Stability of traveling waves with a point vortex

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joint work with
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Introduction

Consider a traveling or steady wave moving through a body of water in $\mathbb{R}^2$.

Shifting to a reference frame moving to the right at the wave speed c, the water occupies the domain

$$\Omega := \{x \in \mathbb{R}^2 : -d < x_2 < \eta(x_1)\},$$

where the air–sea interface S is given as the graph of a smooth free surface profile $\eta = \eta(x_1)$.

The ocean depth is $d \in (0, \infty]$.

$$\begin{cases} (v - c) \cdot N = 0 \\ p = \tau\kappa \end{cases}$$

$x_2 = \eta(x_1)$

$$\begin{cases} (v - c) \cdot \nabla v + \nabla p + g\mathbf{e_2} = 0 \\ \nabla \cdot v = 0. \end{cases}$$

Ω

$x_2 = -d$

$v_2 = 0$

The flow is described by the velocity field $v : \Omega \to \mathbb{R}^2$ and pressure $p : \Omega \to \mathbb{R}$.

Here $g > 0$ is the gravitational constant, $\tau > 0$ is the coefficient of surface tension, κ is the mean curvature, and N is the outward normal.

The vorticity ω is the scalar distribution

$$\omega := \partial_{x_1} v_2 - \partial_{x_2} v_1.$$

Historically, most investigations of water waves have been conducted in the irrotational setting, i.e., with $\omega \equiv 0$.

This is justified on physical grounds (as it propagated by Eulerian flow), but the main appeal is mathematical convenience: if $\omega \equiv 0$, then

$$v = \nabla \varphi, \qquad \Delta \varphi = 0 \text{ in } \Omega$$

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On the other hand, rotational steady waves occur frequently in nature (due to wind forcing, heterogeneous density, etc.) Here, significant progress has been made only recently.

We now enjoy a bounty of existence results for various regimes of rotational waves (gravity waves, stratified waves, waves of infinite depth, waves with critical layers, capillary, and capillary-gravity waves, for example).

Clearly, though, the rotational theory is far less explored than the irrotational.

One common feature of the vast majority of these existence results for rotational waves is that the vorticity is not compactly supported.

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In summary, there is a vast body of work on the irrotational case, and a rapidly growing body of work for the rotational case (where ω does not even vanish at infinity!).

But there is an important middle point: traveling waves where the vorticity is localized.

Recently, with a number of collaborators, I have been investigating various properties of these localized vorticity water waves.

In this talk, I will present some existence results for 2-d traveling waves with point vortices and vortex patches and some ongoing work on stationary waves with exponentially localized vorticity.

The main topic will be the stability of the traveling waves with a point vortex, which is established using a new abstract framework.

Existence theory

Our main objects of interest are traveling waves with a point vortex. This describes the situation where ω is a Dirac δ-measure:

$$\omega = \varepsilon \delta_{\bar{\mathbf{x}}},$$

with $\varepsilon \in \mathbb{R}$ being the vortex strength and $\bar{\mathbf{x}} \subset \Omega$ is the center of the vortex.

We may decompose the velocity field as

$$v = \nabla\Phi + \varepsilon\nabla^{\perp}\Gamma,$$

where Φ is a harmonic function and Γ gives the rotational part of the flow. Indeed, taking the curl of this identity shows that

$$\delta_{\overline{\mathbf{x}}} = \Delta\Gamma,$$

and hence

$$\Gamma = \text{Newtonian potential} + \text{harmonic function}.$$

We choose the harmonic function to counteract the logarithmic growth of the potential at infinity; think of it as a "phantom vortex" outside Ω.

If $\operatorname{dist}(\overline{\mathbf{x}}, S) > 0$, then v can be written as a gradient near the boundary just as in the irrotational regime:

$$v = \nabla\Phi + \varepsilon\nabla\Psi,$$

where $\Psi = \Psi_1 - \Psi_2$ is given by

$$\Psi_1(x) := -\frac{1}{2\pi}\arctan\left(\frac{x_1 - \overline{x}_1}{x_2 - \overline{x}_2}\right)$$
$$\Psi_2(x) := -\frac{1}{2\pi}\arctan\left(\frac{x_1 - \overline{x}_1}{x_2 + \overline{x}_2}\right).$$

Note that Ψ_1 is roughly the harmonic conjugate of the Newtonian potential in $\mathbb{R}^2$. The purpose of the Ψ_2 term is to ensure that $\Psi \in \dot{H}^1$.

The kinematic condition takes the form

$$0 = c\eta' + (-\eta', 1) \cdot \nabla(\Phi + \varepsilon\Psi) \qquad \text{on } S.$$

Likewise, the Bernoulli condition is

$$-c\partial_{x_1}(\Phi + \varepsilon\Psi) + \frac{1}{2}\left|\nabla(\Phi + \varepsilon\Psi)\right|^2 + gx_2 + \tau\kappa = 0 \qquad \text{on } S.$$

Recall that $\tau > 0$ is the coefficient of surface tension and κ is the curvature of the surface.

Following the general strategy of the Zakharov–Craig–Sulem formulation of the time-dependent probelm, let φ be the restriction of Φ to S:

$$\varphi = \varphi(x_1) := \Phi(x_1, \eta(x_1)).$$

Tangential derivatives of Φ can be written in terms of x_1-derivative of φ and η.

To take normal derivatives, we use the Dirichlet–Neumann operator $\mathcal{N}(\eta)$ and its non-normalized counterpart $\mathcal{G}(\eta)$

$$\mathcal{G}(\eta) := \sqrt{1 + (\eta')^2}\mathcal{N}(\eta).$$

Then the Bernoulli condition becomes

$$
\begin{aligned}
0 = -c \left(\varphi' + \varepsilon(1, \eta') \cdot (\nabla \Psi)|_S \right) + \frac{1}{2} \left(\varphi' + \varepsilon(1, \eta') \cdot (\nabla \Psi)|_S \right)^2 \\
- \frac{1}{2(1 + (\eta')^2)} \left(\mathcal{G}(\eta)\varphi + \eta'\varphi' + \varepsilon(1 + (\eta')^2)(\partial_{x_1}\Psi)|_S \right)^2 \\
+ g\eta + \tau\kappa,
\end{aligned}
$$

and the kinematic condition is

$$
0 = c\eta' + \mathcal{G}(\eta)\varphi + \varepsilon(1, \eta') \cdot (\nabla \Psi)|_S.
$$

Finally, we must couple the motion of the point vortex to the flow.

The correct governing equation (obtained by taking the limit as the support of ω shrinks to a point) is to have the center of the vortex $\bar{\mathbf{x}}$ advected by the irrotational part of the flow:

$$c = (\partial_{x_1}\Phi)(\bar{\mathbf{x}}) - \frac{1}{4\pi|\bar{x}_2|}.$$

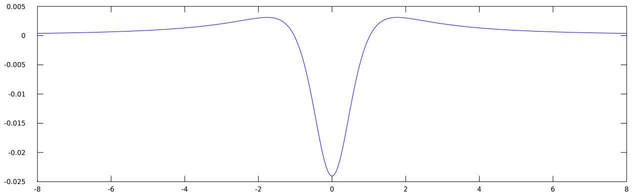
Thus, for traveling waves, the point vortex is stationary in the moving frame.

We will fix its position to be $(0, -a)^T$, where a is the altitude and is treated as a parameter.

The main existence theorem is then the following. For a regularity index $s \geq 3/2$, define

$$\mathbb{W} := H^s_{\mathrm{e}}(\mathbb{R}) \times \left(\dot{H}^s_{\mathrm{o}}(\mathbb{R}) \cap \dot{H}^{\frac{1}{2}}_{\mathrm{o}}(\mathbb{R})\right) \times \mathbb{R}.$$

where the subscripts ‘e’ and ‘o’ denote evenness and oddness in x_1, respectively.



Theorem (Shatah–W.–Zeng, Varholm–Wahlén–W.)

For every $a_0 \in (0, \infty)$, there exists $\varepsilon_0 > 0$, $\alpha_0 > 0$, and C^1 surface

$$\mathcal{S}_{\mathrm{loc}} = \{(\eta(\varepsilon, a), \varphi(\varepsilon, a), c(\varepsilon, a), \varepsilon, a) : |\varepsilon| < \varepsilon_0,\ |a - a_0| < \alpha_0\} \subset \mathbb{W} \times \mathbb{R} \times \mathbb{R}$$

to the traveling capillary-gravity water wave with a point vortex problem. In a sufficiently small neighborhood of 0, $\mathcal{S}_{\mathrm{loc}}$ comprises all solutions.

The proof is an implicit function theorem argument that also furnishes an asymptotic description:

$$\eta(\varepsilon, a) = \frac{\varepsilon^2}{4\pi^2}\left(g - \tau\partial_{x_1}^2\right)^{-1}\left[\frac{x_1^2 + 3a_0^2}{\left(x_1^2 + a_0^2\right)^2}\right] + O\left(|\varepsilon|^3 + |\varepsilon||a - a_0|^2\right)$$

$$\varphi(\varepsilon, a) = O\left(|\varepsilon|^3 + |\varepsilon|^2|a - a_0| + |\varepsilon||a - a_0|^2\right)$$

$$c(\varepsilon, a) = -\frac{\varepsilon}{4\pi a_0} + \frac{\varepsilon(a - a_0)}{4\pi a_0^2} + O\left(|\varepsilon|^3 + |\varepsilon||a - a_0|^2\right).$$

There are a number of other related existence results that we won't discuss in detail today:

- small-amplitude steady capillary-gravity waves with one or more point vortices in finite-depth [Varholm];
- global bifurcation for periodic capillary-gravity waves with a point vortex [Shatah–W.–Zeng];
- small-amplitude traveling capillary-gravity waves with a vortex patch (with generic vorticity distribution in the patch) [Shatah–W.–Zeng]; and
- small-amplitude stationary capillary-gravity waves with exponentially localized vorticity [Ehrnström–W.–Zeng].

Stability theory

Now, we would like to discuss the stability theory for these solutions.

The main machinery for proving this is a generalization of the classical work of Grillakis–Shatah–Strauss on stability of abstract Hamiltonian systems.

Hamiltionian formulation

With that in mind, we must first convince ourselves that this is indeed a Hamiltonian system.

We expect this might be true since the irrotational capillary-gravity water waves problem is Hamiltonian, and the motion of point vortices in the plane is Hamiltonian.

The energy is given by

$$E = E(\eta, \varphi, \overline{\mathbf{x}}, \varepsilon) := K(\eta, \varphi, \overline{\mathbf{x}}, \varepsilon) + V(\eta),$$

where the kinetic energy K is

$$\begin{aligned} K(\eta, \varphi, \overline{\mathbf{x}}, \varepsilon) := \frac{1}{2} \int \varphi \mathcal{G}(\eta) \varphi \, dx_1 + \varepsilon \int \varphi (\partial^\perp \Psi)|_S \, dx_1 \\ + \frac{\varepsilon^2}{2} \int (\partial^\perp \Psi)|_S \Psi|_S \, dx_1 + \frac{\varepsilon^2}{2} \log |2\overline{x}_2|, \end{aligned}$$

and the potential energy V is

$$V(\eta) := \int \left(\frac{1}{2} \eta^2 + \frac{\tau}{g} (\sqrt{1 + \eta_x^2} - 1) \right) \, dx_1.$$

Here $\partial^\perp := -\eta' \partial_{x_1} + \partial_{x_2}$. Finally, the momentum P is given by

$$P(\eta, \varphi, \overline{\mathbf{x}}, \varepsilon) := \varepsilon \overline{x}_2 - \int \eta' (\varphi + \varepsilon \Psi|_S) \, dx_1.$$

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Theorem (Varholm–Wahlén–W.)

The capillary-gravity water wave problem with a point vortex is equivalent to the Hamiltonian equation

$$\frac{du}{dt} = J(u,\varepsilon)\,\mathrm{D}\,E(u,\varepsilon), \qquad u = (\eta, \varphi, \bar{\mathbf{x}})^T,$$

where for each (u,ε), $J(u,\varepsilon)$ *is the skew-symmetric operator*

$$J(u,\varepsilon) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & J_{22} & J_{23} & J_{24} \\ 0 & J_{32} & 0 & 1/\varepsilon \\ 0 & J_{42} & -1/\varepsilon & 0 \end{pmatrix},$$

with entries (denoting $\Theta := \Psi_1 + \Psi_2$*)*

$$\begin{aligned} J_{22} &:= -\varepsilon(\partial_{x_2}\Theta)|_S\langle\cdot,\,(\partial_{x_1}\Psi)|_S\rangle + \varepsilon(\partial_{x_1}\Psi)|_S\langle\cdot,\,(\partial_{x_2}\Theta)|_S\rangle \\ J_{23} &:= -(\partial_{x_2}\Theta)|_S, \qquad J_{24} := \Psi_{x_1}|_S, \\ J_{32} &:= \langle\cdot,\,(\partial_{x_2}\Theta)|_S\rangle, \qquad J_{42} := \langle\cdot,\,-\Psi_{x_1}|_S\rangle. \end{aligned}$$

We can also show that traveling waves are critical points of the augmented Hamiltonian

$$E_c := E - cP.$$

One can think of steady waves as minimizers of the energy E subject to fixed momentum P; the wave speed c is a Lagrange multiplier.

GSS framework

To study the stability of traveling waves (bound states) in a Hamiltonian system of this type, a very powerful tool is the Grillakis–Shatah–Strauss (GSS) method.

An abbreviated statement of their result is the following. Suppose that one has a Hamiltonian system

$$\frac{du}{dt} = J\,\mathrm{D}\,E(u), \qquad u \in C^1(\mathbb{R}_+; \mathbb{X})$$

for which the Cauchy problem is globally well-posed on a Hilbert space $\mathbb{X}$. Here $J : \mathbb{X}^* \to \mathbb{X}$ is a skew symmetric operator that is surjective.

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Assume that the entire system is invariant under translation in some direction.

This symmetry generates an additional conserved quantity P.

Assume that there is a family of traveling waves $\{U_c : |c| < c_0\}$ that are critical points of the augmented Hamiltonian $E_c := E - cP$, and that

$$\operatorname{spec}(D^2 E_c(U_c)) = \{-\mu_c^2\} \cup \{0\} \cup \Sigma,$$

where $-\mu_c^2 < 0$ is simple, and $\Sigma \subset \mathbb{R}_+$ is positively separated from 0.

Finally, define the moment of instability d by

$$d(c) := E(U_c) - cP(U_c).$$

The main conclusion in GSS is that:

- If d is convex at c, the corresponding U_c is orbitally stable.
- Conversely, if d is concave at c, then U_c is orbitally unstable.

Unfortunately, our problem doesn't fit into the hypotheses of the GSS theory.

- J is state-dependent: $J = J(u, \varepsilon)$. Even for a fixed vortex strength, the symplectic structure is not flat.
- J is not surjective.
- There is no global well-posedness theory for the Cauchy problem at the level of the natural energy space. Moreover, the energy E and momentum P aren't even defined unless we assume further regularity.

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These obstructions to applying GSS are quite common in water waves.

- For KdV, $J = \partial_x$, which is not surjective on the natural energy space. Many other examples in dispersive model equations.
- In Benjamin's Hamiltonian formulation of 2-d internal waves, the state variables are the density ρ and "vorticity like quantity"

$$\sigma := \nabla \cdot (\rho \nabla \psi),$$

  where ψ is the stream function, and

$$J(\rho, \sigma) = \begin{pmatrix} 0 & \nabla^\perp \rho \cdot \nabla \\ \nabla^\perp \rho \cdot \nabla & \nabla^\perp \sigma \cdot \nabla \end{pmatrix}.$$

- A mismatch between the energy space and the space where well-posedness has been proved is typical when higher-order energy methods are needed.

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Abstract stability/instability theory

With that in mind, as we studied the stability of traveling waves with a point vortex, we developed an abstract theory that relaxes the assumptions in GSS.

For the purposes of this talk, we will present the general theory while giving as an example the point vortex problem.

We begin with a gradation of Hilbert spaces

$$\mathbb{V} \subset \mathbb{W} \subset \mathbb{X}.$$

Here $\mathbb{X}$ is the energy space where the spectral theory will be formulated.

For the point vortex, we take

$$\mathbb{X} := H^1(\mathbb{R}) \times \dot{H}^{1/2}(\mathbb{R}) \times \mathbb{R}^2,$$

endowed with the natural inner product

$$(u, v)_{\mathbb{X}} := (u_1, v_1)_{H^1(\mathbb{R})} + (|\partial_{x_1}|^{\frac{1}{2}} u_2, |\partial_{x_1}|^{\frac{1}{2}} v_2)_{L^2(\mathbb{R})} + u_3 \cdot v_3.$$

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Think of $\mathbb{W}$ as the well-posedness space.

Assumption 1. The Cauchy problem is locally well-posed for initial data in some open $\mathcal{O} \subset \mathbb{W}$.

For the point vortex problem, we specifically take

$$\mathbb{W} := H^{s+\frac{1}{2}}(\mathbb{R}) \times (\dot{H}^s(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})) \times \mathbb{R}^2,$$

for a fixed $s > 3/2$. This is necessary to ensure, for example, that the Dirichlet–Neumann operator is well-defined.

For $\mathcal{O}$, we take $(\eta, \varphi, \overline{x}) \in \mathbb{W}$ such that the point vortex is positively separated from the free surface.

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For $\mathcal{O}$, we take $(\eta, \varphi, \overline{x}) \in \mathbb{W}$ such that the point vortex is positively separated from the free surface.

Finally, $\mathbb{V}$ is the "very smooth" space. The idea here is that, in order to translate spectral information (which gives control in $\mathbb{X}$) up to the well-posedness space $\mathbb{W}$, we must interpolate with a higher-regularity space.

Assumption 2. For all $v \in \mathbb{V}$, we have the interpolation-type inequality

$$\|v\|_{\mathbb{W}}^3 \lesssim \|v\|_{\mathbb{X}}^2 \|v\|_{\mathbb{V}}.$$

For the point vortex problem, one can take

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We denote by $\mathbb{X}^*$ the dual of $\mathbb{X}$; the natural isomorphism $I : \mathbb{X} \to \mathbb{X}^*$ is then given by

$$\langle Iu, v \rangle := (u, v)_{\mathbb{X}}, \qquad \text{for all } u, v \in \mathbb{X}.$$

For the point vortex problem, I takes the form

$$I = (1 - \partial_{x_1}^2,\, |\partial_{x_1}|,\, \mathrm{Id}),$$

where Id is the 2×2 identity matrix.

We consider abstract Hamiltonian systems of the form

$$\frac{du}{dt} = J(u)\,\mathrm{D}\,E(u), \qquad u|_{t=0} = u_0.$$

Assumption 3. The energy is smooth

$$E \in C^2(\mathcal{O}; \mathbb{R}).$$

Moreover, for each $u \in \mathcal{O}$,

$$J(u) : D(J) \subset \mathbb{X}^* \to \mathbb{X}$$

is a densely defined closed linear operator (with domain $D(J)$ independent of u) that has dense range.

Assumption 4. The system is invariant under a continuous symmetry group $T(\cdot)$. The symmetry group generates a conserved quantity $P \in C^2(\mathcal{O}; \mathbb{R})$. Furthermore, T interacts "nicely" with J.

For the point vortex case, $T(\cdot) : \mathbb{X} \to \mathbb{X}$ is the one-parameter family of densely defined mappings given by

$$T(\sigma) \begin{pmatrix} \eta \\ \varphi \\ \overline{\mathbf{x}} \end{pmatrix} := \begin{pmatrix} \eta(\cdot - \sigma) \\ \varphi(\cdot - \sigma) \\ \overline{\mathbf{x}} + \sigma \mathbf{e}_1 \end{pmatrix}.$$

It is easy to see that

$$E(T(\sigma)u) = E(u), \qquad P(T(\sigma)u) = P(u),$$

for all $\sigma \in \mathbb{R}$, $u \in \mathcal{O}$.

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for all $\sigma \in \mathbb{R}$, $u \in \mathcal{O}$.

By "nice interaction" we mean more precisely that, letting

$$(dT)(\sigma)u := \begin{pmatrix} \eta(\cdot - \sigma) \\ \varphi(\cdot - \sigma) \\ \bar{\mathbf{x}} \end{pmatrix}, \qquad \text{for all } u = (\eta, \varphi, \bar{\mathbf{x}})^T,$$

then $dT(\cdot)$ is an isometry on $\mathbb{X}$, $\mathbb{W}$, and $\mathbb{V}$. It is unitary in an appropriate sense, and commutes with $J(u)$:

$$dT^*(\sigma)J(u)I = J(T(\sigma)u)IdT(\sigma), \qquad \text{for all } \sigma \in \mathbb{R}.$$

We say that $u \in C^1(\mathbb{R}; \mathcal{O})$ is a traveling wave solution of the Hamiltonian system provided that

$$u(t) = T(ct)U,$$

for some $c \in \mathbb{R}$ and stationary $U \in \mathcal{O}$.

Assumption 5. There exists a one-parameter family $\{U_c : |c| < c_0\}$ such that

$$(-c_0, c_0) \ni c \mapsto U_c \in \mathcal{O} \cap \mathbb{V}$$

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For the point vortex problem, we take the family $\mathcal{S}_{\mathrm{loc}}$, fix $0 < |\varepsilon| \ll 1$, and then reparameterize in terms of c, allowing a to vary.

This furnishes a family

$$\Big\{ U_c = (\eta(c), \varphi(c), \bar{\mathbf{x}}(c)) \in \mathcal{O} : |c| \ll 1 \Big\}.$$

It is easy to show that $\{U_c\}$ meets the above hypotheses.

Assumption 6. $\mathrm{D}^2 E_c(U_c) : \mathbb{W} \to \mathbb{W}^*$ extends to a bounded operator $H_c(U_c)\colon \mathbb{X} \to \mathbb{X}^*$ that is self-adjoint in the sense that $I^{-1} H_c(U_c)$ is a bounded symmetric operator on $\mathbb{X}$.

Moreover,

$$\mathrm{spec}(I^{-1} H_c) = \{-\mu_c^2\} \cup \{0\} \cup \Sigma,$$

where $-\mu_c^2 < 0$ is a simple eigenvalue with eigenvector $\chi \in \mathbb{W}$, 0 is a simple eigenvalue generated by $T(\cdot)$, and Σ is a subset of the positive real axis that is bounded away from 0.

We call the set $\{T(\sigma)u : \sigma \in \mathbb{R}\}$ the u-orbit generated by T. Our objective is to prove that these orbits are stable or unstable.

With that in mind, for each $r > 0$, we define the tubular neighborhood of the U_c-orbit generated by T by

$$\mathcal{U}_r^{\mathbb{W}} := \{u \in \mathbb{W} : \inf_{\sigma \in \mathbb{R}} \|u - T(\sigma)U_c\|_{\mathbb{W}} < r\}.$$

Likewise,

$$\mathcal{U}_r^{\mathbb{V}} := \{u \in \mathbb{V} : \inf_{\sigma \in \mathbb{R}} \|u - T(\sigma)U_c\|_{\mathbb{V}} < r\}.$$

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Theorem (Varholm–Wahlén–W.)

For all $c \neq 0$ such that $d''(c) > 0$, U_c is conditionally orbitally stable in following sense.

For any $R > 0$ and $r > 0$, there exists $r_0 = r_0(R, r)$ such that, if

$$u_0 \in \mathcal{U}_{r_0}^{\mathbb{W}} \cap \mathcal{O} \qquad and \qquad \|u_0\|_{\mathbb{V}} < R,$$

then

$$u(t) \in \mathcal{U}_r^{\mathbb{W}} \cap \mathcal{O}$$

for as long as u exists and obeys the bound $\|u(t)\|_{\mathbb{V}} < 2R$.

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Theorem (Varholm–Wahlén–W.)

For all $c \neq 0$ such that $d''(c) < 0$, U_c is conditionally orbitally unstable in following sense.

For all $r > 0$ sufficiently small, one of three alternatives must hold.

- *Globally ill-posed.* $\exists u_0 \in \mathcal{U}_r^{\mathbb{W}} \cap \mathcal{O}$ *and* $t_0 < \infty$ *for which*

$$\lim_{t \to t_0 -} \left[\|u(t)\|_{\mathbb{W}} + \frac{1}{\operatorname{dist}(u(t), \partial \mathcal{O})} \right] = \infty.$$

- *Uncontrolled growth in* $\mathbb{V}$. $\forall R > 0$, $\exists u_0^R \in \mathcal{U}_r^{\mathbb{W}} \cap \mathcal{U}_r^{\mathbb{V}} \cap \mathcal{O}$ *for which* u *exits* $\mathcal{U}_R^{\mathbb{V}}$ *in finite time.*
- *Unstable in* $\mathbb{W}$. $\forall r_0 > 0$, $\exists u_0 \in \mathcal{U}_{r_0}^{\mathbb{W}} \cap \mathcal{O}$ *such that* u *exits* $\mathcal{U}_r^{\mathbb{W}}$ *in finite time.*

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Return to the point vortex problem

With this general theory in hand, let's return to our original question.

Theorem (Varholm–Wahlén–W.)

For all $0 < |c| \ll 1$, $(\eta(c), \varphi(c), \overline{\mathbf{x}}(c))$ is conditionally orbitally stable in the above sense.

To prove this theorem, we must verify the assumptions of the abstract theory and show that

$$d''(c) > 0 \qquad \text{for all } 0 < |c| \ll 1.$$

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Observe that φ occurs quadratically in the energy and can thus be eliminated rather simply:

Fix $(\eta, \bar{\mathbf{x}}, \varepsilon)$ and consider the augmented potential

$$V_c^{\mathrm{aug}} = V_c^{\mathrm{aug}}(\eta, \bar{\mathbf{x}}, \varepsilon) := \min_{\varphi} E_c(\eta, \varphi, \bar{\mathbf{x}}, \varepsilon).$$

One can easily calculate that

$$V_c^{\mathrm{aug}}(\eta, \bar{x}) = E_c(\eta, \varphi_*, \bar{\mathbf{x}}, \varepsilon),$$

where

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It suffices, therefore, to compute the spectrum of $\mathrm{D}^2\, V_c^{\mathrm{aug}}$.

This is a rather lengthy and not-so-trivial calculation that borrows some ideas from Mielke's proof of stability for small-amplitude (irrotational) capillary-gravity solitary waves.

Ultimately we find that, for $0 < |\varepsilon| \ll 1$,

$$\operatorname{spec}\left(\mathrm{D}^2\, V_c^{\mathrm{aug}}\right) = \{-\mu^2\} \cup \{0\} \cup \Sigma,$$

where $-\mu^2 < 0$ is a simple negative eigenvalue, and $\Sigma \subset \mathbb{R}_+$ is positively separated from 0.

Finally, we compute d and show that it is index convex at each c with $0 < |c| \ll 1$.

We hope to apply this framework to a number of other problems.

- New proof of [Bona–Souganidis–Strauss] result on stability/instability of KdV solitons.
- Many other problems in the future (internal waves, vortex patches, other dispersive model equations, . . .).

Thanks for your attention