } |m(\xi,\eta)| \lesssim 2^{-k}.$$

between the smallness of the modulation and the smallness of the depletion factor $\mathfrak{d}$.

We found exactly the same algebraic correlation in the 2d Euler–Maxwell system for electrons (a plasma model).

These energy estimates can be used to control the growth of the high order energy weighted norms

$$\|u\|_{H^{N_0}} \quad \text{and} \quad \|u\|_{H^{N_1}_\Omega} := \sup_{b\in[0,N_1]} \|\Omega^b u\|_{L^2}$$

where $\Omega = x_1\partial_2 - x_2\partial_1$ is the rotation vector-field.

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In our (weighted) case we use Duhamel formula and the concept of space-time resonances (Germain–Masmoudi–Shatah):

$$(\partial_t + i\Lambda)U = \sum_{\pm} \mathcal{N}(U_\pm, U_\pm),$$

where $U_+ = U$, $U_- = \overline{U}$, and the nonlinearities are defined by

$$(\mathcal{F}\mathcal{N}(f,g))(\xi) = \int_{\mathbb{R}^2} \mathfrak{m}(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.$$

With $u(t) = e^{it\Lambda}U(t)$, the Duhamel formula is

$$\widehat{u}(\xi,t) = \widehat{u}(\xi,0) + \sum_{\pm} \int_0^t e^{is\Phi(\xi,\eta)}\mathfrak{m}(\xi,\eta)\widehat{u_\pm}(\xi-\eta,s)\widehat{u_\pm}(\eta,s)\,d\eta ds.$$

Critical points (spacetime resonances): with

$$\Phi(\xi, \eta) = \Lambda(\xi) \pm \Lambda(\eta) \pm \Lambda(\xi - \eta)$$

the set of space-time resonances is

$$\{(\xi, \eta) : \Phi(\xi, \eta) = 0 \text{ and } \nabla_\eta \Phi(\xi, \eta) = 0\}.$$

In our case

$$(\xi, \eta) = (\gamma_1 \omega, \gamma_1 \omega / 2),$$

where $\omega \in \mathbb{S}^1$ and $\gamma_1 = \sqrt{2}$.

If we input Schwartz functions into the Duhamel formula, we get a different type of output,

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We define

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We define

$$\|f\|_Z := \sup_{(k,j)\in\mathcal{J}} \sup_{|\alpha|\leq 50,\, m\leq N_1/2} \|D^\alpha\Omega^m Q_{jk}f\|_{B_j^\sigma},$$

where

$$\|g\|_{B_j^\sigma} := 2^{(1-50\delta)j}2^{-(1/2-49\delta)n}\|A_ng\|_{L^2}.$$

The operators A_n are projection operators relative to the location of the spheres of space-time resonances, $||\xi| - \gamma_1| \approx 2^{-n}$.

Our Z norm depends in a significant way on both the linear part and the quadratic part of the equation. Norms of this type were introduced in work on the Euler–Maxwell equations in 3d.

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$$\|f\|_Z := \sup_{(k,j)\in\mathcal{J}} \sup_{|\alpha|\le 50,\, m\le N_1/2} \|D^\alpha\Omega^m Q_{jk}f\|_{B_j^\sigma},$$

where

$$\|g\|_{B_j^\sigma} := 2^{(1-50\delta)j}2^{-(1/2-49\delta)n}\|A_ng\|_{L^2}.$$

The operators A_n are projection operators relative to the location of the spheres of space-time resonances, $||\xi| - \gamma_1| \approx 2^{-n}$.

Our Z norm depends in a significant way on both the linear part and the quadratic part of the equation. Norms of this type were introduced in work on the Euler–Maxwell equations in 3d.

Elements of the proof:

- time resonance

$$\Phi(\xi, \eta) = 0;$$

- space-time resonance (dispersive analysis)

$$\Phi(\xi, \eta) = 0 \quad \text{and} \quad \nabla_\eta \Phi(\xi, \eta) = 0.$$

- nondegenerate space-time resonance (dispersive analysis)

$$\nabla_\eta^2 \Phi(\xi, \eta) \quad \text{non-singular at space-time resonances.}$$

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- restricted nondegenerate time resonance (energy method)

$$\Phi(\xi,\eta)=0, \qquad \Upsilon(\xi,\eta):=\nabla^2_{\xi,\eta}\Phi(\xi,\eta)\left[\nabla^{\perp}_{\xi}\Phi(\xi,\eta),\nabla^{\perp}_{\eta}\Phi(\xi,\eta)\right]\neq 0.$$

- algebraic correlation between multipliers and modulation

$$\text{if } \ |\Phi(\xi,\eta)|\lesssim 1 \ \text{ then } \ |m(\xi,\eta)|\lesssim (1+|\xi|+|\eta|)^{-1}.$$

- The solution scatters in the Z norm. However,

$$|\widehat{u}(\xi,t)|\gtrsim \varepsilon^2\log(2+t) \qquad \text{if } |\xi|=\gamma_1,$$

for solutions starting from small Schwartz data.

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$$\widetilde{\Phi}(\xi,\eta,\sigma)=\Phi(\xi,\eta)+\Phi(\eta,\sigma)$$

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• Compatible vector-field structures: only certain combinations of vector fields can be propagated through energy estimates:

$$H^{N_0} \cap H^{N_1}_{\Omega}, \qquad N_0 \approx 2N_1 \approx 4000.$$

• Analysis of "almost radial" functions: in the dispersive part of the argument we can pretend that our profiles are almost radial. This leads to "illegal" estimates such as

$$(1+t)\|e^{-it\Lambda}P_{\text{away from }\gamma_0}f\|_{L^\infty} \lesssim \|(1+|x|)^{1/2+}f\|_{L^2}.$$

These are similar to Klainerman–Sobolev inequalities.

• We can see all of these issues in the simpler model

$$(\partial_t + i\Lambda)U = \nabla V \cdot \nabla U + (1/2)\Delta V \cdot U, \qquad U(0) = U_0,$$

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