Convergence of spectral measures and eigenvalue rigidity

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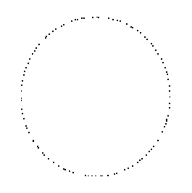

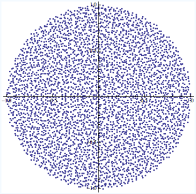

Wigner's Theorem

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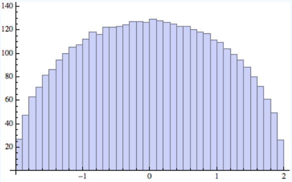
For each $n \in \mathbb{N}$, let $\{Y_i\}_{1 \le i}, \{Z_{ij}\}_{1 \le i < j}$ be independent collections of i.i.d. random variables, with

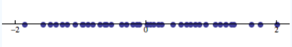
$$\mathbb{E}Y_1 = \mathbb{E}Z_{12} = 0 \qquad \mathbb{E}Z_{12}^2 = 1 \qquad \mathbb{E}Y_1^2 < \infty.$$

Let M_n be the symmetric random matrix with diagonal entries Y_i and off-diagonal entries Z_{ij} or Z_{ji}.

The empirical spectral measure μ_n of $\frac{1}{\sqrt{n}}M_n$ is close, for large n, to the semi-circular law:

$$\frac{1}{2\pi}\sqrt{4 - x^2}\,\mathbb{1}_{|x| \le 2}\,dx.$$





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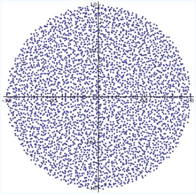
The circular law (Ginibre):

The empirical spectral measure of a large random matrix with i.i.d. Gaussian entries is approximately uniform on a disc.



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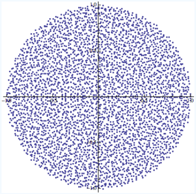
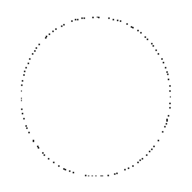


The classical compact groups (Diaconis–Shahshahani):

The empirical spectral measure of a uniform random matrix in $\mathbb{O}(n), \mathbb{U}(n), \mathbb{S}\mathrm{p}(2n)$ is approximately uniform on the unit circle when n is large.



Other examples

Truncations of random unitary matrices (Petz–Reffy):

Let U_m be the upper-left $m \times m$ block of a uniform random matrix in $\mathbb{U}(n)$, and let $\alpha = \frac{m}{n}$.

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$$f_\alpha(z) = \begin{cases} \frac{2(1-\alpha)}{\alpha(1-|z|^2)^2}, & 0 < |z| < \sqrt{\alpha}; \\ 0, & \text{otherwise.} \end{cases}$$

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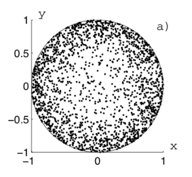

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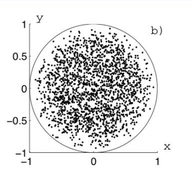

$\alpha = \frac{2}{5}$

Figures from “Truncations of random unitary matrices”, Życzkowski–Sommers, J. Phys. A, 2000

Other examples

Brownian motion on $\mathbb{U}(n)$ (Biane):

Let $\{U_t\}_{t \geq 0}$ be a Brownian motion on $\mathbb{U}(n)$; i.e., a solution to

$$dU_t = U_t dW_t - \frac{1}{2} U_t dt,$$

with $U_0 = I$ and W_t a standard B.M. on $\mathfrak{u}(n)$. There is a deterministic family of measures $\{\nu_t\}_{t \geq 0}$ on the unit circle such that the spectral measure of U_t converges weakly almost surely to ν_t.

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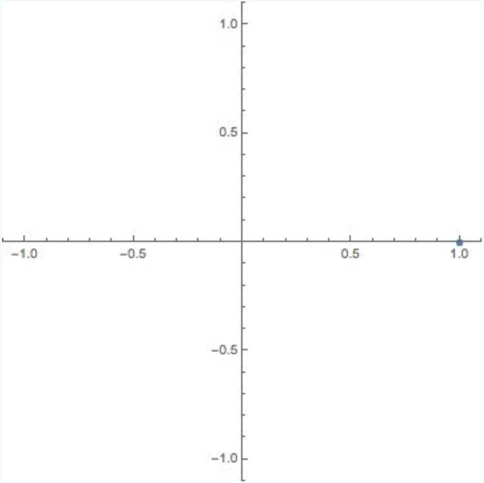

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One may prove that $\mathbb{E}\mu_n \Rightarrow \nu$, possibly via explicit bounds on $d(\mathbb{E}\mu_n, \nu)$ in some metric $d(\cdot, \cdot)$.

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- Convergence weakly in probability or weakly almost surely: for any bounded continuous test function f,

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- The random variable $d(\mu_n, \nu)$:
  Look for ϵ_n such that with high probability (or even probability 1),

$$d(\mu_n, \nu) < \epsilon_n.$$

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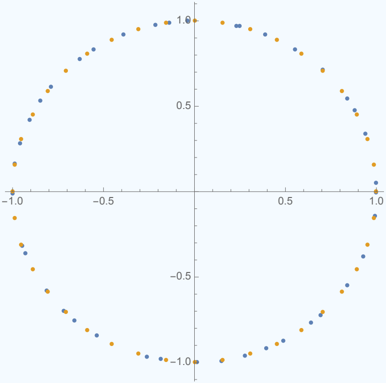

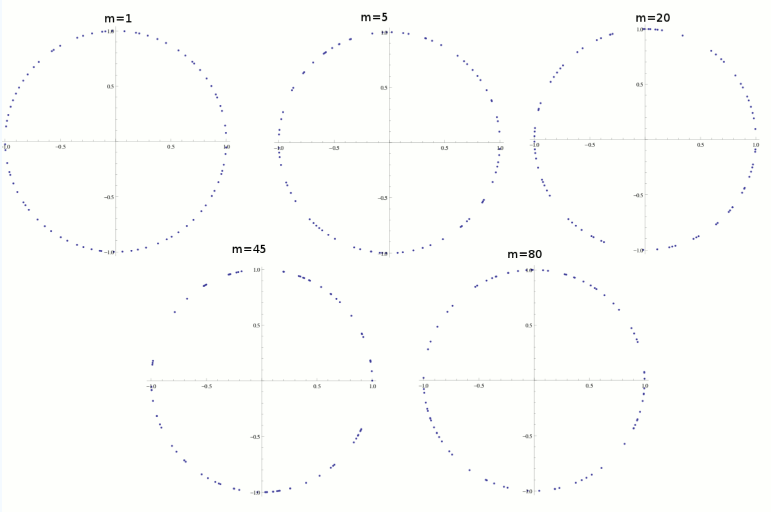

The eigenvalues of U^m for $m = 1, 5, 20, 45, 80$, for U a realization of a random 80×80 unitary matrix.

Theorem (E. M.–M. Meckes)

Let $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ *be the eigenvalue angles of* U^p*, where* U *is a Haar random matrix in* $\mathbb{U}(n)$*. For each* j *and* $t > 0$,

$$\mathbb{P}\left[\left|\theta_j - \frac{2\pi j}{N}\right| > \frac{4\pi}{N} t\right] \leq 4\exp\left[-\min\left\{\frac{t^2}{p\log\left(\frac{N}{p}\right)+1}, t\right\}\right].$$

Concentration of empirical spectral measures

2-D Coulomb gases

Coulomb transport inequality (Chafaï–Hardy–Maïda): Consider the 2-D Coulomb gas model with Hamiltonian

$$H_n(z_1, \ldots, z_n) = -\sum_{j \neq k} \log |z_j - z_k| + n \sum_{j=1}^{n} V(z_j);$$

let μ_V denote the equilibrium measure. There is a constant C_V such that

$$d_{BL}(\mu, \mu_V)^2 \leq W_1(\mu, \mu_V)^2 \leq C_V \left[\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)\right],$$

where $\mathcal{E}_V$ is the modified energy functional

$$\mathcal{E}_V(\mu) = \mathcal{E}(\mu) + \int V d\mu.$$

Truncations of random unitary matrices

Let U be distributed according to Haar measure in $\mathbb{U}(n)$ and let $1 \leq m \leq n$. Let U_m denote the top-left $m \times m$ block of $\sqrt{\frac{n}{m}}U$. The eigenvalue density of U_m is given by

$$\frac{1}{\tilde{c}_{n,m}} \prod_{1 \leq j < k \leq m} |z_j - z_k|^2 \prod_{j=1}^{m} \left(1 - \frac{m}{n}|z_j|^2\right)^{n-m-1} d\lambda(z_1) \cdots d\lambda(z_n),$$

which corresponds to a two-dimensional Coulomb gas with external potential

$$\tilde{V}_{n,m}(z) = \begin{cases} -\frac{n-m-1}{m} \log\left(1 - \frac{m}{n}|z|^2\right). & |z| < \sqrt{\frac{n}{m}}; \\ \infty, & |z| \geq \sqrt{\frac{n}{m}}. \end{cases}$$

Truncations of random unitary matrices

Theorem (M.–Lockwood)

Let $\mu_{m,n}$ *be the spectral measure of the top-left* $m \times m$ *block of* $\sqrt{\frac{n}{m}}U$, *where* U *is a random* $n \times n$ *unitary matrix and* $1 \leq m \leq n - 2\log(n)$. *Let* $\alpha = \frac{m}{n}$, *and let* ν_α *have density*

$$g_\alpha(z) = \begin{cases} \frac{2(1-\alpha)}{(1-\alpha|z|^2)^2}, & 0 < |z| < 1; \\ 0, & \textit{otherwise}. \end{cases}$$

then

$$\mathbb{P}[d_{BL}(\mu_{m,n}, \nu_\alpha) > r] \leq e^{-C_\alpha m^2 r^2 + 2m[log(m) + C'_\alpha]} + e^{-cn},$$

where $C_\alpha = \min\left\{\frac{1}{\log(\alpha^{-1})}, 1\right\}$ *and*

$$C'_\alpha \sim \begin{cases} \log(\frac{1}{\alpha}), & \alpha \to 0; \\ \log(1-\alpha), & \alpha \to 1. \end{cases}$$

Concentration of empirical spectral measures

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If M is an $n \times n$ normal matrix with spectral measure μ_M and $f : \mathbb{C} \to \mathbb{R}$ is 1-Lipschitz, it follows from the Hoffman-Wielandt inequality that

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$\Longrightarrow$ For any reference measure ν,

$$M \mapsto W_1(\mu_M, \nu)$$

is $\frac{1}{\sqrt{n}}$-Lipschitz

Concentration of empirical spectral measures

Many random matrix ensembles satisfy the following concentration property:

Let $F : S \subseteq \mathbb{M}_N \to \mathbb{R}$ be 1-Lipschitz with respect to $\| \cdot \|_{H.S.}$. Then

$$\mathbb{P}\Big[\big|F(M) - \mathbb{E}F(M)\big| > t\Big] \leq Ce^{-cNt^2}.$$

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- Ensembles with matrix density $\propto e^{-N \operatorname{Tr}(u(M))}$, with $u''(x) \geq c > 0$.

Typical vs. average

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In ensembles with the concentration property, $W_1(\mu_n, \nu)$, this means

$$\mathbb{P}[W_1(\mu_n, \nu) > \mathbb{E}W_1(\mu_n, \nu) + t] \leq Ce^{-cN^2t^2}.$$

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$\rightsquigarrow$ To show $W_1(\mu_n, \nu)$ is typically small, it's enough to show that $\mathbb{E}W_1(\mu_n, \nu)$ is small.

Average distance to average

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Under the concentration hypothesis, $\{X_f\}_f$ satisfies a sub-Gaussian increment condition:

$$\mathbb{P}\left[|X_f - X_g| > t\right] \leq 2e^{-\frac{cn^2 t^2}{|f-g|_L^2}}.$$

Dudley's entropy bound together with approximation theory, truncation arguments, etc., can lead to a bound on

$$\mathbb{E} W_1(\mu_n, \mathbb{E}\mu_n) = \mathbb{E}\left(\sup_{|f|_L \leq 1} X_f\right).$$

Theorem (M.–Melcher)

Let μ_t^N *be the spectral measure of* U_t*, where* $\{U_t\}_{t\geq 0}$ *is a Brownian motion on* $\mathbb{U}(n)$ *with* $U_0 = I$*.*

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1. *For any $t, x > 0$,*

$$\mathbb{P}\left(W_1(\mu_t^N, \overline{\mu}_t^N) > c\left(\frac{t}{N^2}\right)^{1/3} + x \right) \leq 2e^{-\frac{N^2 x^2}{t}}.$$

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2. *There are constants c, C such that for $T \geq 0$ and $x \geq c\frac{T^{2/5}\log(N)}{N^{2/5}}$,*

$$\mathbb{P}\left(\sup_{0\leq t\leq T} W_1(\mu_t^N, \nu_t) > x\right) \leq C\left(\frac{T}{x^2}+1\right)e^{-\frac{N^2x^2}{T}}.$$

In particular, with probability one for N sufficiently large

$$\sup_{0\leq t\leq T} W_1(\mu_t^N, \nu_t) \leq c\frac{T^{2/5}\log(N)}{N^{2/5}}.$$

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	$K_N(x,y)$	Λ
GUE	$\sum_{j=0}^{n-1} h_j(x)h_j(y)e^{-\frac{(x^2+y^2)}{2}}$	$\mathbb{R}$
$\mathbb{U}(N)$	$\sum_{j=0}^{N-1} e^{ij(x-y)}$	$[0,2\pi)$
Complex Ginibre	$\frac{1}{\pi}\sum_{j=0}^{N-1} \frac{(z\overline{w})^j}{j!} e^{-\frac{(\lvert z\rvert^2+\lvert w\rvert^2)}{2}}$	$\{\lvert z\rvert = 1\}$

The gift of determinantal point processes

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Theorem (Hough/Krishnapur/Peres/Virág)

Let $K : \Lambda \times \Lambda \to \mathbb{C}$ be the kernel of a determinantal point process, and suppose the corresponding integral operator is self-adjoint, nonnegative, and locally trace-class.

For $D \subseteq \Lambda$, let $\mathcal{N}_D$ denote the number of particles of the point process in D. Then

$$\mathcal{N}_D \stackrel{d}{=} \sum_k \xi_k,$$

where $\{\xi_k\}$ is a collection of independent Bernoulli random variables.

Concentration of the counting function

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Since $\mathcal{N}_D$ is a sum of i.i.d. Bernoullis, Bernstein's inequality applies:

$$\mathbb{P}\left[|\mathcal{N}_D - \mathbb{E}\mathcal{N}_D| > t\right] \leq 2\exp\left(-\min\left\{\frac{t^2}{4\sigma_D^2}, \frac{t}{2}\right\}\right),$$

where $\sigma_D^2 = \mathrm{Var}\,\mathcal{N}_D$.

Rigidity of individual eigenvalues

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$$\rightsquigarrow \quad \mathbb{E}W_1(\mu_N, \nu) \leq \frac{C\sqrt{\log(N)+1}}{N},$$

where ν is the uniform distribution on $\mathbb{S}^1 \subseteq \mathbb{C}$.

Co-authors

- Kathryn Lockwood (Ph.D. student, CWRU):
  truncations of random unitary matrices
- Tai Melcher (UVA):
  Brownian motion on $\mathbb{U}(n)$
- Mark Meckes (CWRU):
  most of the rest