Concentration for Coulomb gases and Coulomb transport inequalities

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Outline of the talk

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- Coulomb gases : definition and known results

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- Concentration inequalities
- Outline of the proof and Coulomb transport inequalities

Coulomb gases ($d \geq 2$)

We consider the Poisson equation

$$\Delta g = -c_d \delta_0.$$

The fundamental solution is given by

$$g(x) := \begin{cases} -\log|x| & \text{for } d = 2, \\ \frac{1}{|x|^{d-2}} & \text{for } d \geq 3. \end{cases}$$

A gas of N particles interacting according to the Coulomb law would have an energy given by

$$H_N(x_1, \dots, x_N) := \sum_{i \neq j} g(x_i - x_j) + N \sum_{i=1}^{N} V(x_i).$$

We denote by $\mathbb{P}^N_{V,\beta}$ the Gibbs measure on $(\mathbb{R}^d)^N$ associated to this energy :

$$\mathrm{d}\mathbb{P}^N_{V,\beta}(x_1, \ldots, x_N) = \frac{1}{Z^N_{V,\beta}} \mathrm{e}^{-\frac{\beta}{2} H_N(x_1,\ldots,x_N)} \mathrm{d}x_1, \ldots, \mathrm{d}x_N$$

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Example (Ginibre) : let M_N be an N by N matrix with iid entries with law $\mathcal{N}_{\mathbb{C}}(0,\frac{1}{N})$, then the eigenvalues have joint law $\mathbb{P}^N_{|x|^2,2}$ with

$$\mathrm{d}\mathbb{P}^N_{|x|^2,2}(x_1,\ldots,x_N) \sim \prod_{i<j} |x_i - x_j|^2 \mathrm{e}^{-N\sum_{i=1}^N |x_i|^2}$$

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One can rewrite

$$\begin{aligned} H_N(x_1, \ldots, x_N) &= N^2 \mathcal{E}_V^{\neq}(\hat{\mu}_N) \\ &:= N^2 \left(\iint_{x \neq y} g(x-y) \hat{\mu}_N(\mathrm{d}x) \hat{\mu}_N(\mathrm{d}y) + \int V(x) \hat{\mu}_N(\mathrm{d}x) \right). \end{aligned}$$

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More generally, one can define, for any $\mu \in \mathcal{P}(\mathbb{R}^d)$,

$$\mathcal{E}_V(\mu) := \iint \left(g(x-y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) \mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

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$$\frac{1}{N^2} \log \mathbb{P}^N_{V,\beta}(\mathrm{d}(\hat{\mu}_N, \mu_V) \geq r) \xrightarrow[N\to\infty]{} -\frac{\beta}{2} \inf_{\mathrm{d}(\mu,\mu_V)\geq r} (\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)).$$

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What about concentration ?

Local behavior extensively using several variations of the concept of renormalized energy (see in particular Simona's talk this morning).

Concentration estimates

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We will consider both the bounded Lipschitz distance d_{BL} and the Wassertein W_1 distance, where we recall that

$$\mathrm{d}_{BL}(\mu,\nu) = \sup_{\substack{\|f\|_\infty \leq 1 \\ \|f\|_{Lip} \leq 1}} \int f \mathrm{d}(\mu-\nu); \; W_1(\mu,\nu) = \sup_{\|f\|_{Lip}\leq 1} \int f \mathrm{d}(\mu-\nu)$$

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Theorem

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Theorem

If V is $\mathcal{C}^2$ and V and ΔV satisfy some growth conditions,then there exist $a>0$, $b\in\mathbb{R}$, $c(\beta)$ such that for all $N\geq 2$ and for all $r>0$,

$$\mathbb{P}^N_{V,\beta}(d(\hat{\mu}_N,\mu_V)\geq r) \leq e^{-a\beta N^2 r^2 + \mathbf{1}_{d=2}\frac{\beta}{4} N\log N + b\beta N^{2-\frac{2}{d}} + c(\beta)N}$$

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- the latter allows to get the almost sure convergence of $W_1(\hat{\mu}_N, \mu_V)$ to zero down to $\beta \simeq \frac{\log N}{N}$
- if the potential is subquadratic, a, b and $c(\beta)$ can be made more explicit.

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- for Ginibre, the constants can be computed explicitely ; improves on previous results based on determinantal structure (can we use the Gaussian nature of the entries ?)
- non optimal local laws can be deduced

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We want to take $A := \{\mathrm{d}(\hat{\mu}_N, \mu_V) \geq r\}$.

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We aim at an inequality of the type : for any $\mu \in \mathcal{P}(\mathbb{R}^d)$,

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This inequality is the Coulomb counterpart of Talagrand $\mathbf{T}_1$ inequality : ν satisfies $\mathbf{T}_1$ iif there exists $C > 0$ such that for any $\mu \in \mathcal{P}(\mathbb{R}^d)$,

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To point out what is specific to the Coulombian nature of the interaction, we will show the following local version of our inequality :

Proposition *For any compact set D of $\mathbb{R}^d$, there exists C_D such that for any $\mu, \nu \in \mathcal{P}(D)$ such that $\mathcal{E}(\mu) < \infty$ and $\mathcal{E}(\nu) < \infty$,*

$$W_1(\mu, \nu)^2 \leq C_D \mathcal{E}(\mu - \nu).$$

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If μ and ν have their support in D, there exists D_+ such that

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But we also have

$$\int \Delta h(x-y) g(y) \mathrm{d}y = \int \Delta g(x-y) h(y) \mathrm{d}y = \Delta U^\eta(x).$$

Therefore, for any Lipschitz function with support in D_+

$$\int f \,\mathrm{d}\eta = -\frac{1}{c_d}\int f(x)\Delta U^\eta(x)\mathrm{d}x = -\frac{1}{c_d}\int \nabla f(x)\cdot\nabla U^\eta(x)\mathrm{d}x$$

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We can now conclude as

$$\begin{aligned} \left| \int \nabla f(x) \cdot \nabla U^\eta(x) \mathrm{d}x \right| &\leq \int_{D_+} |\nabla f| \cdot |\nabla U^\eta| \leq \int_{D_+} |\nabla U^\eta| \\ &\leq \left(\mathrm{vol}(D_+) \int |\nabla U^\eta|^2 \right)^{1/2} . \end{aligned}$$

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But

$$\int|\nabla U^\eta|^2 = c_d\mathcal{E}(\eta).$$

Thank you for your attention !