Maximal Single-Plate Polarization

Let (A, m) be an infinite compact metric space.
$K : A \times A \to (-\infty, \infty]$, **symmetric** and **lower semi-continuous**.
$\omega_N := (x_1, \ldots, x_N) \in A^N$

Definition: *Polarization Constants*

$$P_{K,A}(\omega_N) := \min_{x \in A} \sum_{i=1}^{N} K(x, x_i).$$

The single-plate polarization (Chebyshev) problem: Find

$$\mathcal{P}_K(A, N) := \max_{\omega_N \in A^N} P_{K,A}(\omega_N) = \max_{\omega_N \in A^N} \min_{x \in A} \sum_{i=1}^{N} K(x, x_i)$$

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N-point configurations $\omega_N^* = (x_1^*, \ldots, x_N^*)$ satisfying

$$P_{K,A}(\omega_N^*) = \mathcal{P}_K(A, N)$$

are called ***optimal*** or ***maximal K-polarization configurations***.

Example: Why "Chebyshev" ?

For

$$K(x, y) = \log \frac{1}{|x - y|},$$

$$A = [-1, 1] \subset \mathbb{R},$$

$$\mathcal{P}_K(A, N) = \log \frac{1}{\min_{p(x)=x^N+\dots} \max_{x\in[-1,1]} |p(x)|},$$

where $p(x)$ has all its zeros on $[-1, 1]$.

$$\mathcal{P}_K(A, N) = \log \frac{1}{||T_N(x)||_{[-1,1]}} = (N - 1) \log 2,$$

$T_N(x) = 2^{1-N} \cos(N\theta),\ x = \cos\theta,$ is the monic Chebyshev polynomial of degree N of the first kind. So optimal polarization points are the zeros of $T_N(x)$.

Comparison with Discrete Minimal Energy

K-energy of $\omega_N = (x_1, \ldots, x_N) \in A^N$ is

$$E_K(\omega_N) := \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} K(x_i, x_j) = \sum_{i \neq j} K(x_i, x_j)$$

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Minimal N-point K-energy of the set A is

$$\mathcal{E}_K(A, N) := \inf\{E_K(\omega_N) : \omega_N \in A^N\}$$

If $E_K(\omega_N^*) = \mathcal{E}_K(A, N)$, then ω_N^* is called N-**point** K-**equilibrium configuration** for A or a set of **optimal** K-**energy points**.

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Simple Observation

For every $N \geq 2$,

$$\mathcal{P}_K(A, N) \geq \frac{\mathcal{E}_K(A, N)}{N - 1}.$$

Example: $\mathbb{S}^p = \{x \in \mathbb{R}^{p+1} \; : \; ||x|| = 1\}$

Proposition (Polarization on $\mathbb{S}^p$)

Let $2 \leq N \leq p+1$.

Assume $f : [0,4] \to (-\infty, \infty]$ satisfies $f((0,4]) \subset (-\infty, \infty)$, with f **convex** on $(0,4]$ and is **strictly decreasing** on $[0,4]$.

For $K(\mathbf{x}, \mathbf{y}) = f(|\mathbf{x} - \mathbf{y}|^2)$,

we have that any configuration $\omega_N = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ on $\mathbb{S}^p$ such that $\sum_{i=1}^{N} \mathbf{x}_i = \mathbf{0}$ is optimal for the maximal K-polarization problem on $\mathbb{S}^p$.

Furthermore, $\mathcal{P}_K(\mathbb{S}^p, N) = Nf(2)$.

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Result holds for Riesz s-kernel $K_s(\mathbf{x},\mathbf{y})$ for $s > 0$ and $s = \log$.

For $\mathbb{S}^2$ we know max s-polarization configurations for $N = 1, 2, 3$. Also for $N = 4$, [Y. Su], max polarization points are vertices of inscribed tetrahedron.

Maximal Riesz s-Polarization for $N = 5$ on $\mathbb{S}^2$?

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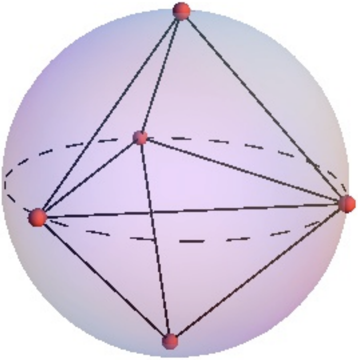
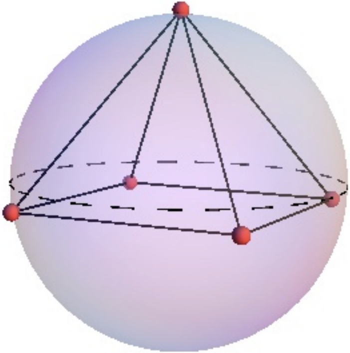
bipyramid

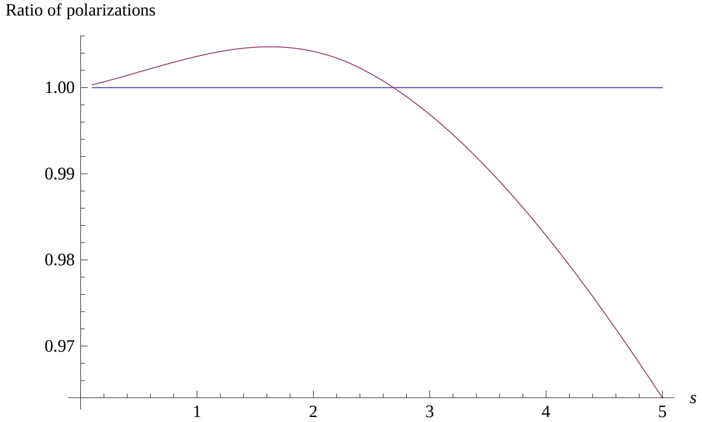


square-base pyramid



Maximal Riesz s-Polarization for $N = 5$ on $\mathbb{S}^2$

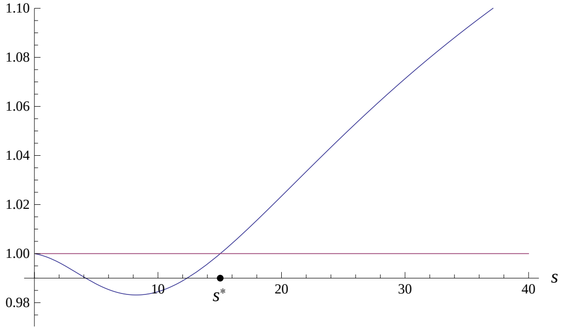
Ratio of s-polar of **optimal sq-base pyramid** to s-polar of **bipyramid**



Square-base pyramid appears optimal for s up to $s \approx 2.69$; thereafter, bipyramid appears optimal.

Compare with Minimal Riesz s-Energy for $N = 5$ on $\mathbb{S}^2$

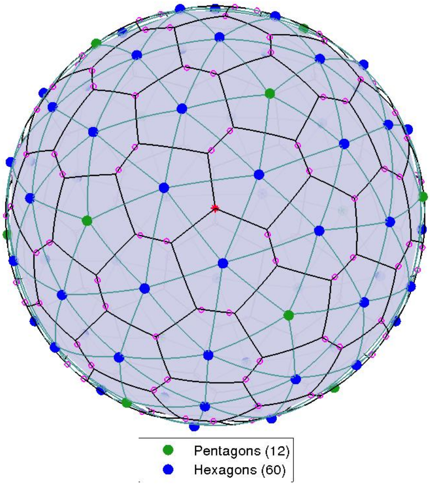
Ratio of s-energy of **bipyramid** to s-energy of **optimal sq-base pyramid**



Melnyk et al (1977) Bipyramid appears optimal for $0 < s < s^*$ where $s^* \approx 15.04808$.

Recently proved by R. Schwartz (over 150 pages + computer assist). Open problem for $s > s^* + \epsilon$

Max Polarization for $N = 72$ and $s = 3$



Example: Unit Ball $\mathcal{B}^p \subset \mathbb{R}^p$

For the Riesz s-kernel $K_s(x, y) = 1/|x - y|^s$,

if $p \geq 3$ and $-2 < s < p - 2$, $s \neq 0$, then

Optimal N-point s-Polarization configurations all lie at the center of $\mathcal{B}^p$

Optimal N-point s-Energy configurations all lie on $\partial\mathcal{B}^p = \mathbb{S}^{p-1}$

ASYMPTOTICS

Connection to Best-Covering as $s \to \infty$

Covering radius of $\omega_N \in A^N$ is given by

$$\rho(\omega_N; A) := \max_{y \in A} \min_{x \in \omega_N} |y - x|.$$

***N*-point covering radius of A** :

$$\rho_N(A) = \inf\{\rho(\omega_N; A) : \omega_N \in A^N\}$$

.

Proposition

For each fixed N, the maximal Riesz s-polarization satisfies

$$\lim_{s \to \infty} \mathcal{P}_s(A, N)^{1/s} = \frac{1}{\rho_N(A)}.$$

Furthermore, every cluster point as $s \to \infty$ of optimal $N-$point s-polarization configurations (ω_N^s) is an $N-$point best-covering configuration.

Asymptotics as $N \to \infty$

A compact, infinite
$\mathcal{M}(A)$ the set of probability measures supported on A.
$K : A \times A \to (-\infty, +\infty]$ symmetric and l.s.c.

Proposition (Polarization) (Ohtsuka)

$$\lim_{N\to\infty} \frac{\mathcal{P}_K(A,N)}{N} = \sup_{\mu\in\mathcal{M}(A)} \inf_{x\in A} \int K(x,y)\, d\mu(y) =: T_K(A).$$

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Proposition(Energy)

$$\lim_{N\to\infty} \frac{\mathcal{E}_K(A,N)}{N^2} = \inf_{\mu\in\mathcal{M}(A)} \iint_{A\times A} K(x,y)\, d\mu(x)d\mu(y) =: W_A(K).$$

$$T_K(A) \geq W_K(A)$$

Non-integrable Riesz Kernels: $s > d = \dim(A)$

Polarization"Poppy-Seed Bagel" Theorem (s>d) (BHRS 2018)

Let $A \subset \mathbb{R}^d$ be an infinite compact set of positive Lebesgue $\mathcal{L}_d$-measure whose boundary has measure zero. If $s > d$, then

$$\lim_{N\to\infty} \frac{\mathcal{P}_s(A,N)}{N^{s/d}} = \frac{\sigma_{s,d}}{\mathcal{L}_d(A)^{s/d}}.$$

Furthermore, every asymptotically maximizing s-polarization sequence of N-point configurations on A is asymptotically uniformly distributed with respect to normalized $\mathcal{L}_d$-measure on A.

$\sigma_{s,1} = 2\zeta(s,1/2) = 2\zeta(s)(2^s - 1)$, for $d \geq 2$, constant $\sigma_{s,d}$ unknown.

Poppy-Seed Theorem for Embedded Sets

Same conclusions hold for any embedded d-dimensional compact C^1-smooth manifold $A \subset \mathbb{R}^p$, $p > d$, with $\mathcal{H}_d(\partial A) = 0$.

Two Special Classes of Integrable Kernels

Theorem(Simanek)

(i) $A \subset \mathbb{R}^t$ compact; $K(x, y) = f(|x - y|)$, where $f \geq 0$ is l.s.c;

(ii) K-energy continuous equilibrium measure μ_A^e is unique ;

(iii) $\mathrm{supp}(\mu_A^e) = A$;

(iv) $U^e(x) := \int_A K(x, y)\, d\mu_A^e(y) = W_K(A)$ everywhere on A.

Then $T_K(A) = W_K(A)$, and for any sequence $\{\omega_N^*\}$ of optimal K-polarization configurations, the associated normalizing counting measures converge weak* to μ_A^e. Furthermore, $\mu_A^e = \mu_A^p$.

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Theorem (Reznikov, S, Vlasiuk)

(i) A is d-regular;
(ii) f is *d-Riesz like* (e.g., t^{-s} for $0 < s < d$).

Then any weak* limit measure of normalized counting measures for optimal K-polarization configurations is an optimal measure for the continuous K-polarization problem.