Lattice coverings

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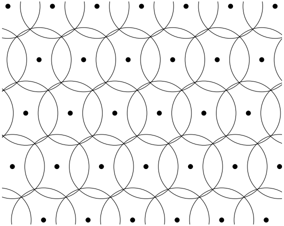
I. Introduction

Lattice coverings

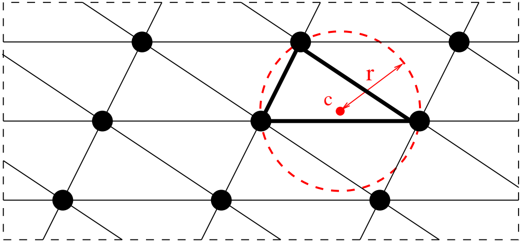
- A lattice $L \subset \mathbb{R}^n$ is a set of the form $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$.
- A covering is a family of balls $B_n(x_i, r)$, $i \in I$ of the same radius r and center x_i such that any $x \in \mathbb{R}^n$ belongs to at least one ball.



- If L is a lattice, the lattice covering is the covering defined by taking the minimal value of $\alpha > 0$ such that $L + B_n(0, \alpha)$ is a covering.

Empty sphere and Delaunay polytopes

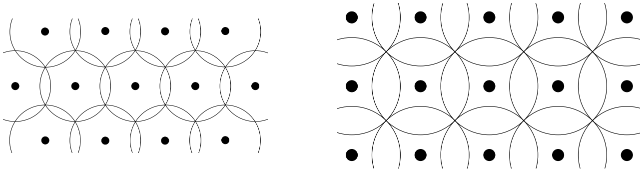
- Def: A sphere $S(c, r)$ of center c and radius r in an n-dimensional lattice L is said to be an empty sphere if:
  (i) $\|v - c\| \geq r$ for all $v \in L$,
  (ii) the set $S(c, r) \cap L$ contains $n + 1$ affinely independent points.
- Def: A Delaunay polytope P in a lattice L is a polytope, whose vertex-set is $L \cap S(c, r)$.



- Delaunay polytopes define a tessellation of the Euclidean space $\mathbb{R}^n$
- Lattice Delaunay polytopes have at most 2^n vertices.

Covering density

- For a lattice L we define the covering radius $\mu(L)$ to be the smallest r such that the family of balls $v + B_n(0, r)$ for $v \in L$ cover $\mathbb{R}^n$.



- The covering density has the expression

$$\Theta(L) = \frac{\mu(L)^n \operatorname{vol}(B_n(0,1))}{\det(L)} \geq 1$$

with

  - $\mu(L)$ being the largest radius of Delaunay polytopes
  - or

$$\mu(L) = \max_{x \in \mathbb{R}^n} \min_{y \in L} \|x - y\|$$

Computing covering density

Known methods:

- For the Leech lattice, the covering density was determined using special enumeration technique of the Delaunay polytopes of maximum radius.
- For the lattice Λ_{23}^* the covering density was computed by considering it as a projection of the Leech lattice.
- The only general technique is to enumerate all the Delaunay polytopes of the lattice.

Algorithm for enumerating the Delaunay polytopes:

- First find one Delaunay polytope by linear programming.
- For each representative of orbit of Delaunay polytope, do the following:
  - Compute the orbits of facets of the polytope (using symmetries, ...).
  - For each facet find the adjacent Delaunay polytope.
  - If not equivalent to a known representative, insert it into the list.
- Finish when all have been treated.

The Niemeier lattices I

- They are the 24-dimensional lattices L with $\det L = 1$, $\langle x, y \rangle \in \mathbb{Z}$, $\|x\|^2 \in 2\mathbb{Z}$. The set of vector of norm 2 is described by a root lattice

nb	root system	Sqr. Cov.	$\mid$ max. Del. $\mid$	$\mid$ Orb. Del. $\mid$
1	D_{24}	3	4096	13
2	$D_{16} + E_8$	3	4096	18
3	$3E_8$	3	4096	4
4	A_{24}	5/2	512	144
5	$2D_{12}$	3	4096	115
6	$A_{17} + E_7$	5/2	240^2, 256^2, 512^2	453
7	$D_{10} + 2E_7$	3	4096	134
8	$A_{15} + D_9$	5/2	240^2, 256^4, 512^3	1526
9	$3D_8$	3	4096	684
10	$2A_{12}$	5/2	512	13853
11	$A_{11} + D_7 + E_6$	23/9	512	11685
12	$4E_6$	8/3	729	250

The Niemeier lattices II

nb	root system	Sqr. Cov.	| max. Del. |	| Orb. Del. |
13	$2A_9 + D_6$	$5/2$	256^3, 512^3	61979
14	$4D_6$	3	256	3605
15	$3A_8$	$\geq 5/2$	512	≥ 182113
16	$2A_7 + 2D_5$	$\geq 5/2$	256^5, 512^4	≥ 237254
17	$4A_6$	$\geq 5/2$	512	≥ 110611
18	$4A_5 + D_4$	$\geq 5/2$	256^2, 512^3	≥ 324891
19	$6D_4$	3	4096	17575
20	$6A_4$	$\geq 5/2$	512	≥ 272609
21	$8A_3$	$\geq 5/2$	256^2, 512^2	≥ 413084
22	$12A_2$	$\geq 8/3$	729	≥ 392665
23	$24A_1$	3	4096	120911

Conjecture (Alahmadi, Deza, DS, Solé, 2018):

- Delaunay polytopes of even unimodular lattices have at most $2^{n/2}$ vertices.
- The Square Covering radius of even unimodular lattices is at most $n/8$.

II. iso-Delaunay domains

Gram matrix formalism

- Denote by S^n the vector space of real symmetric $n \times n$ matrices and $S^n_{>0}$ the convex cone of real symmetric positive definite $n \times n$ matrices.
- Take a basis $(v_1, \dots, v_n)$ of a lattice L and associate to it the Gram matrix $G_{\mathbf{v}} = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n} \in S^n_{>0}$.
- All geometric information about the lattice can be computed from the Gram matrices.
- Lattices up to isometric equivalence correspond to $S^n_{>0}$ up to arithmetic equivalence by $\mathrm{GL}_n(\mathbb{Z})$.
- In practice, Plesken & Souvignier wrote a program isom for testing arithmetic equivalence and a program autom for computing automorphism group of lattices.

Equalities and inequalities

- Take $M = G_v$ with $v = (v_1, \dots, v_n)$ a basis of lattice L.
- If $V = (w_1, \dots, w_N)$ with $w_i \in \mathbb{Z}^n$ are the vertices of a Delaunay polytope of empty sphere $S(c, r)$ then:

$$\|w_i - c\| = r \quad \text{i.e.} \quad {w_i}^T M w_i - 2{w_i}^T Mc + c^T Mc = r^2$$

- Substracting one obtains

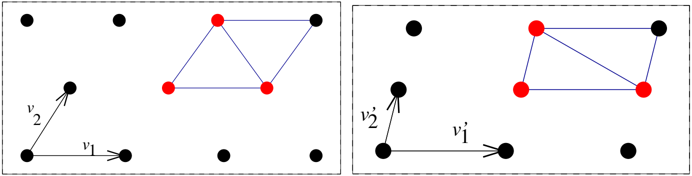
$$\left\{ {w_i}^T M w_i - {w_j}^T M w_j \right\} - 2 \left\{ {w_i}^T - {w_j}^T \right\} Mc = 0$$

- Inverting matrices, one obtains $Mc = \psi(M)$ with ψ linear and so one gets linear equalities on M.
- Similarly $||w - c|| \geq r$ translates into a linear inequality on M: Take $V = (v_0, \dots, v_n)$ a simplex ($v_i \in \mathbb{Z}^n$), $w \in \mathbb{Z}^n$. If one writes $w = \sum_{i=0}^{n} \lambda_i v_i$ with $1 = \sum_{i=0}^{n} \lambda_i$, then one has

$$\|w - c\| \geq r \Leftrightarrow w^T M w - \sum_{i=0}^{n} \lambda_i {v_i}^T M v_i \geq 0$$

Iso-Delaunay domains

- Take a lattice L and select a basis $v_1, \dots, v_n$.
- We want to assign the Delaunay polytopes of a lattice. Geometrically, this means that



are part of the same iso-Delaunay domain.

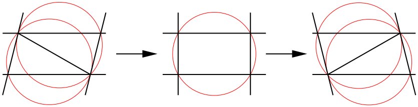
- An iso-Delaunay domain is the assignment of Delaunay polytopes of the lattice.

Primitive iso-Delaunay

- If one takes a generic matrix M in $S^n_{>0}$, then all its Delaunay are simplices and so no linear equality are implied on M.
- Hence the corresponding iso-Delaunay domain is of dimension $\frac{n(n+1)}{2}$, they are called primitive

Equivalence and enumeration

- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$ by arithmetic equivalence and preserve the primitive iso-Delaunay domains.
- Voronoi proved that after this action, there is a finite number of primitive iso-Delaunay domains.
- Bistellar flipping creates one iso-Delaunay from a given iso-Delaunay domain and a facet of the domain. In dim. 2:

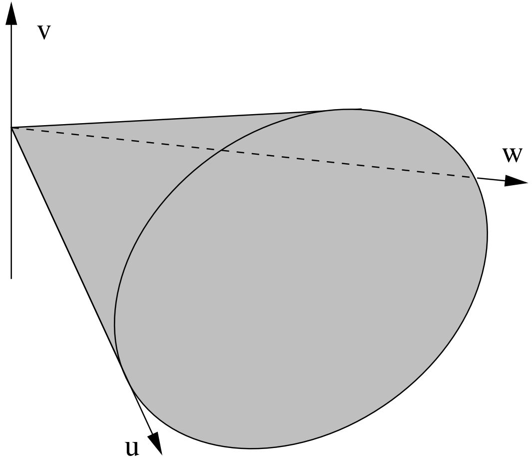


- Enumerating primitive iso-Delaunay domains is done classically:
  - Find one primitive iso-Delaunay domain.
  - Find the adjacent ones and reduce by arithmetic equivalence.

  The algorithm is graph traversal and iteratively finds all the iso-Delaunay up to equivalence.

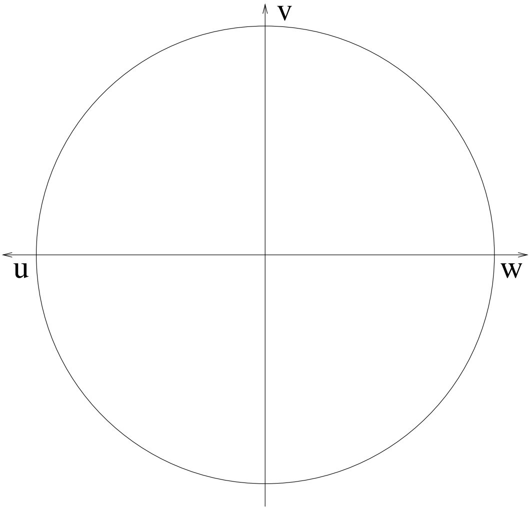
The partition of $S^2_{>0} \subset \mathbb{R}^3$ I

$\begin{pmatrix} u & v \\ v & w \end{pmatrix} \in S^2_{>0}$ if and only if $v^2 < uw$ and $u > 0$.



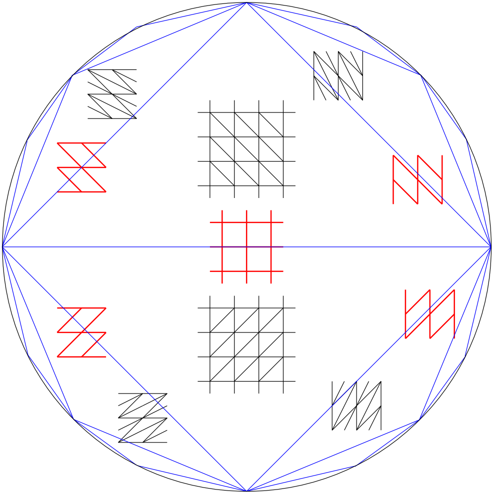
The partition of $S^2_{>0} \subset \mathbb{R}^3$ II

We cut by the plane $\mathrm{u} + \mathrm{w} = 1$ and get a circle representation.



The partition of $S^2_{>0} \subset \mathbb{R}^3$ III

Primitive iso-Delaunay domains in $S^2_{>0}$:



Enumeration results

Dimension	Nr. L-type	Nr. primitive
1	1	1
2	2	1
3	5 Fedorov, 1885	1 Fedorov, 1885
4	52 Delaunay & Shtogrin 1973	3 Voronoi, 1905
5	110244 MDS, AG, AS & CW, 2016	222 Engel & Gr. 2002
6	?	$\geq 2.10^8$ Engel, 2013

- Partition in Iso-Delaunay domains is just one example of polyhedral partition of $S^n_{\geq 0}$.
- There are some other theories if we fix only the edges of the Delaunay polytopes (C-type, Baranovski & Ryshkov 1975).

III. SDP optimization

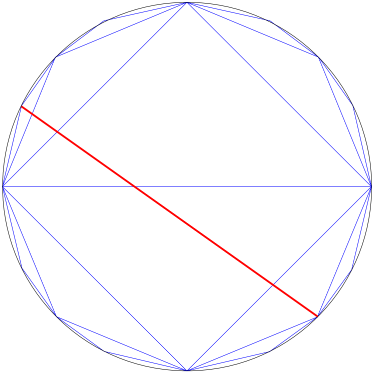
SDP for coverings

- Fix a primitive iso-Delaunay domain, i.e. a collection of simplexes as Delaunay polytopes $D_1, \dots, D_m$.
- Thm (Minkowski): The function $-\log\det(M)$ is strictly convex on $S^n_{>0}$.
- Solve the problem
  - M in the iso-Delaunay domain (linear inequalities),
  - the Delaunay D_i have radius at most 1 (semidefinite condition by Delaunay, Dolbilin, Ryshkov & Shtogrin, 1970).,
  - minimize $-\log\det(M)$ (strictly convex).
- Thm: Given an iso-Delaunay domain LT, there exist a unique lattice, which minimize the covering density over LT.
- The above problem is solved by the interior point methods implemented in MAXDET by Vandenberghe, Boyd & Wu. This approach was introduced in F. Vallentin, thesis, 2003.
- This allows to solve the lattice covering problem for $n \leq 5$.

Packing covering problem

- The packing-covering problem consists in optimizing the quotient
$$\frac{\Theta(L)}{\alpha(L)}$$
with $\alpha(L)$ the packing density.
- There is a SDP formulation of this problem (Schürmann & Vallentin, 2006) for a given iso-Delaunay domain with Delaunay $D_1, \dots, D_m$:
Solve the problem for (α, M):
  - M in the iso-Delaunay domain (linear inequalities),
  - the Delaunay D_i have radius at most 1.
  - $\alpha \leq M[x]$ for all edges x of Delaunay polytope D_i.
  - maximize α
- The problem is solved for $n \leq 5$ (Horvath, 1980, 1986).
- Dimension $n \geq 6$ are open.
- E_8 is conjectured to be a local optimum.

IV. $S^n_{>0}$-spaces



$S^n_{>0}$-spaces

- A $S^n_{>0}$-space is a vector space $\mathcal{SP}$ of S^n, which intersect $S^n_{>0}$.
- We want to describe the Delaunay decomposition of matrices $M \in S^n_{>0} \cap \mathcal{SP}$.
- Motivations:
  - The enumeration of iso-Delaunay is done up to dimension 5 but higher dimension are very difficult.
  - We hope to find some good covering by selecting judicious $\mathcal{SP}$. This is a search for best but unproven to be optimal coverings.
- A iso-Delaunay in $\mathcal{SP}$ is an open convex polyhedral set included in $S^n_{>0} \cap \mathcal{SP}$, for which every element has the same Delaunay decomposition.
- Possible choices of spaces (typically we want dimension at most 4):
  - Space of forms invariant under a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$.
  - Lower dimensional space and a lamination.
  - A form A and a rank 1 form defined by a shortest vector of A.

$S^n_{>0}$-space theory

- Relevant group is $\mathrm{Aut}(\mathcal{S}P) = \{g \in \mathrm{GL}_n(\mathbb{Z}) \text{ s.t. } g\mathcal{S}Pg^T = \mathcal{S}P\}$.
- For a finite group $G \subset \mathrm{GL}_n(\mathbb{Z})$ of space

$$\mathcal{S}P(G) = \left\{A \in S^n \text{ s.t. } gAg^T = A \text{ for } g \in G\right\}$$

  we have $\mathrm{Aut}(\mathcal{S}P(G)) = Norm(G, \mathrm{GL}_n(\mathbb{Z}))$ (Zassenhaus) and a finite number of iso-Delaunay domains.
- There exist some $S^n_{>0}$-spaces having a rational basis and an infinity of iso-Delaunay domains. Example by Yves Benoist:

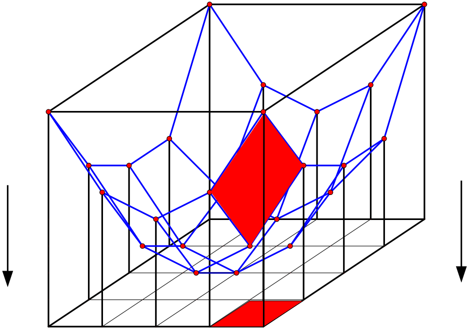
$$\mathcal{S}\mathcal{P} = \mathbb{R}(x^2 + 2y^2 + z^2) + \mathbb{R}(xy)$$

- Another finiteness case is for spaces obtained from $\mathrm{GL}_n(R)$ with R number ring.
- We can have dead ends if a facet of an $\mathcal{S}P$ iso-Delaunay domains does not intersect $S^n_{>0}$.
- In practice we often do the computation and establish finiteness ex-post facto.

Lifted Delaunay decomposition

- The Delaunay polytopes of a lattice L correspond to the facets of the convex cone $\mathcal{C}(L)$ with vertex-set:

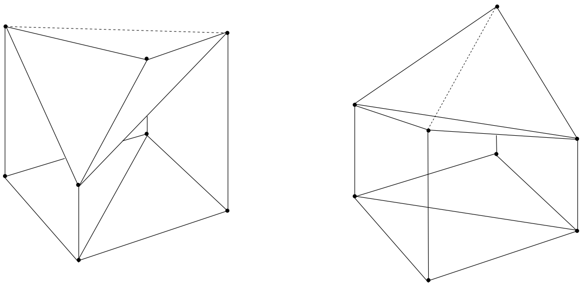
$$\{(x, ||x||^2) \text{ with } x \in L\} \subset \mathbb{R}^{n+1} .$$



- See Edelsbrunner & Shah, 1996.

Generalized bistellar flips

- The “glued” Delaunay form a Delaunay decomposition for a matrix M in the $(\mathcal{SP}, L)$-iso-Delaunay satisfying to $f(M) = 0$.
- The flipping break those Delaunays in a different way.
- Two triangulations of $\mathbb{Z}^2$ correspond in the lifting to:



- The polytope represented is called the repartitioning polytope. It has two partitions into Delaunay polytopes.
- The lower facets correspond to one tesselation, the upper facets to the other tesselation.

Enumeration technique

- Find a primitive $(\mathcal{SP}, L)$-iso-Delaunay domain, insert it to the list as undone.
- Iterate
  - For every undone primitive $(\mathcal{SP}, L)$-iso-Delaunay domain, compute the facets.
  - Eliminate redundant inequalities.
  - For every non-redundant inequality realize the flipping, i.e. compute the adjacent primitive $(\mathcal{SP}, L)$-iso-Delaunay domain. If it is new, then add to the list as undone.
- See for full details DS, Vallentin, Schürmann, 2008.
- Then we solve the SDP problem on all the obtained primitive iso-Delaunay domains and get the get covering density in the subspace.

Best known lattice coverings

d	lattice / covering density Θ		
1	$\mathbb{Z}^1$ 1	13	L_{13}^c (DSV) 7.762108
2	A_2^* (Kershner) 1.209199	14	L_{14}^c (DSV) 8.825210
3	A_3^* (Bambah) 1.463505	15	L_{15}^c (DSV) 11.004951
4	A_4^* (Delaunay & Ryshkov) 1.765529	16	A_{16}^* 15.310927
5	A_5^* (Ryshkov & Baranovski) 2.124286	17	A_{17}^9 (DSV) 12.357468
6	L_6^c (Vallentin) 2.464801	18	A_{18}^* 21.840949
7	L_7^c (Schürmann & Vallentin) 2.900024	19	A_{19}^{10} (DSV) 21.229200
8	L_8^c (Schürmann & Vallentin) 3.142202	20	A_{20}^7 (DSV) 20.366828
9	L_9^c (DSV) 4.268575	21	A_{21}^{11} (DSV) 27.773140
10	L_{10}^c (DSV) 5.154463	22	Λ_{22}^* (Smith) $\leq$ 27.8839
11	L_{11}^c (DSV) 5.505591	23	Λ_{23}^* (Smith, MDS) 15.3218
12	L_{12}^c (DSV) 7.465518	24	*Leech* 7.903536

- For $n \leq 5$ the results are definitive.
- The lattices A_n^r for r dividing $n+1$ are the Coxeter lattices. They are often good coverings and they are used for perturbations.
- For dimensions 10 and 12 we use laminations over Coxeter lattices of dimension 9 and 11.
- Leech lattice is conjecturally optimal (it is local optimal Schürmann & Vallentin, 2005)

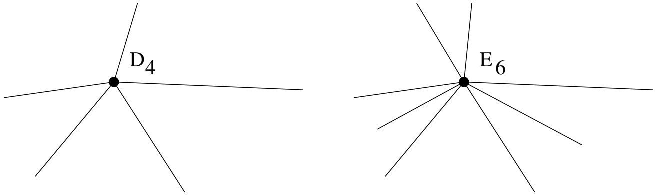
Periodic coverings

- For general point sets the problem is nonlinear and the above formalism does not apply.
- If we fix a number of translation classes

$$(c_1 + \mathbb{Z}^n) \cup \cdots \cup (c_M + \mathbb{Z}^n)$$

  and vary the quadratic form then we get some iso-Delaunay domains.
- If the c_i are rational then we have finiteness of the number of iso-Delaunay domains.
- If the quadratic form belong to a $S^n_{>0}$-space and c_i are rational then finiteness is independent of the c_i.
- Maybe one can get periodic covering for $n \leq 5$ better than lattice coverings.

V. Covering maxima, pessima and their characterization



Perfect Delaunay polytopes

Instead of considering the whole Delaunay tesselation, one alternative viewpoint is to consider a single Delaunay polytope.

- Def: A finite set $S \subset \mathbb{Z}^n$ is a perfect Delaunay polytope if
  - S is the vertex set of a Delaunay polytope for $Q_0 \in S^n_{>0}$.
  - The quadratic forms making S a Delaunay are positive multiple of Q_0.
- A perfect n-dimensional Delaunay polytope has at least $\binom{n+2}{2} - 1$ vertices. There is only one way to embed it as a Delaunay polytope of a lattice.
- Perfect Delaunay can be pretty wild (DS & Rybnikov, 2014):
  - They do not necessarily span the lattice.
  - A lattice can have several perfect Delaunay polytopes.
  - Automorphism group of lattice can be larger than the perfect Delaunay.
- For a given polytope P with *vert* $P \subset \mathbb{Z}^n$ the set of quadratic forms having P as a Delaunay is the interior of a polyhedral cone.

Enumeration results for perfect Delaunay and simplices

- The opposite of a perfect Delaunay is a Delaunay simplex which has just $n+1$ vertices.
- It turns out the right space for studying a single Delaunay polytopes is the Erdahl cone defined as

$$Erdahl(n) = \{f \in E_2(n) \text{ s.t. } f(x) \geq 0 \text{ for } x \in \mathbb{Z}^n\}$$

  with $E_2(n)$ the space of quadratic functions on $\mathbb{R}^n$.
- Known results:

dim.	Nr. Perf. Del.	Nr. Del. simplex
1	1 ([0, 1])	1
2,3,4	0	1
5	0	2
6	1 (*Sch*) (Deza & D., 2004)	3
7	2 (*Gos*, ER_7) (DS, 2017)	11 (DS, 2017)
8	$\geq$ 26 (DS, Erdahl, Rybnikov 2007)	?
9	$\geq$ 100000	?

Covering Maxima and Eutacticity

- A given lattice L is called a covering maxima if for any lattice L' near L we have $\Theta(L') < \Theta(L)$.
- Def: Take a Delaunay polytope P for a quadratic form Q of center c_P and square radius μ_P. P is called eutactic if there are $\alpha_v > 0$ so that

$$\left\{\begin{array}{rcl} 1 & = & \sum_{v\in \text{vert } P} \alpha_v, \\ 0 & = & \sum_{v\in \text{vert } P} \alpha_v(v-c_P), \\ \frac{\mu_P}{n} Q^{-1} & = & \sum_{v\in \text{vert } P} \alpha_v(v-c_P)(v-c_P)^T. \end{array}\right.$$

- Thm: For a lattice L the following are equivalent:
  - L is a covering maxima
  - Every Delaunay polytope of maximal circumradius of L is perfect and eutactic.
- It is an analogue of a similar result for perfect forms by Voronoi.
- See DS, Schürmann, Vallentin, 2012.

The infinite series

Thm (DSV, 2012): For any $n \geq 6$ there exist one lattice $L(DS_n)$ which is a covering maxima.
There is only one orbit of perfect Delaunay polytope $P(DS_n)$ of maximal radius in $L(DS_n)$.

- We have

$$|vert(P(DS_n))| = \begin{cases} 1 + 2(n-1) + 2^{n-2} & \text{if } n \text{ is even} \\ 4(n-1) + 2^{n-2} & \text{if } n \text{ is odd} \end{cases}$$

- We have $L(DS_6) = \mathsf{E}_6$ and $L(DS_7) = \mathsf{E}_7$.
- Conj: $L(DS_n)$ has the maximum covering density among all n-dim. covering maxima.
  If true this would imply Minkowski conjecture by Shapira, Weiss, 2017.
- Conj: Among all perfect Delaunay polytopes, $P(DS_n)$ has
  - maximum number of vertices,
  - maximum volume.

Pessimum and Morse function property

- For a lattice L let us denote $D_{crit}(L)$ the space of direction d of deformation of L such that Θ increases in the direction d.
- Def: A lattice L is said to be a covering pessimum if the space D_{crit} is of measures 0.
- Thm (DSV, 2012): If the Delaunay polytopes of maximum circumradius of a lattice L are eutactic and are not simplices then L is a pessimum.

name	# vertices	# orbits Delaunay polytopes
$\mathbb{Z}^n$	2^n	1
D_4	8	1
D_n ($n \geq 5$)	2^{n-1}	2
E_6^*	9	1
E_7^*	16	1
E_8	16	2
K_{12}	81	4

- Thm (DSV, 2012): The covering density function $Q \mapsto \Theta(Q)$ is a topological Morse function if and only if $n \leq 3$.