Transverse stability of periodic waves in water-wave models

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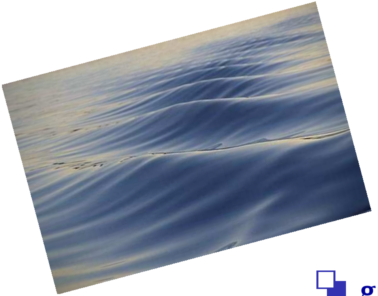
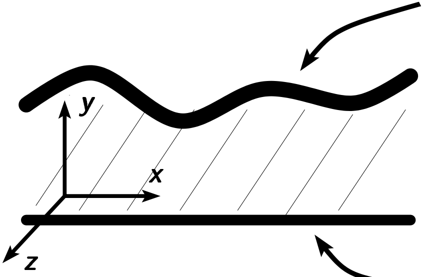
Water-wave problem



- **gravity/gravity-capillary waves**
  - *three-dimensional inviscid fluid layer*
  - *constant density*
  - *gravity/gravity and surface tension*
  - *irrotational flow*

Water-wave problem



$\boxed{y = h + \eta(x, z, t)}$

(free surface)

$\boxed{y = 0}$ *(flat bottom)*

Domain

$$\boxed{D_\eta = \{(x, y, z) \ : \ x, z \in \mathbb{R}, \ \ y \in (0, h + \eta(x, z, t))\}}$$

- depth at rest h

Euler equations

- **Laplace's equation**

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \textit{in } D_\eta$$

- **boundary conditions**

$$\begin{aligned} \phi_y &= 0 \quad \textit{on } y = 0 \\ \eta_t &= \phi_y - \eta_x\phi_x - \eta_z\phi_z \quad \textit{on } y = h + \eta \\ \phi_t &= -\frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) - g\eta + \frac{\sigma}{\rho}\mathcal{K} \quad \textit{on } y = h + \eta \end{aligned}$$

- **velocity potential** ϕ; **free surface** $h + \eta$
- mean curvature $\mathcal{K} = \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x + \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z$
- **parameters** ρ, g, σ, h

Euler equations

- **moving coordinate system,** *speed* $-c$
- **dimensionless variables**
  - *characteristic length* h
  - *characteristic velocity* c

- **parameters**
  - *inverse square of the Froude number* $\boxed{\alpha = \dfrac{gh}{c^2}}$
  - *Weber number* $\boxed{\beta = \dfrac{\sigma}{\rho h c^2}}$

Euler equations

$$\boxed{\begin{array}{l} \qquad\quad \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \textit{for} \quad 0 < y < 1 + \eta \\ \phi_y = 0 \quad \textit{on} \quad y = 0 \\ \phi_y = \eta_t + \eta_x + \eta_x \phi_x + \eta_z \phi_z \quad \textit{on} \quad y = 1 + \eta \\ \phi_t + \phi_x + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + \alpha \eta - \beta \mathcal{K} = 0 \quad \textit{on} \quad y = 1 + \eta \end{array}}$$

Euler equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \textit{for} \quad 0 < y < 1 + \eta$$

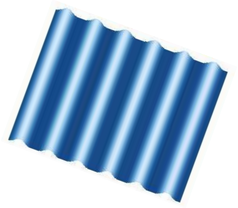
$$\phi_y = 0 \quad \textit{on} \quad y = 0$$

$$\phi_y = \eta_t + \eta_x + \eta_x \phi_x + \eta_z \phi_z \quad \textit{on} \quad y = 1 + \eta$$

$$\phi_t + \phi_x + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + \alpha\eta - \beta\,\mathcal{K} = 0 \quad \textit{on} \quad y = 1 + \eta$$

- **difficulties**
  - variable domain (free surface)
  - nonlinear boundary conditions
- **very rich dynamics**
  - symmetries, Hamiltonian structures
  - many particular solutions

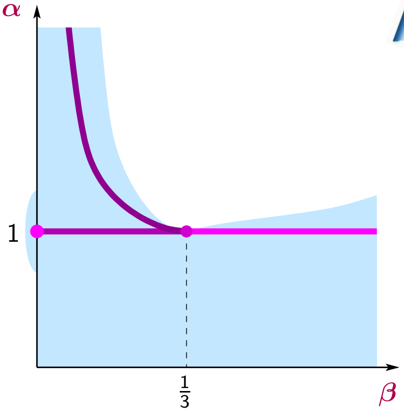
Focus on ...

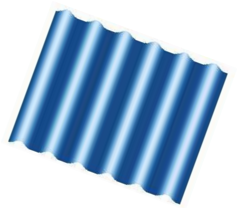


- traveling periodic 2D waves
- transverse stability/instability
- analytical results
- long-wave models

Two-dimensional periodic waves

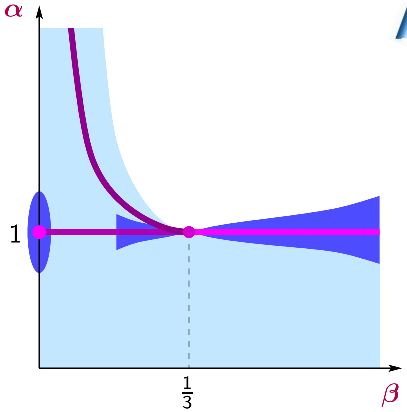
⇝ **exist in different parameter regimes**

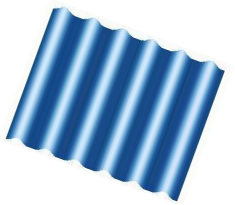




Two-dimensional periodic waves

⇝ **transverse (in)stability**



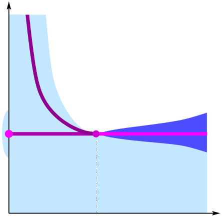


Large surface tension

- **transverse linear instability**
  - *longitudinal co-periodic perturbations*
  - *transverse periodic perturbations*

- **Euler equations**

  [H., 2015]

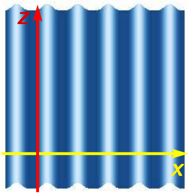


Transverse instability problem

Transverse spatial dynamics

$$\boxed{U_z = DU_t + F(U)}$$

- $U(x, z, t)$, D *linear operator,* F *nonlinear map*
- **a periodic wave $U_*(x)$ is an equilibrium**



Transverse linear instability

Transverse spatial dynamics

$$\boxed{U_z = DU_t + F(U)}$$

$U_*(x)$ is **transversely linearly unstable** if the linearized system

$$\boxed{U_z = DU_t + \mathcal{L}U}, \quad \boxed{\mathcal{L} = F'(U_*)}$$

possesses a solution of the form $\boxed{U(x,z,t) = \mathrm{e}^{\lambda t} V_\lambda(x,z)}$

with $\lambda \in \mathbb{C}$, $Re\lambda > 0$, V_λ bounded function.

HYPOTHESES

1. the **system** $\boxed{U_z = DU_t + F(U)}$ is **reversible/Hamiltonian**;
2. the **linear operator** $\boxed{\mathcal{L} = F'(U_*)}$ possesses **a pair of simple purely imaginary eigenvalues** $\pm \mathrm{i}\kappa_*$;
3. the operators $\mathbf{D}$ and $\mathcal{L}$ are closed in $\mathcal{X}$ with $D(\mathcal{L}) \subset D(\mathbf{D})$;

Main result

Theorem

1. *For any* $\boldsymbol{\lambda} \in \mathbb{R}$ *sufficiently small, the linearized system*

$$\boxed{U_z = DU_t + \mathcal{L}U}$$

possesses a solution of the form $\boxed{U(\cdot, z, t) = \mathrm{e}^{\lambda t} V_\lambda(\cdot, z)}$
with $V_\lambda(\cdot, z) \in D(\mathcal{L})$ **a periodic function in** z.

2. U_* **is transversely linearly unstable**.

[Godey, 2016; *see also* **Rousset & Tzvetkov, 2010]**

Hamiltonian formulation of the 3D problem:

$$\boxed{U_z = DU_t + F(U)}$$

- boundary conditions

$$\boxed{\phi_y = b(U)_t + g(U)} \quad \textit{on} \quad y = 0, 1$$

(*e.g.* [Groves, H., Sun, 2002])

Periodic waves

$\boxed{\beta > \frac{1}{3}, \quad \alpha = 1 + \epsilon}$, ϵ *small*

model: **Kadomtsev-Petviashvili-I equation** $\rightsquigarrow$ **instability**

$$\partial_x \partial_t u = \partial_x \partial_x (\partial_x^2 u + u + \frac{1}{2} u^2) - \partial_y^2 u$$

[H.; Johnson & Zumbrun; Hakkaev, Stanislavova & Stefanov, ...]

Periodic waves

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$$\partial_x \partial_t u = \partial_x \partial_x (\partial_x^2 u + u + \frac{1}{2} u^2) - \partial_y^2 u$$

[H.; Johnson & Zumbrun; Hakkaev, Stanislavova & Stefanov, ...]

- *The Euler equations possess a* **one-parameter family of symmetric periodic waves**

$$\boxed{\eta_{\epsilon,a}(x) = \varepsilon p_a(\varepsilon^{1/2} x, \varepsilon), \quad \varphi_{\epsilon,a}(x) = \varepsilon^{1/2} q_a(\varepsilon^{1/2} x, \varepsilon)}$$

$p_a(\xi, 0) = \partial_\xi q_a(\xi, 0)$, $p_a(\xi, 0)$ *satisfies the* **Korteweg de Vries equation**

[Kirchgässner, 1989]

Linearized system

- **linearized system (rescaled)**

$$\boxed{U_z = D_\varepsilon U_t + DF_\varepsilon(u_a)U}$$

  - **boundary conditions**

$$\boxed{\phi_y = Db_\varepsilon(u_a)U_t + Dg_\varepsilon(u_a)U} \quad \textit{on} \quad y = 0, 1$$

Linearized system

linearized system (rescaled)

$$\boxed{U_z = D_\varepsilon U_t + DF_\varepsilon(u_a)U}$$

- **boundary conditions**

$$\boxed{\phi_y = Db_\varepsilon(u_a)U_t + Dg_\varepsilon(u_a)U} \quad on \quad y = 0, 1$$

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linear operator $\boxed{\mathcal{L}_\varepsilon := DF_\varepsilon(u_a)}$

- **boundary conditions**

$$\boxed{\phi_y = Dg_\varepsilon(u_a)U} \quad on \quad y = 0, 1$$

- space of symmetric functions ($x \to -x$)

$$\boxed{\mathcal{X}_s = H^1_e(0, 2\pi) \times L^2_e(0, 2\pi) \times H^1_o((0, 2\pi) \times (0, 1)) \times L^2_o((0, 2\pi) \times (0, 1))}$$

Linear operator $\mathcal{L}_\varepsilon$

$$\boxed{\mathcal{L}_\varepsilon = \mathcal{L}_\varepsilon^0 + \mathcal{L}_\varepsilon^1} \qquad \mathcal{L}_\varepsilon^0 \begin{pmatrix} \eta \\ \omega \\ \phi \\ \xi \end{pmatrix} = \begin{pmatrix} \dfrac{\omega}{\beta} \\ -\varepsilon k_a^2 \beta \eta_{xx} + (1+\epsilon)\eta - k_a \phi_x|_{y=1} \\ \xi \\ -\varepsilon k_a^2 \phi_{xx} - \phi_{yy} \end{pmatrix}, \quad \mathcal{L}_\varepsilon^1 \begin{pmatrix} \eta \\ \omega \\ \phi \\ \xi \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ G_1 \\ G_2 \end{pmatrix}$$

$$g_1 = \frac{(1+\varepsilon k_a^2 \eta_{ax}^2)^{1/2}}{\beta} \left(\omega + \frac{1}{1+\varepsilon \eta_a} \int_0^1 y \phi_{ay} \xi \, dy \right) - \frac{\omega}{\beta}$$

$$\begin{aligned} g_2 = \int_0^1 \Bigg\{ & \varepsilon k_a^2 \phi_{ax} \phi_x - \frac{\phi_{ay}\phi_y}{(1+\varepsilon\eta_a)^2} + \frac{\varepsilon \phi_{ay}^2 \eta}{(1+\varepsilon \eta_a)^3} - \frac{\varepsilon^3 k_a^2 y^2 \eta_{ax}^2 \phi_{ay} \phi_y}{(1+\varepsilon\eta_a)^2} - \frac{\varepsilon^3 k_a^2 y^2 \eta_{ax} \phi_{ay}^2 \eta_x}{(1+\varepsilon \eta_a)^2} + \frac{\varepsilon^3 k_a^2 y^2 \eta_{ax}^2 \phi_{ay}^2 \eta}{(1+\varepsilon\eta_a)^3} \\ & + \left[\varepsilon k_a^2 y \phi_{ay} \phi_x + \varepsilon k_a^2 y \phi_{ax} \phi_y - \frac{2\varepsilon^2 k_a^2 y^2 \eta_{ax} \phi_{ay} \phi_y}{1+\varepsilon\eta_a} - \frac{\varepsilon^2 k_a^2 y^2 \phi_{ay}^2 \eta_x}{1+\varepsilon \eta_a} + \frac{\varepsilon^3 k_a^2 y^2 \eta_{ax} \phi_{ay}^2 \eta}{(1+\varepsilon\eta_a)^2} \right]_x \Bigg\} dy \\ & + \varepsilon k_a^2 \beta \eta_{xx} - \varepsilon k_a^2 \beta \left[\frac{\eta_x}{(1+\varepsilon^3 k_a^2 \eta_{ax}^2)^{3/2}} \right]_x \end{aligned}$$

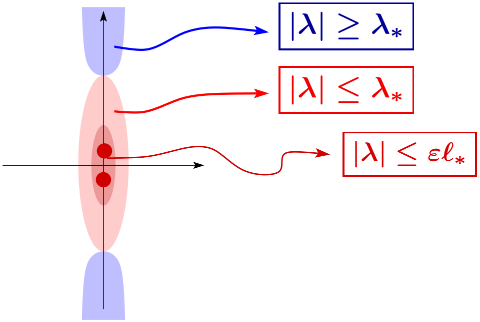
$$G_1 = -\frac{\varepsilon \eta_a \xi}{1+\varepsilon\eta_a} + \frac{(1+\varepsilon^3 k_a^2 \eta_{ax}^2)^{1/2}}{\beta(1+\varepsilon\eta_a)} \left(\omega + \frac{1}{1+\varepsilon\eta_a} \int_0^1 y\phi_{ay}\xi \, dy \right) y \phi_{ay}$$

$$\begin{aligned} G_2 = & \left[\frac{\varepsilon \eta_a \phi}{(1+\varepsilon\eta_a)} + \frac{\varepsilon \phi_a \eta}{(1+\eta_a)^2} \right]_{yy} - \varepsilon^2 k_a^2 [\eta_a \phi_x + \phi_{ax}\eta - y\phi_{ay}\eta_x - y\eta_{ax}\phi_y]_x \\ & + \varepsilon^2 k_a^2 \left[y \eta_{ax}\phi_x + y\phi_{ax}\eta_x + \frac{\varepsilon^2 y^2 \eta_{ax}^2 \phi_{ay} \eta}{(1+\varepsilon\eta_a)^2} - \frac{\varepsilon y^2 \eta_{ax}^2 \phi_y}{1+\varepsilon\eta_a} - \frac{2\varepsilon y^2 \eta_{ax} \phi_{ay} \eta_x}{1+\varepsilon\eta_a} \right]_y \end{aligned}$$

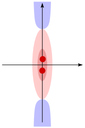
Check hypotheses ...

Main difficulty: spectrum of $\mathcal{L}_\varepsilon$ ...

- *operator with compact resolvent* $\longrightarrow$ *pure point spectrum*
- **spectral analysis**



Key step



Reduction to a scalar operator $\mathcal{B}_{\varepsilon,\ell}$ *in* $L^2_o(0,2\pi)$

- **scaling** $\boxed{\lambda = \varepsilon\ell, \quad \omega = \varepsilon\tilde{\omega}, \quad \xi = \varepsilon\tilde{\xi}}$
- **decomposition** $\boxed{\phi(x,y) = \phi_1(x) + \phi_2(x,y)}$
- $\boxed{\lambda = \varepsilon\ell}$ **eigenvalue** **iff** $\boxed{\mathcal{B}_{\varepsilon,\ell}\phi_1 = 0}$

$$\boxed{\mathcal{B}_{\varepsilon,\ell}\phi_1 = \left(\beta - \frac{1}{3}\right) k_a^4 \phi_{1xxxx} - k_a^2 \phi_{1xx} + \ell^2(1+\epsilon)\phi_1 - 3k_a^2(P_a\phi_{1x})_x + \ldots}$$

$$\omega = \frac{\beta}{(1+\epsilon^3\eta_x^{\star 2})^{1/2}}(\eta^\dagger + \mathrm{i}k\eta) - \frac{1}{1+\epsilon\eta^\star}\int_0^1 y\Phi_y^\star \xi dy,$$

$$\xi = (1+\epsilon\eta^\star)(\Phi^\dagger + \mathrm{i}k\Phi) - \epsilon y \Phi_y^\star(\eta^\dagger + \mathrm{i}k\eta)$$

$$\frac{(1+\epsilon)}{\epsilon^2}\eta - \frac{1}{\epsilon^2}\Phi_x|_{y=1} - \frac{1}{\epsilon}\beta\eta_{xx} - \mathrm{i}k\beta(h_1^\epsilon + \mathrm{i}k\eta) = h_2^\epsilon$$

$$-\frac{1}{\epsilon}\Phi_{xx} - \frac{1}{\epsilon^2}\Phi_{yy} - \mathrm{i}k(H_1^\epsilon + \mathrm{i}k\Phi) = H_2^\epsilon,$$

$$h_2^\epsilon = \omega^\dagger - g_2^\epsilon,$$

$$H_2^\epsilon = \xi^\dagger - G_2^\epsilon$$

$$h_1^\epsilon = \frac{\omega}{\beta} - \mathrm{i}k\eta$$

$$= -\frac{1}{\beta(1+\epsilon\eta^\star)}\int_0^1 y\Phi_y^\star[-\epsilon y\Phi_y^\star(\mathrm{i}k\eta + \eta^\dagger) + (1+\epsilon\eta^\star)(\mathrm{i}k\Phi + \Phi^\dagger)]dy$$

$$+ \left(\frac{1}{(1+\epsilon^3\eta_x^{\star 2})^{1/2}} - 1\right)\mathrm{i}k\eta + \frac{\eta^\dagger}{(1+\epsilon^3\eta_x^{\star 2})^{1/2}},$$

$$H_1^\epsilon = \xi - \mathrm{i}k\Phi$$

$$= (1+\epsilon\eta^\star)\Phi^\dagger + \mathrm{i}k\epsilon\eta^\star\Phi - \epsilon y\Phi_y^\star(\eta^\dagger + \mathrm{i}k\eta).$$

$$B^\epsilon(\eta,\Phi) = -\epsilon\eta_x + B_0^\epsilon + B_1^\epsilon,$$

$$B_0^\epsilon = \left.\frac{\epsilon\eta^\star\Phi_y}{1+\epsilon\eta^\star} + \frac{\epsilon\Phi_y^\star\eta}{(1+\epsilon\eta^\star)^2}\right|_{y=1},$$

$$B_1^\epsilon = \epsilon^2\eta_x^\star\Phi_x + \epsilon^2\Phi_x^\star\eta_x + \frac{\epsilon^4\eta_x^{\star 2}\Phi_y^\star\eta}{(1+\epsilon\eta^\star)^2} - \frac{\epsilon^3\eta_x^{\star 2}\Phi_y}{1+\epsilon\eta^\star} -$$

$$-\hat\Phi_{yy} + q^2\hat\Phi = \epsilon^2(\hat H_2^\epsilon + \mathrm{i}k\hat H_1^\epsilon), \qquad 0 < y < 1$$

$$\hat\Phi_y = 0,\ y = 0$$

$$\hat\Phi_y - \frac{\epsilon\mu^2\hat\Phi}{1+\epsilon+\beta q^2} = -\frac{\epsilon^3\mathrm{i}\mu(\hat h_2^\epsilon + \mathrm{i}k\beta\hat h_1^\epsilon)}{1+\epsilon+\beta q^2} + \hat B_0^\epsilon + \hat B_1^\epsilon,\ y = 1$$

$$G(y,\zeta) = \begin{cases} \dfrac{\cosh qy}{\cosh q}\,\dfrac{(1+\epsilon+\beta q^2)\cosh q(1-\zeta) + (\epsilon\mu^2/q)}{q^2 - (1+\epsilon+\beta q^2)q\tanh q -} \\[2ex] \dfrac{\cosh q\zeta}{\cosh q}\,\dfrac{(1+\epsilon+\beta q^2)\cosh q(1-y) + (\epsilon\mu^2/q)}{q^2 - (1+\epsilon+\beta q^2)q\tanh q -} \end{cases}$$

$$
\begin{aligned}
\hat{\Phi}_1 \quad &= \quad \frac{1+\epsilon}{\epsilon^2(k^2(1+\epsilon)+\mu^2+(\beta-1/3)\mu^4)} \times \left\{ \int_0^1 \epsilon^2(\hat{\xi}^\dagger - \mathrm{i}\mu \hat{G}^\epsilon_{2,2} + \mathrm{i}k\hat{H}^\epsilon_1) d\zeta - \epsilon q^2 \int_0^1 \hat{p}^\epsilon_2 d\zeta \right. \\
&\qquad \left. - \frac{\epsilon^3 \mathrm{i}\mu(\hat{h}^\epsilon_2 + \mathrm{i}k\beta\hat{h}^\epsilon_1)}{1+\epsilon+\beta q^2} + \frac{\epsilon^2\mu^2\hat{p}^\epsilon_2|_{\zeta=1}}{1+\epsilon+\beta q^2} \right\}, \\
\hat{\Phi}_2 \quad &= \quad -\int_0^1 G_1(\hat{\xi}^\dagger - \mathrm{i}\mu\hat{G}^\epsilon_{2,2} + \mathrm{i}k\hat{H}^\epsilon_1)d\zeta - \int_0^1 G_{1\zeta}\hat{G}^\epsilon_{2,1}d\zeta + \int_0^1 (\epsilon k^2+\mu^2)G_1\hat{p}^\epsilon_2 d\zeta + \epsilon\hat{p}^\epsilon_2 \\
&\qquad - G_1|_{\zeta=1}\left(-\frac{\epsilon\mathrm{i}\mu(\hat{h}^\epsilon_2 + \mathrm{i}k\beta\hat{h}^\epsilon_1)}{1+\epsilon+\beta q^2} + \frac{\mu^2\hat{p}^\epsilon_2|_{\zeta=1}}{1+\epsilon+\beta q^2} \right),
\end{aligned}
$$

$$
\hat{\Phi} = -\int_0^1 G\epsilon^2(\hat{\xi}^\dagger - \imath\mu\hat{G}^\epsilon_{2,2} + \imath k\hat{H}^\epsilon_1)d\zeta - \int_0^1 G_\zeta\epsilon^2\hat{G}^\epsilon_{2,1}d\zeta + \frac{\epsilon^3\imath\mu G|_{\zeta=1}(\hat{h}^\epsilon_2+\imath k\beta\hat{h}^\epsilon_1)}{1+\epsilon+\beta q^2} + \int_0^1 \epsilon q^2 G\hat{p}^\epsilon_2 d\zeta + \epsilon\hat{p}^\epsilon_2 - \frac{\epsilon^2\mu}{1}
$$

$$
\begin{aligned}
&\int_0^1 \epsilon q^2 G\hat{p}^\epsilon_2 d\zeta + \epsilon\hat{p}^\epsilon_2 - \frac{\epsilon^2\mu^2 G|_{\zeta=1}\hat{p}^\epsilon_2|_{\zeta=1}}{1+\epsilon+\beta q^2} \\
&\qquad = \quad \int_0^1 G\epsilon\hat{p}^\epsilon_{2\zeta\zeta}d\zeta - \epsilon G|_{\zeta=1}\hat{p}^\epsilon_{2\zeta}|_{\zeta=1} = \int_0^1 G\epsilon^2(\hat{G}^\epsilon_{2,0})_{\zeta\zeta}d\zeta - G|_{\zeta=1}\hat{B}^\epsilon_0,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Phi}_1 + \hat{\Phi}_2 = &-\int_0^1 G\epsilon^2(\hat{\xi}^\dagger - \imath\mu\hat{G}^\epsilon_{2,2} + \imath k\hat{H}^\epsilon_1)d\zeta - \int_0^1 G_\zeta\epsilon^2\hat{G}^\epsilon_{2,1}d\zeta \\
&+ \frac{\epsilon^3\imath\mu G|_{\zeta=1}(\hat{h}^\epsilon_2+\imath k\beta\hat{h}^\epsilon_1)}{1+\epsilon+\beta q^2} + \int_0^1 \epsilon q^2 G\hat{p}^\epsilon_2 d\zeta + \epsilon\hat{p}^\epsilon_2 - \frac{\epsilon^2\mu^2 G|_{\zeta=1}\hat{p}^\epsilon_2|_{\zeta=1}}{1+\epsilon+\beta q^2},
\end{aligned}
$$

$$\imath\mu\hat{h}_2^\epsilon \quad = \quad \imath\mu\hat{\omega}^\dagger + \imath\mu\mathcal{F}\left[-\frac{1}{\epsilon^2}\int_0^1\left\{\epsilon\Phi_x^\star\Phi_x - \frac{\Phi_y^\star\Phi_y}{(1+\epsilon\eta^\star)^2} + \frac{\epsilon\Phi_y^{\star 2}\eta}{(1+\epsilon\eta^\star)^3} - \frac{\epsilon^3y^2\eta_x^{\star 2}\Phi_y^\star\Phi_y}{(1+\epsilon\eta^\star)^2} - \frac{\epsilon^3y^2\eta_x^\star\Phi_y^{\star 2}\eta_x}{(1+\epsilon\eta^\star)^2} + \frac{\epsilon^4}{(1}\right.\right.$$

$$+\frac{\mu^2}{\epsilon}\mathcal{F}\left[\int_0^1\left\{y\Phi_y^\star\Phi_x + y\Phi_x^\star\Phi_y - \frac{2\epsilon y^2\eta_x^\star\Phi_y\Phi_y^\star}{1+\epsilon\eta^\star} - \frac{\epsilon y^2\Phi_y^{\star 2}\eta_x}{1+\epsilon\eta^\star} + \frac{\epsilon^2y^2\eta_x^\star\Phi_y^{\star 2}\eta}{(1+\epsilon\eta^\star)^2}\right\}dy\right] - \frac{\beta\mu^2}{\epsilon}\mathcal{F}\left[\frac{}{(1+}\right.$$

$$\mathcal{F}^{-1}\left[\frac{\epsilon\imath\mu\hat{h}_2^\epsilon}{1+\epsilon+\beta q^2}\right] = -\mathcal{F}^{-1}\left[\frac{1}{1+\epsilon+\beta q^2}\mathcal{F}[(\Phi_{1x}^\star\Phi_{1x})_x]\right] + \mathcal{F}^{-1}\left[\frac{\mu^2}{1+\epsilon+\beta q^2}\mathcal{F}\left[\int_0^1 y\Phi_x^\star\Phi_{2y}\,dy\right]\right]$$

$$+\left\{\mathcal{F}^{-1}\left[-\frac{1}{1+\epsilon+\beta q^2}\mathcal{F}\left[\int_0^1\left(\Phi_{2x}^\star\Phi_{1x} + \Phi_x^\star\Phi_{2x} - \frac{\Phi_y^\star\Phi_y}{\epsilon(1+\epsilon\eta^\star)^2} + \frac{\Phi_y^{\star 2}\eta}{(1+\epsilon\eta^\star)^3}\right.\right.\right.\right.$$

$$\left.\left. - \frac{\epsilon^2y^2\eta_x^{\star 2}\Phi_y^\star\Phi_y}{(1+\epsilon\eta^\star)^2} - \frac{\epsilon^2y^2\eta_x^\star\Phi_y^{\star 2}\eta_x}{(1+\epsilon\eta^\star)^2} + \frac{\epsilon^3y^2\eta_x^\star\Phi_y^{\star 2}\eta}{(1+\epsilon\eta^\star)^3}\right)dy\right]$$

$$-\frac{\imath\mu}{1+\epsilon+\beta q^2}\mathcal{F}\left[\int_0^1\left(y\Phi_y^\star\Phi_x - \frac{2\epsilon y^2\eta_x^\star\Phi_y^\star\Phi_y}{1+\epsilon\eta^\star} - \frac{\epsilon y^2\Phi_y^{\star 2}\eta_x}{1+\epsilon\eta^\star} + \frac{\epsilon^2y^2\eta_x^\star\Phi_y^{\star 2}\eta}{(1+\epsilon\eta^\star)^2}\right)dy\right]$$

$$\left.\left.\left.+\frac{\beta\imath\mu}{1+\epsilon+\beta q^2}\mathcal{F}\left[\frac{\eta_x}{(1+\epsilon^3\eta_x^{\star 2})^{3/2}} - \eta_x\right]\right]\right\}_x + \mathcal{F}^{-1}\left[\frac{\epsilon\imath\mu\hat{\omega}^\dagger}{1+\epsilon+\beta q^2}\right]$$

$$= \quad -\mathcal{F}^{-1}\left[\frac{1}{1+\epsilon+\beta q^2}\mathcal{F}[(\Phi_{1x}^\star\Phi_{1x})_x]\right] + \mathcal{F}^{-1}\left[\frac{\mu^2}{1+\epsilon+\beta q^2}\mathcal{F}\left[\int_0^1 y\Phi_x^\star\Phi_{2y}\,dy\right]\right]$$

$$+(\mathcal{L}(\epsilon\Phi_{1x},\Phi_{2x},\Phi_{2y},\epsilon^2\eta,\epsilon^4\eta_x))_x + \epsilon^{-1/2}(\mathcal{L}(\epsilon\Phi_x,\epsilon^2\Phi_{2y},\epsilon^4\eta,\epsilon^3\eta_x))_x + \epsilon^{1/2}\mathcal{L}(\omega^\dagger),$$

$$\mathcal{F}^{-1}\left[\frac{\mu^2}{1+\epsilon+\beta q^2}\mathcal{F}\left[\int_0^1 y\Phi_x^\star\Phi_{2y}\,dy\right]\right]=\mathcal{F}^{-1}\left[\frac{\mu^2}{1+\epsilon+\beta q^2}\mathcal{F}\left[\Phi_{1x}^\star\Phi_2|_{y=1}-\int_0^1\Phi_{1x}^\star\Phi_2\,dy+\int_0^1 y\Phi_{2x}^\star\Phi_{2}\right.\right.$$

$$\begin{aligned}
&= \left[\mathcal{F}^{-1}\left[\frac{\mu^{1/2}}{1+\epsilon+\beta q^2}\mu^{1/2}\mathcal{F}[\Phi_{1x}^\star\Phi_2|_{y=1}]-\frac{1}{1+\epsilon+\beta q^2}\int_0^1(\Phi_{1x}^\star\Phi_2)_x dy \quad +\frac{\mu}{1+\epsilon+\beta q^2}\int_0^1 y\Phi_{2x}^\star\Phi_{2y}\,dy\right]\right]_x \\
&= \epsilon^{-1/4}(\mathcal{L}(\Phi_2))_x+(\mathcal{L}(\Phi_2,\Phi_{2x},\epsilon^{1/2}\Phi_{2y}))_x,
\end{aligned}$$

$$\mathcal{F}^{-1}\left[\frac{\epsilon\imath\mu\hat{h}_2^\epsilon}{1+\epsilon+\beta q^2}\right]=-\mathcal{F}^{-1}\left[\frac{1}{1+\epsilon+\beta q^2}\mathcal{F}[(\Phi_{1x}^\star\Phi_{1x})_x]\right]+\epsilon^{-1/4}(\mathcal{L}(\Phi_2))_x$$

$$+\,\epsilon^{-1/2}(\mathcal{L}(\epsilon\Phi_x,\epsilon^2\Phi_{2y},\epsilon^4\eta,\epsilon^3\eta_x)_x+(\mathcal{L}(\epsilon\Phi_{1x},\Phi_2,\Phi_{2x},\Phi_{2y},\epsilon^2\eta,\epsilon^4\eta_x)_x+\mathcal{H}.$$

$$\begin{aligned}
\mathcal{F}^{-1}\left[\frac{\epsilon\imath\mu.\imath k\hat{h}_1^\epsilon}{1+\epsilon+\beta q^2}\right] &= (\mathcal{L}(\Phi_2,\epsilon^2\eta))_x+\epsilon^2k^2(\mathcal{L}(\Phi_1))_x+\mathcal{H},\ \mathcal{F}^{-1}\left[\frac{\mu^2\hat{p}_2^\epsilon|_{\zeta=1}}{1+\epsilon+\beta q^2}\right]=\epsilon^{-1/4}(\mathcal{L}(\Phi_2,\epsilon\eta)) \\
\mathcal{F}^{-1}\left[(\epsilon k^2+\mu^2)\int_0^1\hat{p}_2^\epsilon\,d\zeta\right] &= k^2\mathcal{L}(\epsilon\Phi_2,\epsilon^2\eta)+(\mathcal{L}(\Phi_2,\Phi_{2x},\epsilon\eta,\epsilon\eta_x))_x
\end{aligned}$$

$$\int_0^1(\xi^\dagger-(G_{2,2}^\epsilon)_x+\imath kH_1^\epsilon)d\zeta=(\eta^\star\Phi_{1x})_x+(\Phi_{1x}^\star\eta)_x+(\mathcal{L}(\Phi_{2x},\Phi_{2y},\epsilon\eta,\epsilon\eta_x))_x$$

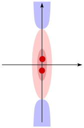
$$(\beta-1/3)\Phi_{1xxxx}-\Phi_{1xx}+k^2(1+\epsilon)\Phi_1=(\eta^\star\Phi_{1x})_x+(\Phi_{1x}^\star\eta)_x+\mathcal{F}^{-1}\left[\frac{1}{1+\epsilon+\beta q^2}\mathcal{F}[(\Phi_{1x}^\star\Phi_{1x})_x]\right]$$

$$+\,(\mathcal{L}(\epsilon^{1/2}\Phi_{1x},\epsilon^{-1/4}\Phi_2,\Phi_{2x},\Phi_{2y},\epsilon^{3/4}\eta,\epsilon\eta_x))_x+k^2[\mathcal{L}(\epsilon\Phi_1,\epsilon\Phi_2,\epsilon^2\eta)+\epsilon^2\mathcal{L}(\Phi_1)_x]+\mathcal{H},$$

$$\eta=\mathcal{F}^{-1}\left[\frac{\imath\mu\hat{\Phi}_1}{1+\epsilon+\beta q^2}\right]+\mathcal{L}(\epsilon\Phi_{1x},\epsilon^{3/4}\Phi_2,\Phi_{2x},\Phi_{2y},\epsilon^3\eta,\epsilon^{7/2}\eta_x)+k^2\epsilon^3\mathcal{L}(\Phi_1)+\mathcal{H}.$$

Locate eigenvalues

$$\boxed{|\lambda| \le \varepsilon \ell_*}$$

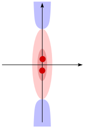


- **two simple eigenvalues $\pm i\varepsilon\kappa_\varepsilon$**
  - $\mathcal{B}_{\varepsilon,\ell}$ small relatively bounded perturbation of $\mathcal{B}_{0,\ell}$

$$\boxed{\mathcal{B}_{0,\ell} = k_a^2 \partial_x \, \mathcal{A} \, \partial_x + \ell^2} \quad \boxed{\mathcal{A} = \left(\beta - \frac{1}{3}\right) k_a^2 \partial_{xx} - 1 - 3P_a}$$

Locate eigenvalues

$\boxed{|\lambda| \leq \varepsilon \ell_*}$



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- spectrum of $\partial_x \mathcal{A} \partial_x$ is known (KP-I):

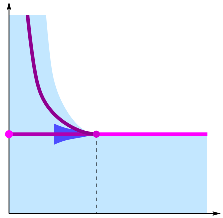
  one simple negative eigenvalue $-\omega_a^2$

- perturbation arguments . . . . . .

Critical surface tension

- **transverse linear instability**
  - *longitudinal co-periodic perturbations*
  - *transverse periodic perturbations*
- **5th order KP model**

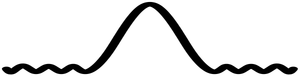
  [H. & Wahlén, 2017]



A 5th order KP model

$$\boxed{\partial_t \partial_x u = \partial_x^2 \left(\partial_x^4 u + \partial_x^2 u + \frac{1}{2} u^2 \right) + \partial_y^2 u}$$

- **traveling generalized solitary waves**



  - *solutions of the Kawahara equation*

$$\boxed{\partial_t u = \partial_x \left(\partial_x^4 u + \partial_x^2 u - cu + \frac{1}{2} u^2 \right)}$$

Periodic traveling waves

small periodic traveling waves: *a two-parameter family*

$$\boxed{\varphi_{a,c}(x) = p_{a,c}(k_{a,c}x)}$$

- depend analytically upon $(a,c) \in (-a_0, a_0) \times (-c_0, c_0)$
- $k_{a,c} = k_0(c) + c\widetilde{k}(a,c)$,

  $k_0(c) = \left(\frac{1+\sqrt{1+4c}}{2}\right)^{1/2}$, $\widetilde{k}(a,c) = \sum_{n\geq 1} \widetilde{k}_{2n}(c)a^{2n}$

- $p_{a,c}(z) = ac\cos(z) + c\sum_{m,n} \widetilde{p}_{n,m}(c)\mathrm{e}^{\mathrm{i}(n-m)z}a^{n+m}$,

  $(n, m \geq 0,\ n+m \geq 2,\ n-m \neq \pm 1)$

- explicit Taylor expansions for $\widetilde{k}(a,c)$, $\widetilde{p}_{n,m}(c)$

[Lombardi, 2000]

Transverse instability problem

- **one-dimensional periodic wave u_***
- **u_* is transversely linearly unstable** *if the linearized equation*

$$\boxed{\partial_t \partial_x u = \partial_x^2 \left(\partial_x^4 u + \partial_x^2 u - cu + u_* u \right) + \partial_y^2 u}$$

possesses a solution of the form $\boxed{u(t,x,y) = \mathrm{e}^{\lambda t} v(x,y)}$,
for some $\mathrm{Re}\,\lambda > 0$

(v belongs to the set of the allowed perturbations)

Transverse instability problem

- **linearized equation**

$$\boxed{\partial_t \partial_x u = \mathcal{A}_* u + \partial_y^2 u}, \qquad \boxed{\mathcal{A}_* = \partial_x^2 \left(\partial_x^4 + \partial_x^2 - c + u_* \right)}$$

  - **Fourier transform in y**

$$\boxed{\partial_t \partial_x u = \mathcal{A}_* u - \omega^2 u}$$

Transverse instability problem

- **linearized equation**

$$\boxed{\partial_t\partial_x u = \mathcal{A}_* u + \partial_y^2 u}, \quad \boxed{\mathcal{A}_* = \partial_x^2\left(\partial_x^4 + \partial_x^2 - c + u_*\right)}$$

  - **Fourier transform in y**

$$\boxed{\partial_t\partial_x u = \mathcal{A}_* u - \omega^2 u}$$

- **u_* is transversely unstable if** *there exists a solution of the form* $\boxed{u(t,x) = \mathrm{e}^{\lambda t} v(x)}$, *for some* $\mathrm{Re}\,\lambda > 0$, *and* $\omega \in \mathbb{R}^*$

  - $v \in H$, a space of functions depending upon the longitudinal spatial variable x, e.g., $H = L^2(\mathbb{R})$ or $H = L^2(0, L)$, and

$$\boxed{\lambda\partial_x v = \mathcal{A}_* v - \omega^2 v}$$

Transverse instability problem

- for some $\boxed{\mathrm{Re}\,\lambda > 0 \text{ and } \omega \in \mathbb{R}^*}$, there exists a solution

$$\boxed{\lambda \partial_x v = \mathcal{A}_* v - \omega^2 v}, \quad \boxed{v \in H}$$

  - u_* **is transversely spectrally unstable** if the linear operator $\lambda\partial_x - \mathcal{A}_* + \omega^2$ is not invertible in H

u_* is **transversely spectrally unstable** if the spectrum of the linear operator $\lambda\partial_x - \mathcal{A}_*$ contains a negative value $-\omega^2 < 0$ for some $\mathrm{Re}\,\lambda > 0$.

Transverse instability problem

$\boldsymbol{u}_*$ is **transversely spectrally unstable** if the **spectrum** of the linear operator $\boldsymbol{\lambda\partial_x - \mathcal{A}_*}$ contains a negative value $\boldsymbol{-\omega^2 < 0}$ for some $\mathbf{Re}\,\boldsymbol{\lambda} > \mathbf{0}$.

- if $-\omega^2$ is an **isolated eigenvalue** then $\boldsymbol{u}_*$ is *transversely linearly unstable*

- if $-\omega^2$ belongs to the **essential spectrum**

  $\sigma_{\text{ess}}(\lambda\partial_x - \mathcal{A}_*) = \{\nu \in \mathbb{C}\ ;\ \lambda\partial_x - \mathcal{A}_* - \nu$ is not Fredholm with index $0\}$

  then $\boldsymbol{u}_*$ is *transversely essentially unstable*

Periodic waves

- **small periodic waves:** $\boxed{\varphi_{a,c}(x) = p_{a,c}(k_{a,c}x)}$
  - **scaling:** $z = k_{a,c}x, \quad \lambda = k_{a,c}\Lambda$
- **rescaled operator**

$$\boxed{\Lambda\partial_z - \mathcal{B}_{a,c}, \quad \mathcal{B}_{a,c} = \partial_z^2(k_{a,c}^4\partial_z^4 + k_{a,c}^2\partial_z^2 - c + p_{a,c})}$$

with 2π-*periodic coefficients*

Co-periodic perturbations

$$\boxed{\Lambda\partial_z - \mathcal{B}_{a,c}, \quad \mathcal{B}_{a,c} = \partial_z^2(k_{a,c}^4\partial_z^4 + k_{a,c}^2\partial_z^2 - c + p_{a,c})}$$

closed operator in $H = L^2(0, 2\pi)$

Theorem

1. *the linear operator* $\Lambda\partial_z - \mathcal{B}_{a,c}$ *acting in* $L^2(0, 2\pi)$ *has a* **simple negative eigenvalue**.
2. *the periodic wave* $\varphi_{a,c}$ *is* **transversely linearly unstable with respect to co-periodic longitudinal perturbations.**

Proof

$$\boxed{\Lambda\partial_z - \mathcal{B}_{a,c}, \quad \mathcal{B}_{a,c} = \partial_z^2(k_{a,c}^4\partial_z^4 + k_{a,c}^2\partial_z^2 - c + p_{a,c})}$$

- **show that $\mathcal{B}_{a,c}$ has a simple positive eigenvalue**
  - the operator $\Lambda\partial_z - \mathcal{B}_{a,c}$ is real
  - perturbation argument: the negative eigenvalue of $-\mathcal{B}_{a,c}$ persists for small real Λ
- **(point) spectrum of $\mathcal{B}_{a,c}$?**

Proof

Spectrum of $\boxed{\mathcal{B}_{a,c} = \partial_z^2(k_{a,c}^4\partial_z^4 + k_{a,c}^2\partial_z^2 - c + p_{a,c})}$

use perturbation arguments: small a *and* c

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use perturbation arguments: small $\boldsymbol{a}$ *and* $\boldsymbol{c}$

- $\boxed{a = 0, c = 0}$

$$\mathcal{B}_{0,0} = \partial_z^2\left(\partial_z^4 + \partial_z^2\right), \quad \sigma(\mathcal{B}_{0,0}) = \{-n^2(n^4 - n^2),\ n \in \mathbb{Z}\}$$

  - $\mathbf{0}$ is a triple eigenvalue
  - all other eigenvalues are negative

Proof

Spectrum of $\boxed{\mathcal{B}_{a,c} = \partial_z^2(k_{a,c}^4\partial_z^4 + k_{a,c}^2\partial_z^2 - c + p_{a,c})}$

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- spectral decomposition for small a and c

$$\sigma(\mathcal{B}_{a,c}) = \sigma_1(\mathcal{B}_{a,c}) \cup \sigma_2(\mathcal{B}_{a,c})$$

  - $\sigma_1(\mathcal{B}_{a,c}) \subset V$, V neighborhood of 0
  - $\sigma_2(\mathcal{B}_{a,c}) \subset \{\nu \in \mathbb{C}\ ;\ \mathrm{Re}\,\nu < -m\}$

Proof

Spectrum of $\boxed{\mathcal{B}_{a,c} = \partial_z^2(k_{a,c}^4\partial_z^4 + k_{a,c}^2\partial_z^2 - c + p_{a,c})}$**:**

locate the small eigenvalues

Spectrum of $\boxed{\mathcal{B}_{a,c} = \partial_z^2(k_{a,c}^4\partial_z^4 + k_{a,c}^2\partial_z^2 - c + p_{a,c})}$**:**

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- $\boxed{a = 0}$

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- $\boxed{a = 0}$

$$\sigma(\mathcal{B}_{0,c}) = \{-n^2(k_0^2n^4 - k_0^2n^2 - c),\ n \in \mathbb{Z}\}$$

  - 0 is a triple eigenvalue
  - all other eigenvalues are negative

- $\boxed{a \neq 0}$
  - use symmetries and show that 0 is a double eigenvalue
  - third eigenvalue: compute an expansion for small a, ...

$$\ldots, \quad \nu_{a,c} = a^2c^2\left(\frac{1}{4X_2} + O(a^2 + c^2)\right) > 0$$

$\square$

Consequences

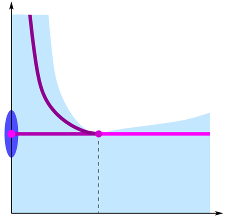
- *implies* **essential transverse instability of periodic waves** *with respect to localize perturbations*
- *implies* **essential transverse instability of generalized solitary waves** *with respect to localize perturbations*

- *extend to* **Euler equations . . . ?**

Zero surface tension

- **transverse spectral stability**
  - *fully localized/bounded perturbations*
- **KP-II equation**

[H., Li, & Pelinovsky, 2017]



Count unstable eigenvalues

- **Hamiltonian structure:** *linear operator of the form* $\boxed{\mathcal{J}\mathcal{L}}$
  - $\mathcal{J}$ skew-adjoint operator
  - $\mathcal{L}$ self-adjoint operator

- **Under suitable conditions:**

$$\boxed{n_u(\mathcal{J}\mathcal{L}) \leq n_s(\mathcal{L})}$$

  - $n_u(\mathcal{J}\mathcal{L}) =$ number of unstable eigenvalues of $\mathcal{J}\mathcal{L}$
  - $n_s(\mathcal{L}) =$ number of negative eigenvalues of $\mathcal{L}$

[well-known result, extensively used in stability problems . . .]

[does not work very well for periodic waves . . .]

An extended eigenvalue count

- **Hamiltonian structure:** *linear operator of the form* $\boxed{\mathcal{J}\mathcal{L}}$
  - $\mathcal{J}$ skew-adjoint operator
  - $\mathcal{L}$ self-adjoint operator
- **There exists a self-adjoint operator** $\mathcal{K}$ *such that*

$$\boxed{(\mathcal{J}\mathcal{L})(\mathcal{J}\mathcal{K}) = (\mathcal{J}\mathcal{K})(\mathcal{J}\mathcal{L})}$$

- **Under suitable conditions:**

$$\boxed{n_u(\mathcal{J}\mathcal{L}) \leq n_s(\mathcal{K})}$$

  - $n_u(\mathcal{J}\mathcal{L})$ = number of unstable eigenvalues of $\mathcal{J}\mathcal{L}$
  - $n_s(\mathcal{K})$ = number of negative eigenvalues of $\mathcal{K}$

Stability of periodic waves

- **classical result:** *allows to show (orbital) stability of periodic waves with respect to co-periodic perturbations*

- **particular case $n_s(\mathcal{K}) = 0$:** *used to show nonlinear (orbital) stability of periodic waves with respect to subharmonic perturbations (for the KdV and NLS equations)*

  [Deconinck, Kapitula, 2010; Gallay, Pelinovsky, 2015]

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[Deconinck, Kapitula, 2010; Gallay, Pelinovsky, 2015]

- **key step: construction of a nonnegative operator $\mathcal{K}$**
  - relies upon the existence of a conserved higher-order energy functional (due to integrability)

KP-II EQUATION

- **Kadomtsev-Petviashivili equation**

$$\boxed{(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0}$$

- **one-parameter family of one-dimensional periodic traveling waves** (up to symmetries)

$$\boxed{u(x,t) = \phi_c(x + ct)}$$

  - **speed** $c > 1$
  - **2π-periodic, even profile** ϕ_c satisfying the KdV equation

$$\boxed{v''(x) + cv(x) + 3v^2(x) = 0}$$

  - **known explicitly!**

Linearized equation

linearized KP-II equation

$$\boxed{(w_t + w_{xxx} + cw_x + 6(\phi_c(x)w)_x)_x + w_{yy} = 0}$$

- 2π-periodic coefficients in x
- Ansatz

$$\boxed{w(x,y,t) = e^{\lambda t + ipy} W(x), \quad \lambda \in \mathbb{C},\ p \in \mathbb{R}}$$

linearized equation for $W(x)$

$$\boxed{\lambda W_x + W_{xxxx} + cW_{xx} + 6(\phi_c(x)W)_{xx} - p^2 W = 0}$$

Spectral stability problem

- **linearized equation for $W(x)$**

$$\boxed{\lambda W_x + W_{xxxx} + cW_{xx} + 6(\phi_c(x)W)_{xx} - p^2 W = 0}$$

- *the periodic wave ϕ_c is* **spectrally stable iff** *the linear operator*

$$\boxed{\mathcal{A}_{c,p}(\lambda) = \lambda\partial_x + \partial_x^4 + c\partial_x^2 + 6\partial_x^2(\phi_c(x)\,\cdot) - p^2}$$

is invertible for $\mathbf{Re}\,\boldsymbol{\lambda} > \mathbf{0}$.

- **2D bounded perturbations: space $C_b(\mathbb{R})$** and $p \in \mathbb{R}$.
- continuous spectrum . . .

Floquet/Bloch decomposition

- $\mathcal{A}_{c,p}(\lambda)$ *is invertible in* $C_b(\mathbb{R})$ *iff the operators*

$$\boxed{\mathcal{A}_{c,p}(\lambda,\gamma) = \lambda(\partial_x + i\gamma) + (\partial_x + i\gamma)^4 + c(\partial_x + i\gamma)^2 + 6(\partial_x + i\gamma)^2(\phi_c(x)\,\cdot) - p^2}$$

are invertible in $L^2_{per}(0, 2\pi)$, *for any* $\gamma \in [0, 1)$.

- $\boxed{\gamma \in (0,1)}$: **study the spectrum of the operator**

$$\boxed{\mathcal{B}_{c,p}(\gamma) = -(\partial_x + i\gamma)^3 - c(\partial_x + i\gamma) - 6(\partial_x + i\gamma)(\phi_c(x)\,\cdot) + p^2(\partial_x + i\gamma)^{-1}}$$

- $\boxed{\gamma = 0}$: **restrict to functions with zero mean**

Counting criterion

- apply the counting criterion to

$$\boxed{\mathcal{B}_{c,p}(\gamma) = \mathcal{J}(\gamma)\mathcal{L}_{c,p}(\gamma)}$$

  - **skew-adjoint operator** $\boxed{\mathcal{J}(\gamma) = (\partial_x + i\gamma)}$
  - **self-adjoint operator**

$$\boxed{\mathcal{L}_{c,p}(\gamma) = -(\partial_x + i\gamma)^2 - c - 6\phi_c(x) + p^2(\partial_x + i\gamma)^{-2}}$$

- construct **positive commuting operators** $\boxed{\mathcal{K}_{c,p}(\gamma)}$
  - find **commuting operators** $\mathcal{M}_{c,p}(\gamma)$
  - show that suitable linear combination of $\mathcal{M}_{c,p}(\gamma)$ and $\mathcal{L}_{c,p}(\gamma)$ **is a positive operator**

Commuting operators

- **natural candidate:** use a higher-order conserved functional
  - resulting operator satisfies the commutativity relation
  - cannot obtain positive operators ...

Commuting operators

- **natural candidate:** use a higher-order conserved functional
  - resulting operator satisfies the commutativity relation
  - cannot obtain positive operators ...

- **second option:** use the operators from the KdV equation
  - $p = 0$ corresponds to the KdV equation
  - decompose:

    $$\boxed{\mathcal{L}_{c,p} = \mathcal{L}_{\mathrm{KdV}} + p^2 \mathcal{L}_{\mathrm{KP}}, \ \mathcal{M}_{c,p} = \mathcal{M}_{\mathrm{KdV}} + p^2 \mathcal{M}_{\mathrm{KP}}}$$

  - $\mathcal{M}_{\mathrm{KdV}}$ is obtained from a higher order conserved functional:

    $$\boxed{\mathcal{M}_{\mathrm{KdV}} = \partial_x^4 + 10\partial_x \phi_c(x)\partial_x - 10c\phi_c(x) - c^2}$$

  - compute $\mathcal{M}_{\mathrm{KP}}$ directly from the commutativity relation:

    $$\boxed{\mathcal{M}_{\mathrm{KP}} = \frac{5}{3}\left(1 + c\partial_x^{-2}\right)}$$

Main result

- **Transverse spectral stability of periodic waves** (with respect to bounded perturbations):
  - *there exist constants* $\boldsymbol{b}$ *such that the operators* $\boxed{\mathcal{K}_{c,p,b}(\gamma) = \mathcal{M}_{c,p}(\gamma) - b\mathcal{L}_{c,p}(\gamma)}$ **are positive**[1]
  - *the commutativity relation holds*
  - *the general counting criterion implies that the spectra of* $\boxed{\mathcal{B}_{c,p}(\gamma) = \mathcal{J}(\gamma)\mathcal{L}_{c,p}(\gamma)}$ **are purely imaginary**

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- *the commutativity relation holds*
- *the general counting criterion implies that the spectra of* $\boxed{\mathcal{B}_{c,p}(\gamma) = \mathcal{J}(\gamma)\mathcal{L}_{c,p}(\gamma)}$ **are purely imaginary**

Consequence: **transverse linear stability of the periodic waves** *with respect to doubly periodic perturbations*

Many open problems . . .

- **water waves:** *other parameter regimes, other types of waves (solitary waves, three-dimensional waves) . . .*
- **periodic waves:** *nonlinear stability with respect to localized perturbations (KdV equation?) . . .*