Spectral gap in bipartite biregular graphs and applications

Ioana Dumitriu

Department of Mathematics
University of Washington (Seattle)

Joint work with Gerandy Brito and Kameron Harris

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1. Intro: expanders and bipartite graphs
2. Random Bipartite Biregular Graphs are almost Ramanujan
3. Applications

Definitions

- A simple bipartite graph consists of a set of vertices partitioned into two classes, and a set of edges which occur solely between the classes.
- Sometimes denoted as $G = (X, Y, E)$, where X, Y are vertex classes and E is the set of edges.
- Notation: $|X| = m$, $|Y| = n$.

Definitions

- A biregular bipartite graph has the property that all vertices in the same class have the same degree
- Notation: $|X| = m$, $|Y| = n$, d_1 for the common degree of class X, d_2 for the degree of class Y.
- Note that $md_1 = nd_2$.

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- A number of important and interesting classes of graphs are bipartite and some are biregular (trees, even cycles, median graphs, hypercubes).
- Applications include projective geometry (Levi graphs), coding theory (yielding factor codes and Tanner codes, more on that later), computer science (Petri nets, assignment problems, community detection), signal processing (matrix completion).

Adjacency matrix

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$$A = \left[\begin{array}{cc} 0 & X \\ X^T & 0 \end{array} \right] .$$

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- As a consequence, their spectrum is symmetric.

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- Of particular interest in CS and coding theory (from mixing to design of error-correcting codes)
- Random regular graphs (uniformly distributed) are classical (and best-known) examples of such expanders; expanding properties characterized by the *spectral gap* of the adjacency matrix.
- Uniform distribution important in making assertions like “almost all regular graphs”

Random regular graphs

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- $\lambda_1 = d$ (trivial)
- Quantity of interest is the *second largest eigenvalue*, defined as $\eta = \max\{|\lambda_2|, |\lambda_n|\}$.

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- Work on lower bounding η also by Alon-Boppana ('86), upper bounding η by Friedman ('03). Uniformly random regular graphs are *almost Ramanujan*, i.e.,

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  a.a.s. as $n \to \infty$.
- Recently, Bordenave ('15) tightened Friedman's proof to $\eta = 2\sqrt{d-1} + o(1)$.

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- Again, uniform distribution important.

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- Matching upper bound: work by Brito, D., Harris (2018).

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- Instead, use the configuration model (Bender, Canfield '78, Bollobas '80)
  - "asymptotically uniform" (contiguous to the uniform one), anything happening a.a.s. in configuration model happens a.a.s. in the uniform one

Main Result

- Let $G(d_1, d_2, m, n)$ be a random bipartite graph generated with the configuration model.
- Largest modulus eigenvalues are $\pm\lambda = \pm\sqrt{(d_1 - 1)(d_2 - 1)}$. What is the third largest?

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- Proof follows in the footsteps of Bordenave ('15)

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  - Not here. Sadly, $\tilde{A}^m$ is too hard to work with (too much "chaff")

Non-backtracking matrix

- Idea: Examine instead the "non-backtracking" matrix B, whose rows/columns indexed by *edges*, and $B_{ef} = 1$ iff $e = (v_1, v_2)$, $f = (v_2, v_3)$ with $v_1 \neq v_3$. Non-symmetric!

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- Can relate the eigenvalues of B to those of the adjacency matrix A via the Ihara-Bass formula

$$\det(B - \lambda I) = (\lambda^2 - 1)^{|E|-n} \det(D - \lambda A + \lambda^2 I) ,$$

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  with $|E|$ = number of edges, D the diagonal matrix of degrees.
- Spectral gap for B may yield spectral gap for A (works here).

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- The rest is (roughly) sophisticated circuit-counting.

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- This, together with non-backtracking feature, helps with circuit-counting.

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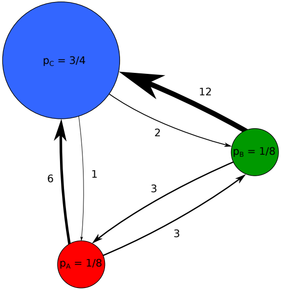
- Same here about multiple repetitions, but no exact cancellation for edges appearing only once; finer estimates needed due to lack of independence. Still, doable.

□

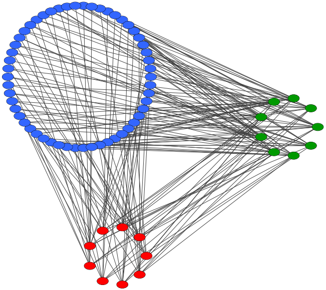
Applications for RBBG: community detection

– frame graphs: given a small, edge-weighted graph, use it to define community structure in a larger, random graph. Each graph is represented by a vertex, the weights in the frame define the number of edges between classes. Quasi-regular.

A Frame



B Random regular frame graph



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- Using a very general theorem of Meila '15 (under certain conditions, the highest eigenvalues of the random graphs are those of the frame), we concluded that **community detection is possible in such graphs** (removing assumptions).

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- Using a very general theorem of Meila '15 (under certain conditions, the highest eigenvalues of the random graphs are those of the frame), we concluded that **community detection is possible in such graphs** (removing assumptions).
- Conditions not optimal, but a starting point for further study.

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- Expander codes (Tanner codes) introduced in Tanner, ’62
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- Linear error-correcting codes whose parity-check matrix encoded in an expander graph
- Using Tanner '81, Janwa and Lal '03, one may construct codes with decent relative minimum distance and rate by using bipartite biregular graphs.

Applications for RBBG: matrix completion

– Idea: given Y a large matrix with "low complexity" (e.g. sparse, low-rank, etc.) observe some of Y's entries, and based on them find Y' such that $||Y - Y'||$ is small (or even 0) in some norm $|| \cdot ||$. (Netflix problem; Amazon, etc.)

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- Recent idea: sample entries according to a random regular graph (Heiman et al '14, Bhojanapalli and Jain '14, Gamarnik et al '17).
- If one uses a RBBG instead (simple-mindedly), improvement in bounds by a factor of 2 (as compared to Heiman et al. '14; studying Gamarnik et al. '17). Possibly more?...