Randomness in $\mathbb{C}^2$ and Pluripotential Theory

Outline

1. Zeros of univariate random polynomials $p : \mathbb{C} \to \mathbb{C}$ and potential theory; recent results of Bloom-Dauvergne
2. Random polynomials $p : \mathbb{C}^2 \to \mathbb{C}$ and random polynomial mappings $\mathbf{F} = (p, q) : \mathbb{C}^2 \to \mathbb{C}^2$ and pluripotential theory; recent results of Bayraktar
3. Generalizations/modifications and open questions

Kac-Hammersley polynomials

Consider *random polynomials* $p_n(z) = \sum_{j=0}^{n} a_j z^j$ where the coefficients $a_0, \ldots, a_n$ are i.i.d. complex Gaussian random variables with $\mathbf{E}(a_j) = \mathbf{E}(a_j a_k) = 0$ and $\mathbf{E}(a_j \bar{a}_k) = \delta_{jk}$. Thus we get a probability measure $Prob_n$ on $\mathcal{P}_n$, the polynomials of degree at most n, identified with $\mathbb{C}^{n+1}$, where, for $G \subset \mathbb{C}^{n+1}$,

$$Prob_n(G) = \frac{1}{\pi^{n+1}} \int_G e^{-\sum_{j=0}^{n} |a_j|^2} dm(a_0) \cdots dm(a_n)$$

where dm =Lebesgue measure on $\mathbb{C}$.

Asymptotic expectation

Write $p_n(z) = a_n \prod_{j=1}^{n}(z - \zeta_j)$ and call $\widetilde{Z}_{p_n} := \frac{1}{n}\sum_{j=1}^{n} \delta_{\zeta_j}$ the *normalized zero measure* of p_n. Note

$$\widetilde{Z}_{p_n} = \Delta \frac{1}{n} \log |p_n|$$

where (ignore 2π) $\Delta \log |z| = \delta_0$.
What can we say about asymptotics of $\mathbf{E}(\widetilde{Z}_{p_n})$ as $n \to \infty$?
Here, $\mathbf{E}(\widetilde{Z}_{p_n})$ is a measure defined, for $\psi \in C_c(\mathbb{C})$, as

$$\left(\mathbf{E}(\widetilde{Z}_{p_n}), \psi\right)_{\mathbb{C}} := \int_{\mathbb{C}^{n+1}} (\widetilde{Z}_{p_n}, \psi)_{\mathbb{C}} \, dProb_n(\mathbf{a^{(n)}})$$

where $\mathbf{a^{(n)}} = (a_0, ..., a_n)$ and $(\widetilde{Z}_{p_n}, \psi)_{\mathbb{C}} = \frac{1}{n}\sum_{j=1}^{n} \psi(\zeta_j)$.

Key idea: Reproducing kernel and monomials

Note that $\{z^j\}_{j=0,\ldots,n} := \{b_j(z)\}_{j=0,\ldots,n}$ form an orthonormal basis for $\mathcal{P}_n$ in $L^2(\mu_{S^1})$ where $\mu_{S^1} = \frac{1}{2\pi} d\theta$ on $S^1 = \{z : |z| = 1\}$.

Proposition. $\lim_{n\to\infty} \mathbf{E}(\widetilde{Z}_{p_n}) = \mu_{S^1}$.

$$S_n(z,w) := \sum_{j=0}^{n} b_j(z)\overline{b_j(w)} = \sum_{j=0}^{n} z^j \bar{w}^j$$

is the reproducing kernel for point evaluation at z on $\mathcal{P}_n$. On the diagonal $w = z$, we have $S_n(e^{i\theta}, e^{i\theta}) = n + 1$ and

$$K_n(z) := S_n(z,z) = \sum_{j=0}^{n} |z|^{2j} = \frac{1 - |z|^{2n+2}}{1 - |z|^2} \text{ Thus:}$$

$$\frac{1}{2n}\log K_n(z) = \frac{1}{2n}\log\frac{1-|z|^{2n+2}}{1-|z|^2} \to \log^+|z| = \max[0, \log|z|]$$

locally uniformly on $\mathbb{C}$. Note that $\Delta\log^+|z| = \mu_{S^1}$; thus

$$\Delta\Big(\frac{1}{2n}\log K_n(z)\Big) \to \mu_{S^1}.$$

Write $|p_n(z)| = |\sum_{j=0}^n a_j b_j(z)| =: | < \mathbf{a^{(n)}}, \mathbf{b^{(n)}}(z) >_{\mathbb{C}^{n+1}} |$

$$= K_n(z)^{1/2}| < \mathbf{a^{(n)}}, \mathbf{u^{(n)}}(\mathbf{z}) >_{\mathbb{C}^{n+1}} |$$

where

$$\mathbf{u^{(n)}}(\mathbf{z}) := \frac{\mathbf{b^{(n)}}(z)}{||\mathbf{b^{(n)}}(z)||} = \frac{\mathbf{b^{(n)}}(z)}{K_n(z)^{1/2}}.$$

Use $|p_n(z)| = K_n(z)^{1/2}| < \mathbf{a^{(n)}}, \mathbf{u^{(n)}(z)} >_{\mathbb{C}^{n+1}} |$:

For $\psi \in C_c(\mathbb{C})$ (recall $\widetilde{Z}_{p_n} = \Delta \frac{1}{n} \log |p_n|$)

$$(\mathbf{E}(\widetilde{Z}_{p_n}), \psi)_{\mathbb{C}} = \int_{\mathbb{C}^{n+1}} (\Delta \frac{1}{n} \log |p_n(z)|, \psi(z))_{\mathbb{C}} \; dProb_n(\mathbf{a^{(n)}})$$

$$= \int_{\mathbb{C}^{n+1}} (\Delta \frac{1}{2n} \log K_n(z), \psi(z))_{\mathbb{C}} dProb_n(\mathbf{a^{(n)}})$$

$$+ \int_{\mathbb{C}^{n+1}} (\Delta \frac{1}{n} \log | < \mathbf{a^{(n)}}, \mathbf{u^{(n)}(z)} >_{\mathbb{C}^{n+1}} |, \psi(z))_{\mathbb{C}} \; dProb_n(\mathbf{a^{(n)}}).$$

The first term (deterministic) goes to $\int_{S^1} \psi d\mu_{S^1}$ as $n \to \infty$ and the second term can be rewritten:

$$\int_{\mathbb{C}^{n+1}} (\frac{1}{n} \log | < \mathbf{a^{(n)}}, \mathbf{u^{(n)}(z)} >_{\mathbb{C}^{n+1}} |, \Delta\psi(z))_{\mathbb{C}} \; dProb_n(\mathbf{a^{(n)}})$$

$$= \int_{\mathbb{C}} \Delta\psi(z)\Big[\frac{1}{n}\int_{\mathbb{C}^{n+1}} \log|<\mathbf{a^{(n)}}, \mathbf{u^{(n)}}(\mathbf{z})>_{\mathbb{C}^{n+1}}| \, dProb_n(\mathbf{a^{(n)}})\Big] dm(z)$$

(Fubini). By unitary invariance of $dProb_n(\mathbf{a^{(n)}})$,

$$I_n(\mathbf{u^{(n)}}(\mathbf{z})) := \int_{\mathbb{C}^{n+1}} \log|<\mathbf{a^{(n)}}, \mathbf{u^{(n)}}(\mathbf{z})>_{\mathbb{C}^{n+1}}| \, dProb_n(\mathbf{a^{(n)}})$$

$$= \int_{\mathbb{C}^{n+1}} \frac{1}{\pi^{n+1}} \log|<\mathbf{a^{(n)}}, \mathbf{u^{(n)}}(\mathbf{z})>_{\mathbb{C}^{n+1}}| e^{-\sum_{j=0}^{n}|a_j|^2} dm(a_0)\cdots dm(a_n)$$

$$= \frac{1}{\pi}\int_{\mathbb{C}} \log|a_0| e^{-|a_0|^2} dm(a_0) = \mathbf{E}(\log|a_0|) \text{ (let } \mathbf{u^{(n)}}(\mathbf{z}) \to (1,0,...,0))$$

is a constant for unit vectors $\mathbf{u^{(n)}}(\mathbf{z})$, independent of n (and z). Thus the second term in $\big(\mathbf{E}(\widetilde{Z}_{p_n}), \psi\big)_{\mathbb{C}}$ is $0(1/n)$ and

$$\lim_{n\to\infty} \mathbf{E}(\widetilde{Z}_{p_n}) = \mu_{S^1}.$$

Remarks

1. Clearly "wiggle room" for improvement: more general random coefficients than normalized complex Gaussian
2. Generalizations to random polynomials $\sum_{j=0}^{n} a_j b_j(z)$
3. "Harder" probabilistic results involve analyzing

$$K_n(z) = S_n(z,z) = \sum_{j=0}^{n} |b_j(z)|^2$$

   and off-diagonal asymptotics of $S_n(z,w)$
4. Sequences vs. arrays of i.i.d. random variables

$$\sum_{j=0}^{n} a_j b_j(z) \text{ vs. } \sum_{j=0}^{n} a_j^{(n)} b_j(z).$$

5. Weighted case: $\sum_{j=0}^{n} a_j^{(n)} b_j^{(n)}(z)$

For $K \subset \mathbb{C}$ compact, we define

$$V_K(z) := \sup\{u(z) : u \in L(\mathbb{C}),\ u \leq 0 \text{ on } K\}$$

$$= \sup\{\frac{1}{deg(p)} \log|p(z)| : p \in \cup_n \mathcal{P}_n,\ ||p||_K \leq 1\}$$

where $L(\mathbb{C}) = \{u \in SH(\mathbb{C}) : u(z) - \log|z| = 0(1),\ |z| \to \infty\}$. For $K = S^1$, $V_{S^1}(z) = \log^+|z|$. If V_K is continuous, defining

$$\phi_n(z) := \sup\{|p(z)| : p \in \mathcal{P}_n,\ ||p||_K \leq 1\}, \text{ we have}$$

$$\frac{1}{n}\log\phi_n(z) \to V_K(z) \text{ locally uniformly on } \mathbb{C}.$$

Let $\mu_K := \Delta V_K$.

Let $p_{\mu_K}(z) := \int_K \log \frac{1}{|z-\zeta|} d\mu_K(\zeta)$ so $\Delta p_{\mu_K} = -\mu_K$ and

$$I(\mu_K) = \int_K p_{\mu_K}(z) d\mu_K(z) = \inf_{\mu \in \mathcal{M}(K)} I(\mu)$$

where $I(\mu) = \int_K \int_K \log \frac{1}{|z-\zeta|} d\mu(z) d\mu(\zeta)$. Then

$$V_K(z) = I(\mu_K) - p_{\mu_K}(z) \text{ so } \Delta V_K = \mu_K.$$

We can recover V_K and μ_K via L^2-methods. Note if τ is a measure on K such that

$$||p||_K \leq M_n ||p||_{L^2(\tau)} \text{ for all } p \in \mathcal{P}_n,$$

then (exercise!) the best constant is given by

$$M_n = \max_{z \in K} K_n(z)^{1/2} = \max_{z \in K} (\sum_{j=0}^{n} |b_j(z)|^2)^{1/2}$$

where $\{b_j\}_{j=0}^n$ form an orthonormal basis for $\mathcal{P}_n$ in $L^2(\tau)$.

Relate K_n, ϕ_n: $\frac{1}{n+1} \leq \frac{K_n(z)}{\phi_n(z)^2} \leq M_n^2(n+1)$

The right-hand inequality is from $||p||_K \leq M_n ||p||_{L^2(\tau)}$; the left-hand inequality uses the reproducing property of $S_n(z,w)$. If (K,τ) is (BM) i.e., $M_n^{1/n} \to 1$, this shows

$$\frac{1}{2n} \log K_n(z) \asymp \frac{1}{n} \log \phi_n(z) \asymp V_K(z).$$

Indeed:

If V_K is continuous, then (BM) for (K,τ) is *equivalent* to

$$\lim_{n\to\infty} \frac{1}{2n} \log K_n(z) = V_K(z) \text{ locally uniformly on } \mathbb{C}.$$

Hence

$$\Delta \frac{1}{2n} \log K_n(z) \to \mu_K.$$

Summary

Thus, what we have *really* proved is the following:

Theorem

Let τ be a (BM) measure on a compact set K with V_K continuous. Consider random polynomials of the form $p_n(z) = \sum_{j=0}^{n} a_j b_j(z)$ where $\{b_j(z)\}_{j=0,\ldots,n}$ form an orthonormal basis for $\mathcal{P}_n$ in $L^2(\tau)$ and $a_0, \ldots, a_n$ are i.i.d. complex Gaussian random variables with $\mathbf{E}(a_j) = \mathbf{E}(a_j a_k) = 0$ and $\mathbf{E}(a_j \bar{a}_k) = \delta_{jk}$. Then

$$\lim_{n\to\infty} \mathbf{E}(\widetilde{Z}_{p_n}) = \mu_K.$$

Note *any* (BM) measure yields the same limit measure μ_K (this is a type of "universality"). "Same" result in weighted case ($b_j^{(n)}$ change with n); limit $\mu_{K,Q}$. Conclusion: limit depends on *basis*.

Further questions on random polynomials

The method above was used (and generalized) by Bloom, Shiffman, Zelditch (and others).

We briefly address the following questions:

1. What can we say about generic convergence of the (random) sequence of subharmonic functions $\{\frac{1}{n}\log|p_n|\}$?
2. Can we allow more general coefficients than i.i.d. complex Gaussian?

We write $\mathcal{P}$ for the space of sequences of random polynomials; note if we consider random polynomials $p_n \in \mathcal{P}_n$ as

$$p_n(z) = \sum_{j=0}^{n} a_j^{(n)} b_j(z), \;\; a_j^{(n)} \text{ i.i.d}$$

then

$$\mathcal{P} := \otimes_{n=1}^{\infty}(\mathcal{P}_n, Prob_n) = \otimes_{n=1}^{\infty}(\mathbb{C}^{n+1}, Prob_n).$$

Also (relevant for weighted case) can have $b_j^{(n)}(z)$.

General coefficients:

The following is due to Ibragimov/Zaporozhets (2013):

Theorem

For random Kac polynomials of the form $p_n(z) = \sum_{j=0}^{n} a_j z^j$ with a_j i.i.d., $\mathbf{E}(\log(1 + |a_j|)) < \infty$ is a necessary and sufficient condition for

$$\widetilde{Z}_{p_n} = \Delta(\frac{1}{n} \log |p_n|) \to \frac{1}{2\pi} d\theta \text{ amost surely in } \mathcal{P}.$$

Kabluchko/Zaporozhets (2014) considered p. s. of random analytic functions of the form $G_n(z) = \sum_{j=0}^{n} a_j f_{n,j} z^j$ with deterministic coefficients $\{f_{n,j}\}$ satisfying certain hypotheses to get conv. in prob. to a target measure. We discuss recent generalizations by Tom BLOOM and Duncan DAUVERGNE (2018).

Let a_j be i.i.d. complex random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For $\epsilon > 0$, $n \in \mathbb{Z}^+$, let

$$\Omega_{n,\epsilon} := \{\omega \in \Omega : |a_j(\omega)| \le e^{\epsilon n},\ j = 0, ..., n\}.$$

$$\mathbf{E}(\log(1 + |a_j|)) < \infty \iff \forall \epsilon,\ \sum_{n=0}^{\infty} \mathbf{P}(\Omega_{n,\epsilon}^c) < \infty.$$

$$\mathbf{P}(|a_j| > e^{|z|}) = o(1/|z|) \Rightarrow \forall \epsilon,\ \lim_{n\to\infty} \mathbf{P}(\Omega_{n,\epsilon}^c) = 0.$$

When does $\widetilde{Z}_{p_n} \to \mu_K$ a.s.? In probability? This latter means for any open set U in the space of prob. measures on $\mathbb{C}$ with $\mu_K \in U$, we have $\mathbf{P}(\widetilde{Z}_{p_n} \in U) \to 0$ as $n \to \infty$.

Bloom-Dauvergne conv. in prob. result

Let τ be a (BM) measure on a compact set K with V_K ctn. Consider random polynomials of the form $p_n(z) = \sum_{j=0}^n a_j b_j(z)$ where $\{b_j\}_{j=0,\ldots,n}$ form an orthonormal basis for $\mathcal{P}_n$ in $L^2(\tau)$.

Theorem

For random polynomials of the form $p_n(z) = \sum_{j=0}^n a_j b_j(z)$, if $\mathbf{P}(|a_j| > e^{|z|}) = o(1/|z|)$ then

$$\widetilde{Z}_{p_n} = \Delta(\frac{1}{n}\log|p_n|) \to \mu_K \text{ in probability.}$$

Moreover, for Kac polynomials $\sum_{j=0}^n a_j z^j$, the condition $\mathbf{P}(|a_j| > e^{|z|}) = o(1/|z|)$ is necessary and sufficient for $\widetilde{Z}_{p_n} \to \mu_{S^1} = \frac{1}{2\pi}d\theta$ in probability.

Bloom-Dauvergne a.s. result

Let $\{f_{n,j}\}$ be deterministic coefficients satisfying certain hypotheses and

$$V(z) := \lim_{n\to\infty} \frac{1}{n} \log\Big(\sum_{j=0}^{n} |f_{n,j}||z|^j\Big) \text{ loc. unif.}$$

Theorem

For random polynomials of the form $p_n(z) = \sum_{j=0}^{n} a_j f_{n,j} z^j$, if $\mathbf{E}(\log(1+|a_j|)) < \infty$ then a.s.

$$\widetilde{Z}_{p_n} = \Delta(\frac{1}{n} \log |p_n|) \to \Delta V.$$

Note $f_{n,j} \equiv 1,\ \forall j, n$ give Kac poly.'s (and $V(z) = \log^+ |z|$).

Sufficiency for $\widetilde{Z}_{p_n} \to \mu_K$ a.s., in probability

Sufficiency for $\widetilde{Z}_{p_n} \to \mu_K$ a.s.:

1. a.s. $\{|p_n|\}$ (or $\{\log|p_n|\}$) locally bounded above
2. a.s., $\limsup_{n\to\infty} \frac{1}{n} \log|p_n(z)| \leq V_K(z)$, all z
3. for each z_j in a countable dense set $\{z_j\}$,
   $\lim_{n\to\infty} \frac{1}{n} \log|p_n(z_j)| = V_K(z_j)$ a.s.

Sufficiency for $\widetilde{Z}_{p_n} \to \mu_K$ in probability:

1. For any subsequence $Y \subset \mathbb{Z}^+$ there is a further subsequence Y_0 such that, a.s., $\{|p_n|\}_{n\in Y_0}$ is locally bounded above and $\limsup_{n\in Y_0} \frac{1}{n} \log|p_n(z)| \leq V_K(z)$, all z
2. for each z_j in a countable dense set $\{z_j\}$,
   $\lim_{n\to\infty} \frac{1}{n} \log|p_n(z_j)| = V_K(z_j)$ in probability

Condition $\mathbf{E}(\log(1+|a_j|)) < \infty$ gives UPPER BOUND on full sequence (for a.s.) while Condition $\mathbf{P}(|a_j| > e^{|z|}) = o(1/|z|)$ gives UPPER BOUND on subsequence (for conv. in prob.)

Lower bound on $\{\frac{1}{n} log|p_n|\}$

Need lower bound to show $\lim_{n\to\infty} \frac{1}{n}\log|p_n(z_j)| = V_K(z_j)$ on countable dense set a.s. or in probability. This is the hard part; we just make a remark.

1. For conv. in prob.: Use Kolmogorov-Rogozin inequality on concentration function of sum $\mathbf{X}_1 + \cdots \mathbf{X}_n$ of random variables to get conv. in prob. of $\frac{1}{n}\log|p_n| \to V_K$ at all but a countable set of points. Here, for $\mathbf{X}$ r.v.,

$$\mathcal{Q}(\mathbf{X}; r) := \sup\{z \in \mathbb{C} : \mathbf{P}(\mathbf{X} \in B(z, r))\}$$

   is concentration fcn. of $\mathbf{X}$. (Idea to use Kolmogorov-Rogozin inequality due to Ibragimov/Zaporozhets).
2. For a.s. result: Use version of "small ball probability" result of Nguyen-Vu for complex-valued random variables.

Remark on modes of convergence and on to $\mathbb{C}^2$

Let τ be a (BM) measure on $K \subset \mathbb{C}$ with V_K ctn. Consider random polynomials of the form $p_n(z) = \sum_{j=0}^{n} a_j^{(n)} b_j^{(n)}(z)$ where $\{b_j^{(n)}(z)\}_{j=0,\ldots,n}$ form o.n. basis for $\mathcal{P}_n$ in $L^2(\tau)$. Let $\{a_j^{(n)}\}$ i.i.d. such that (e.g., std. complex Gaussian) a.s. in $\mathcal{P}$

$$\left(\limsup_{n\to\infty} \frac{1}{n} \log |p_n(z)|\right)^* = V_K(z)$$

pointwise for all $z \in \mathbb{C}$ ($u^*(z) := \limsup_{\zeta\to z} u(\zeta)$).Then

1. $\frac{1}{n} \log |p_n| \to V_K$ in $L^1_{loc}(\mathbb{C})$ a.s. $\mathcal{P}$; hence
2. $\widetilde{Z}_{p_n} = \Delta(\frac{1}{n} \log |p_n|) \to \mu_K = \Delta V_K$ a.s. $\mathcal{P}$ (Δ linear operator).

Onto $\mathbb{C}^2$

Let's work in $\mathbb{C}^2$ with variables $z = (z_1, z_2)$. For a polynomial

$$p(z) = \sum_{j+k=0}^{n} a_{jk} z_1^j z_2^k \in \mathcal{P}_n,$$

the zero set $Z_p = \{z \in \mathbb{C}^2 : p(z) = 0\}$ is a one-dimensional (complex) analytic (algebraic) variety – unbounded.
Given *two* polynomials $p_1(z)$ and $p_2(z)$ in $\mathcal{P}_n$, consider

1. the polynomial mapping $\mathbf{F}(z) := (p_1(z), p_2(z)) : \mathbb{C}^2 \to \mathbb{C}^2$ and
2. the common zeros of p_1 and p_2:

$$Z_{\mathbf{F}} := \{z \in \mathbb{C}^2 : p_1(z) = p_2(z) = 0\}.$$

By Bertini/Bezout, generically $Z_{\mathbf{F}}$ consists of n^2 points.
Example: If $p_1(z) = z_1^n - 1$ and $p_2(z) = z_2^n - 1$, then

$$Z_{\mathbf{F}} = \{(e^{2\pi ij/n}, e^{2\pi ik/n}) : \; j, k = 0, ..., n-1\}.$$

We study (normalized versions of) Z_p *and/or* $Z_{\mathbf{F}}$*. Consider*

$$\frac{1}{n}\log|p| \ \textit{and/or}\ \frac{1}{n}\log||\mathbf{F}||$$

where $||\mathbf{F}||^2 = |p_1|^2 + |p_2|^2$*.* For u a real or complex-valued function on a domain D in $\mathbb{C}^2$, we write the 1−form

$$du = \sum_{j=1}^{2} \frac{\partial u}{\partial z_j} dz_j + \sum_{j=1}^{2} \frac{\partial u}{\partial \overline{z}_j} d\overline{z}_j =: \partial u + \overline{\partial} u$$

as the sum of a form ∂u of *bidegree* $(1,0)$ and a form $\overline{\partial} u$ of bidegree $(0,1)$ where

$$\frac{\partial u}{\partial z_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}\right);\ \frac{\partial u}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j}\right);$$

and we have

$$dz_j = dx_j + idy_j;\ d\overline{z}_j = dx_j - idy_j.$$

For a complex-valued $f \in C^1(D)$, we say f is holomorphic in D if $\overline{\partial} f = 0$ in D ($\iff$ f is separately holomorphic in z_1 and z_2).

We also define

$$d^c u := i(\overline{\partial} u - \partial u).$$

Note that if $u \in C^2(D)$, the linear operator

$$dd^c u = 2i\partial\overline{\partial}u = 2i \sum_{j,k=1}^{2} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k$$

($(1,1)-$form) so that the coefficients of the $2-$form $dd^c u$ give the entries of the 2×2 *complex Hessian matrix*

$$H(u) := [\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}]_{j,k=1}^{2},$$

of u. Elementary linear algebra shows that the nonlinear operator

$$(dd^c u)^2 := dd^c u \wedge dd^c u = c_2 \det H(u) dV$$

where $dV = (\frac{1}{2i})^2 dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2$ is the volume form on $\mathbb{C}^2$ and c_2 is a dimensional constant.

A function $u : D \to [-\infty, +\infty)$ defined on a domain $D \subset \mathbb{C}^2$ is *plurisubharmonic* (psh) in D if

1. u is uppersemicontinuous on D and
2. $u|_{D\cap l}$ is subharmonic (shm) on components of $D \cap l$ for each complex line (one-dimensional (complex) affine space) l.

For $u \in C^2(D)$, u is psh in D if and only if $H(u) = [\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}]_{j,k=1}^2$ is positive semi-definite; thus $(dd^c u)^2$ is a positive measure. If f is holomorphic in D, $u = \log|f|$ is psh in D. In particular, $\log|p|$ is psh in $\mathbb{C}^2$ for any polynomial p. For $p_n \in \mathcal{P}_n$,

$$\widetilde{Z}_{p_n} := dd^c(\frac{1}{n}\log|p_n|) \text{ (can't take } dd^c(\cdot)^2\text{!!)}$$

is the *normalized zero current* of p_n ($(1,1)-$form with dist. coeff.). Example: If $p_n(z) = z_1^n$, then $\widetilde{Z}_{p_n}$ is the *current of integration* on the variety $\{z \in \mathbb{C}^2 : z_1 = 0\}$. Note this is *unbounded*.

For a polynomial mapping $\mathbf{F}(z) := (p_1(z), p_2(z)) : \mathbb{C}^2 \to \mathbb{C}^2$ with $p_1, p_2 \in \mathcal{P}_n$, the zero set

$$Z_{\mathbf{F}} := \{z \in \mathbb{C}^2 : p_1(z) = p_2(z) = 0\}$$

generically consists of n^2 distinct points and "generically" one can define the *normalized zero current for* **F** as

$$\widetilde{Z}_{\mathbf{F}} := dd^c(\frac{1}{n}\log|p_1|) \wedge dd^c(\frac{1}{n}\log|p_2|)$$

$$= (dd^c\frac{1}{n}\log||\mathbf{F}_n||)^2 = \left(dd^c\frac{1}{2n}\log[|p_1|^2 + |p_2|^2]\right)^2 .$$

Example: If $p_1(z) = z_1^n - 1$ and $p_2(z) = z_2^n - 1$, then

$$\widetilde{Z}_{\mathbf{F}} = \frac{1}{n^2}\sum_{j,k=0}^{n-1} \delta_{(e^{2\pi ij/n}, e^{2\pi ik/n})}.$$

Follows from $(dd^c[\frac{1}{2}\log(|z_1|^2 + |z_2|^2)])^2 = \delta_{(0,0)}$.

Generalization of V_K

The definition of V_K and BM measure are the "same" as in $\mathbb{C}$, e.g., for $K \subset \mathbb{C}^2$ *nonpluripolar*,

$$V_K(z) := \sup\{u(z) : u \in L(\mathbb{C}^2),\ u \le 0 \text{ on } K\}$$

$$= \sup\{\frac{1}{deg(p)} \log|p(z)| : p \in \cup_n \mathcal{P}_n,\ ||p||_K \le 1\}$$

where $L(\mathbb{C}^2) = \{u \in PSH(\mathbb{C}^2) : u(z) - \log|z| = 0(1),\ |z| \to \infty\}$. Let τ be a BM measure on K; let $\{b_{jk}^{(n)}\}$ be an orthonormal basis for $L^2(\tau)$ and consider random polynomials

$$p(z) = \sum_{j+k=0}^{n} a_{jk}^{(n)} b_{jk}^{(n)}(z) \in \mathcal{P}_n$$

where $a_{jk}^{(n)}$ are i.i.d. complex random variables. Let $m_n = \dim \mathcal{P}_n = \binom{n+2}{2}$ and

$$\mathcal{P} := \otimes_{n=1}^{\infty}(\mathbb{C}^{m_n}, Prob_{m_n}),\ \mathcal{F} := \otimes_{n=1}^{\infty}((\mathbb{C}^{m_n})^2, (Prob_{m_n})^2).$$

Almost sure convergence

Theorem

For $a_{jk}^{(n)}$ i.i.d. complex random variables with "tail hyp." consider sequences of random polynomials $\{p_n\} \in \mathcal{P}$ and sequences of random polynomial mappings $\mathbf{F}_n = (p_n^{(1)}, p_n^{(2)}) \in \mathcal{F}$. Then a.s. we have both (i.e., in $\mathcal{P}$ or in $\mathcal{F}$)

$$\lim_{n\to\infty} \frac{1}{n} \log |p_n| = V_K \textit{ ptwse. \& in } L^1_{loc}(\mathbb{C}^2) \textit{ and}$$

$$\lim_{n\to\infty} \frac{1}{n} \log ||\mathbf{F}_n|| = V_K \textit{ ptwse. \& in } L^1_{loc}(\mathbb{C}^2) \textit{ hence}$$

$$\lim_{n\to\infty} dd^c\left(\frac{1}{n} \log ||\mathbf{F}_n||\right)$$

$$= \lim_{n\to\infty} dd^c\left(\frac{1}{n} \log |p_n|\right) = dd^c V_K$$

as positive currents (recall dd^c is a linear operator).

General coeff.: Bloom-Dauvergne in $\mathbb{C}^2$

Theorem

Let $K \subset \mathbb{C}^2$ with V_K continuous. For sequences of random polynomials $\{p_n = \sum_{j+k=0}^{n} a_{jk} b_{jk}(z)\}$ where a_{jk} i.i.d. with $\mathbf{P}(|a_{jk}| > e^{|z|}) = o(1/|z|^2)$, $\{b_{jk}\}$ o.n. for $L^2(\tau)$ (τ BM),

$$\frac{1}{n} \log |p_n| \to V_K \text{ in prob. in } L^1_{loc}(\mathbb{C}^2) \text{ and}$$

$$dd^c\left(\frac{1}{n} \log |p_n|\right) \to dd^c V_K \text{ in prob..}$$

They also prove a result on a.s. convergence for the 2-d Kac ensemble (here $b_{jk}(z) = z_1^j z_2^k$) under the hypothesis

$$\mathbf{E}(\log(1 + |a_{jk}|))^2 < \infty.$$

$dd^c V_K$ vs. $\mu_K := (dd^c V_K)^2$

For K not pluripolar, $dd^c V_K$ generically has unbounded support;

$$\mu_K := (dd^c V_K)^2$$

is the $\mathbb{C}^2-$analogue of the equilibrium measure and is supported in K. We have an asymptotic expectation result with tail hyp. on $a_{jk}^{(n)}$ using the "probabilistic Poincare-Lelong formula":

$$\mathbf{E}(\widetilde{Z}_{\mathbf{F}_n}) := \mathbf{E}(\frac{1}{n} dd^c \log |p_n^{(1)}| \wedge \frac{1}{n} dd^c \log |p_n^{(2)}|)$$

$$= \mathbf{E}(\frac{1}{n} dd^c \log |p_n^{(1)}|) \wedge \mathbf{E}(\frac{1}{n} dd^c \log |p_n^{(2)}|);$$

i.e., when $n \to \infty$, $\mathbf{F}_n = (p_n^{(1)}, p_n^{(2)})$,

$$\mathbf{E}(\widetilde{Z}_{\mathbf{F}_n}) = \mathbf{E}(\widetilde{Z}_{p_n^{(1)}}) \wedge \mathbf{E}(\widetilde{Z}_{p_n^{(2)}}) \to \left(dd^c V_K(z)\right)^2.$$

The fact that $\widetilde{Z}_{\mathbf{F}_n} \to (dd^c V_K)^2$ as positive measures a.s. in $\mathcal{F}$ is a deeper result of T. Bayraktar (IUMJ, 2016).

Random polynomial mappings in $\mathbb{C}^2$: Modification

For $K \subset \mathbb{C}^2$ compact, we know

$$V_K(z_1, z_2) = \sup\{\frac{1}{deg(p)} \log|p(z_1, z_2)| : ||p||_K \leq 1\}$$

$$= \sup\{\frac{1}{2deg(P)} \log[|p_1(z_1, z_2)|^2 + |p_2(z_1, z_2)|^2] : ||p_i||_K \leq 1, \ i = 1, 2\}$$

where $deg(p_1) = deg(p_2) =: deg(P)$ $(P := (p_1, p_2))$.

Definition

For $K_1, K_2 \subset \mathbb{C}^2$ compact with V_{K_1}, V_{K_2} ctn.,

$$U_{K_1,K_2}(z_1, z_2) :=$$

$$\sup\{\frac{1}{2deg(P)} \log[|p_1(z_1, z_2)|^2 + |p_2(z_1, z_2)|^2] : ||p_i||_{K_i} \leq 1\}.$$

We have $U_{K_1,K_2} = \max[V_{K_1}, V_{K_2}]$ in all of $\mathbb{C}^2$.

Let $\{p_\nu^{(n)}\}_{|\nu|\leq n}$ be an o.n. basis of $\mathcal{P}_n$ in $L^2(\mu_1)$ where μ_1 is a BM measure on K_1 and let $\{q_\nu^{(n)}\}_{|\nu|\leq n}$ be an o.n. basis of $\mathcal{P}_n$ in $L^2(\mu_2)$ where μ_2 is a BM measure on K_2. Consider random polynomial mappings of degree at most n of the form

$$\mathbf{H_n}(z) := (H_n^{(1)}(z), H_n^{(2)}(z)) \text{ where}$$

$$H_n^{(1)}(z) = \sum_{|\nu|\leq n} a_\nu^{(n)} p_\nu^{(n)}(z),\ \ H_n^{(2)}(z) = \sum_{|\nu|\leq n} b_\nu^{(n)} q_\nu^{(n)}(z)$$

and $a_\nu^{(n)}, b_\nu^{(n)}$ are i.i.d. complex random variables with a distribution satisfying mild tail probability requirements. Identify this more general $\mathcal{F}$ with $\otimes_{n=1}^{\infty}((\mathbb{C}^{m_n})^2, (Prob_{m_n})^2)$.

Theorem

Almost surely in $\mathcal{F}$ we have

$$\big(\limsup_{n\to\infty} \frac{1}{2n} \log[|H_n^{(1)}(z)|^2 + |H_n^{(2)}(z)|^2\big)^*$$

$$= \max[V_{K_1}(z), V_{K_2}(z)]$$

pointwise for all $(z) \in \mathbb{C}^2$ and a.s.

$$\frac{1}{2n} \log[|H_n^{(1)}(z)|^2 + |H_n^{(2)}(z)|]^2 \to \max[V_{K_1}(z), V_{K_2}(z)]$$

in $L^1_{loc}(\mathbb{C}^2)$. Hence (ddᶜ linear operator) a.s.

$$dd^c\big(\frac{1}{2n} \log[|H_n^{(1)}(z)|^2 + |H_n^{(2)}(z)|]^2\big)$$

$$\to dd^c\big(\max[V_{K_1}(z), V_{K_2}(z)]\big)$$

as positive currents (same result in weighted case).

However, from Bayraktar's results, we have a.s. in $\mathcal{F}$

$$(dd^c \frac{1}{2n} \log[|H_n^{(1)}|^2 + |H_n^{(2)}|^2])^2 \to dd^c V_{K_1} \wedge dd^c V_{K_2}. \qquad (1)$$

Indeed, it is relatively straightforward to deduce

$$\mathbf{E}\big((dd^c \frac{1}{2n} \log[|H_n^{(1)}|^2 + |H_n^{(2)}|^2])^2\big) \to dd^c V_{K_1} \wedge dd^c V_{K_2}$$

$$\text{from } \mathbf{E}(dd^c \frac{1}{n} \log|H_n^{(j)}|) \to dd^c V_{K_j},\ j = 1, 2$$

and the probabilistic Poincaré-Lelong formula. The previous theorem "suggests" this limit might instead be

$$(dd^c \max[V_{K_1}, V_{K_2}])^2.$$

1. $L^1_{loc}(\mathbb{C}^2)$ convergence is not sufficient to conclude Monge-Ampère convergence!
2. No Monge-Ampère convergence theorems for non-locally bounded fcn's.

These currents (here, pos. measures) are generally much different:

$$(dd^c \max[u,v])^2 = dd^c \max[u,v] \wedge dd^c(u+v) - dd^c u \wedge dd^c v.$$

In general, both $\mathrm{supp}(dd^c u \wedge dd^c v)$ and $\mathrm{supp}(dd^c \max[u,v])^2$ are unbounded – and difficult to compute.

Thus: Once $K_1 \neq K_2$, positive probability some "zeros" go to infinity!

Remark. $K \to K_1, K_2$ changes o.n. basis, i.e., different for $H_n^{(1)}$ and $H_n^{(2)}$.

Hard to calculate $dd^c V_{K_1} \wedge dd^c V_{K_2}$.

Example: Two balls

For $u(z_1, z_2) := \frac{1}{2}\log^+(|z_1|^2 + |z_2|^2)$ and $v(z_1, z_2) := \frac{1}{2}\log^+(|z_1 - a|^2 + |z_2|^2)$ in $\mathbb{C}^2$, two extremal functions for unit balls about $(0, 0)$ and $(a, 0)$, outside of the union of these balls the density of $dd^c u \wedge dd^c v$ is (modulo a constant)

$$\frac{|a|^2|z_2|^2}{(|z_1|^2 + |z_2|^2)^2(|z_1 - a|^2 + |z_2|^2)^2}$$

while $dd^c u \wedge dd^c v = 0$ on the interior of the union. In particular:

1. this density is positive everywhere outside of the union of the balls (off $z_2 = 0$);
2. this density goes to 0 everywhere outside of the union of the balls as $a \to 0$; and
3. the integral of this density outside of the union of the balls goes to 0 as $a \to 0$ (because of 2. and the fact this "total mass" is uniformly bounded (by one, say) for all a).

Another modification: $P-$extremal functions

Given a convex body $P \subset (\mathbb{R}^+)^2$, for $n = 1, 2, \ldots$ define

$$Poly(nP) := \{ \sum_{J \in nP \cap (\mathbb{Z}^+)^2} c_J z^J = \sum_{(j_1, j_2) \in nP \cap (\mathbb{Z}^+)^2} c_{j_1 j_2} z_1^{j_1} z_2^{j_2} : c_J \in \mathbb{C} \}.$$

Example: $P_q := \{(x_1, x_2) \in (\mathbb{R}^+)^2 : (x_1^q + x_2^q)^{1/q} \leq 1\}$.

For $K \subset \mathbb{C}^2$ compact, define the $P-$extremal function

$$V_{P,K}(z) = \sup\{u(z) : u \in L_P(\mathbb{C}^2),\ u \leq 0 \text{ on } K\}$$

$$= \lim_{n \to \infty} [\sup\{\frac{1}{n} \log |p_n(z)| : p_n \in Poly(nP),\ ||p_n||_K \leq 1\}]$$

where $L_P(\mathbb{C}^2) = \{u \in PSH(\mathbb{C}) : u(z) - H_P(z) = 0(1),\ |z| \to \infty\}$,

$$H_P(z) := \sup_{J \in P} \log |z^J| := \sup_{J \in P} \log[|z_1|^{j_1} |z_2|^{j_2}]$$

(*logarithmic indicator function*). For $K = T$, the unit torus,

$$V_{P,T}(z) = H_P(z) = \max_{J \in P} \log |z^J|.$$

Random $Poly(nP)$ polynomials in $\mathbb{C}^2$

Let μ be a BM measure for $K \subset \mathbb{C}^2$, $\{p_\alpha\}$ an o.n. basis for $Poly(nP)$ in $L^2(\mu)$. Consider random $Poly(nP)$ polynomials $P_n(z) = \sum_{\alpha \in nP} a_\alpha^{(n)} p_\alpha(z)$ (where $a_\alpha^{(n)}$ are i.i.d. complex-valued random variables) and random polynomial mappings $\mathbf{F_n}(z) = (P_n(z), Q_n(z))$. We get a probability measure $Prob_n$ on $\mathcal{F}_n$, the random polynomial mappings with $P_n, Q_n \in Poly(nP)$. Identify $\mathcal{F}_n$ with $\mathbb{C}^{d_n} \times \mathbb{C}^{d_n}$ where $d_n = \dim Poly(nP)$. Given $\mathbf{F_n} \in \mathcal{F}_n$, let

$$\widetilde{Z}_{\mathbf{F_n}} := (dd^c \frac{1}{n} \log ||\mathbf{F_n}||)^2 = (dd^c[\frac{1}{2n} \log(|P_n|^2 + |Q_n|^2)])^2.$$

For generic $\mathbf{F_n}$, $\widetilde{Z}_{\mathbf{F_n}}$ is, up to a constant, the normalized zero measure on the (finite) zero set $\{P_n = Q_n = 0\}$.

Bayraktar (MMJ, 2017), to explain S-Z 2004, proved that

$$\lim_{n\to\infty} \mathbf{E}(\widetilde{Z}_{\mathbf{F_n}}) = (dd^c V_{P,K})^2.$$

as measures. Forming the product probability space of sequences of random polynomial mappings

$$\mathcal{P} := \otimes_{n=1}^{\infty}(\mathcal{F}_n, Prob_n) = \otimes_{n=1}^{\infty}(\mathbb{C}^{d_n} \times \mathbb{C}^{d_n}, Prob_n),$$

almost surely (a.s.) in $\mathcal{P}$ (mild tail hyp.) we have

$$\frac{1}{n}\log||\mathbf{F_n}|| = \frac{1}{2n}\log(|P_n|^2 + |Q_n|^2) \to V_{P,K}(z)$$

pointwise in $\mathbb{C}^2$ and in $L^1_{loc}(\mathbb{C}^2)$. Moreover, a.s. in $\mathcal{P}$ we have

$$(dd^c\frac{1}{n}\log||\mathbf{F_n}||)^2 = (dd^c[\frac{1}{2n}\log(|P_n|^2 + |Q_n|^2)])^2 \to (dd^c V_{P,K})^2.$$

as measures. Call $\mu_{P,K} := (dd^c V_{P,K})^2$.

Example: The torus T and P_q

Let $T = S^1 \times S^1 = \{(z_1, z_2) : |z_1| = |z_2| = 1\}$. We know that

$$V_{P,T}(z_1, z_2) = H_P(\log^+ |z_1|, \log^+ |z_2|).$$

Let P_q be the portion of the l_q−ball in $(\mathbb{R}^+)^2$ (so $Poly(nP_q)$ spaces vary with q). For any $1 \leq q \leq \infty$, we have

$$V_{P_q,T}(z_1, z_2) = [(\log^+ |z_1|)^{q'} + (\log^+ |z_2|)^{q'}]^{1/q'},$$

$1/q + 1/q' = 1$. By invariance under $(z_1, z_2) \to (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$, $\mu_{P_q,T}$ is a multiple of Haar measure on T: $\mu_{P_q,T}(T) = 2Vol(P_q)$.

Corollary

With $K = T$, for $P = P_q$, $\mathbf{E}(\widetilde{Z}_{\mathbf{F_n}}) \to \mu_{P_q,T}$ with analogous statements for the a.s. results (normalized monomial basis).

Thus only total mass of target measure changes.

Example: $B_2 := \{(z_1, z_2) : |z_1|^2 + |z_2|^2 \leq 1\}$ and P_q

Here, $V_{P_1,B_2} = V_{B_2} = \frac{1}{2}\log^+(|z_1|^2 + |z_2|^2)$ and μ_{P_1,B_2} is normalized surface area measure on ∂B_2. On the other hand:

Theorem

For V_{P_∞,B_2}, we have μ_{P_∞,B_2} is a multiple of Haar measure on the torus $\{|z_1| = |z_2| = 1/\sqrt{2}\}$.

Corollary

With $K = B_2$, for

1. $P = P_1 = \Sigma$, $\mathbf{E}(\widetilde{Z}_{\mathbf{F_n}}) \to \mu_{P_1,B_2}$, *normalized surface area measure on ∂B_2; while for*
2. $P = P_\infty$, $\mathbf{E}(\widetilde{Z}_{\mathbf{F_n}}) \to \mu_{P_\infty,B_2}$, *a multiple of Haar measure on the torus $\{|z_1| = |z_2| = 1/\sqrt{2}\}$*

with analogous statements for the a.s. results.

Question: As q varies from $q = 1$ to $q = \infty$, μ_{P_q,B_2} varies from normalized surface area measure on ∂B_2 (3-d support) to a multiple of Haar measure on the torus $\{|z_1| = |z_2| = 1/\sqrt{2}\}$ (2-d support). Thus there *must* be a "discontinuity" of

$$S_q := \text{supp}(\mu_{P_q,B_2})$$

for some q. Does this happen at $q = \infty$ or does S_q shrink gradually from $q = 1$ to $q = \infty$?

Remark. $K, P \to K, P'$ modifies $Poly(nP) \to Poly(nP')$; e.g., $Poly(nP_q)$ spaces vary with q. Here $\text{supp}(\mu_{P,K}), \text{supp}(\mu_{P',K})$ stay in K. Similar for weighted extremal fcn. if modify weight.

Problem 1: Compute more examples of $dd^c V_{K_1} \wedge dd^c V_{K_2}$.

Problem 2: Compute more examples of $\mu_{P,K} := (dd^c V_{P,K})^2$.