Lattices from equiangular tight frames

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Computational Challenges in the Theory of Lattices
ICERM Brown
April 23 - 27, 2018

Lattices: basic notions

A **lattice** $\Lambda \subset \mathbb{R}^k$ of rank m, $1 \leq m \leq k$, is a free $\mathbb{Z}$-module of rank m, which is the same as a discrete co-compact subgroup of $V := \operatorname{span}_{\mathbb{R}} \Lambda$. If $m = k$, i.e. $V = \mathbb{R}^k$, we say that Λ is a lattice of **full rank** in $\mathbb{R}^k$. Hence

$$\Lambda = \operatorname{span}_{\mathbb{Z}}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_m\} = A\mathbb{Z}^m,$$

where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in \mathbb{R}^k$ are $\mathbb{R}$-linearly independent **basis** vectors for Λ and $A = (\boldsymbol{a}_1 \ \dots \ \boldsymbol{a}_m)$ is the corresponding $k \times m$ basis matrix.

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The **determinant** of Λ is

$$\det \Lambda := \sqrt{\det(A^\top A)},$$

which is equal to the volume (quotient Lebesgue measure) of V/Λ.

Lattices: minimal vectors

Minimal norm of a lattice Λ is

$$|\Lambda| = \min\left\{\|\boldsymbol{x}\| : \boldsymbol{x} \in \Lambda \setminus \{\mathbf{0}\}\right\},$$

where $\|\ \|$ is Euclidean norm. The set of **minimal vectors** of Λ is

$$S(\Lambda) = \{\boldsymbol{x} \in \Lambda : \|\boldsymbol{x}\| = |\Lambda|\}.$$

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- If $\operatorname{rk} \Lambda > 4$, a strictly stronger condition is that Λ is **generated by minimal vectors**, i.e. $\Lambda = \operatorname{span}_{\mathbb{Z}} S(\Lambda)$.
- It has been shown by Conway & Sloane (1995) and Martinet & Schürmann (2011) that there are lattices of rank ≥ 10 generated by minimal vectors which do not contain a **basis of minimal vectors**.

Lattices: eutaxy and perfection

Let $m = \operatorname{rk} \Lambda$ and

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Lattices: eutaxy and perfection

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This lattice is called **eutactic** if there exist positive real numbers $c_1, \dots, c_n$ such that

$$\|\boldsymbol{v}\|^2 = \sum_{i=1}^{n} c_i (\boldsymbol{v}, \boldsymbol{x}_i)^2$$

for every vector $\boldsymbol{v} \in \operatorname{span}_{\mathbb{R}} \Lambda$, where $(\cdot, \cdot)$ is the usual inner product. If $c_1 = \dots = c_n$, we say that Λ is **strongly eutactic**.

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This lattice is called **perfect** if the set of symmetric matrices

$$\{\boldsymbol{x}_i \boldsymbol{x}_i^\top : \boldsymbol{x}_i \in S(\Lambda)\}$$

spans the space of $m \times m$ symmetric matrices.

Packing density

The **packing density** of a lattice Λ of rank m is defined as

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Theorem 1 (G. Voronoi, 1908)

A lattice is extremal if and only if it is perfect and eutactic.

What is a frame?

A spanning set $\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$, $n \geq k$, is called a **frame** if there exist constants $\gamma_1, \gamma_2 \in \mathbb{R}$ such that for every $\boldsymbol{x} \in \mathbb{R}^k$,

$$\gamma_1 \|\boldsymbol{x}\|^2 \leq \sum_{j=1}^{n} (\boldsymbol{x}, \boldsymbol{f}_j)^2 \leq \gamma_2 \|\boldsymbol{x}\|^2,$$

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- A frame is called **unit** if $\|\boldsymbol{f}_j\| = 1$ for every $1 \leq j \leq n$.
- A frame is called **tight** if $\gamma_1 = \gamma_2$.
- A frame is called **equiangular** if $|(\boldsymbol{f}_i, \boldsymbol{f}_j)| = c$ for all $1 \leq i \neq j \leq n$, for some constant $c \in [0, 1]$,

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ETFs generalize the notion of an orthonormal basis, while redundancy of an overdetermined spanning set allows for better recovery of information in case of errors: we can think of "coordinates" with respect to such an overdetermined set as extra "frequencies" that can help recover information in case of erasures in transmission.

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ETFs are extensively used in coding theory and data compression, among many other areas – they are important tools of Applied Harmonic Analysis.

An example

Here is a (3, 2)-ETF $\mathcal{F} := \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix} \right\}$:

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Notice that $\pm\mathcal{F} = S(\Lambda_h)$, the set of minimal vectors of the hexagonal lattice $\Lambda_h = \begin{pmatrix} 0 & \sqrt{3}/2 \\ 1 & 1/2 \end{pmatrix} \mathbb{Z}^2$.

Coherence of an ETF

Given a unit ETF

$$\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\} \subset \mathbb{R}^k,$$

the common value $c = c(\mathcal{F}) = |(\boldsymbol{f}_i, \boldsymbol{f}_j)|$ for all $1 \leq i \neq j \leq n$ is called the **coherence** of this frame. This is absolute value of the cosine of the angle between distinct frame vectors.

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Notice that $c \in [0, 1]$, and the closer is c to 0, the closer is $\mathcal{F}$ to being an orthogonal basis. Ideally, we want $n = |\mathcal{F}|$ large while c is small: these are naturally conflicting goals, which often result in an optimization problem. Frames with small coherence are often referred to as *incoherent*.

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Remark 1

This is the only context known to me where being incoherent is good.

Properties of ETFs

Let $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$ be an (n, k) real unit ETF. Here are some of its fundamental properties.

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$$c(\mathcal{F}) = \sqrt{\frac{n-k}{k(n-1)}}.$$

3. (Neumann) If $n \neq 2k$, then $1/c(\mathcal{F})$ is an odd integer.
4. The "tightness" constant $\gamma_1 = \gamma_2$ is easily computed to be $\frac{k}{n}$.

Lattices generate tight frames

Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$ be a finite subset of the unit sphere Σ_{k-1} in $\mathbb{R}^k$. It is called a **spherical t-design** for an integer $t \geq 1$ if for every real polynomial p in k variables of degree $\leq t$,

$$\int_{\Sigma_{k-1}} p(\boldsymbol{y})\ d\nu(\boldsymbol{y}) = \frac{1}{n} \sum_{i=1}^{n} p(\boldsymbol{x}_i),$$

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Theorem 3 (follows from Holmes & Paulsen, 2004)

X as above is a spherical 2-design iff it is a unit tight frame and $\sum_{i=1}^{n} \boldsymbol{x}_i = \boldsymbol{0}$.

Do ETFs generate lattices?

Let $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$ be a (n, k)-ETF, and define

$$\Lambda(\mathcal{F}) = \operatorname{span}_{\mathbb{Z}} \mathcal{F}.$$

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Question 1

When is $\Lambda(\mathcal{F})$ a lattice? If it is a lattice, what are its properties?

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Theorem 4 (Böttcher, F., Garcia, Maharaj, Needell (2016))

1. *$\Lambda(\mathcal{F})$ is a lattice if and only if $c = \sqrt{\frac{n-k}{k(n-1)}}$ is rational.*
2. *If $\Lambda(\mathcal{F})$ is a lattice, it is of full rank.*
3. *If $\Lambda(\mathcal{F})$ is a lattice and*

$$S(\Lambda(\mathcal{F})) = \{\pm \boldsymbol{f}_1, \dots, \pm \boldsymbol{f}_n\},$$

then $\Lambda(\mathcal{F})$ is strongly eutactic.

Irrational example

Let $p = \frac{1+\sqrt{5}}{2}$ and let $\mathcal{F}$ be the set of columns of the matrix

$$\frac{1}{\sqrt{1+p^2}} \begin{pmatrix} 0 & 0 & 1 & -1 & p & p \\ 1 & -1 & p & p & 0 & 0 \\ p & p & 0 & 0 & 1 & -1 \end{pmatrix}.$$

This is a $(6,3)$ real unit ETF with *irrational* coherence $1/\sqrt{5} \approx 0.4472$. By Dirichlet's theorem in Diophantine Approximations, there exist infinitely many relatively prime integers a, b such that

$$\left| \frac{a}{b} + p \right| \leq \frac{1}{b^2}.$$

Taking a linear combination of the vectors of $\mathcal{F}$ with coefficients

$$a+b,\ a-b,\ b,\ b,\ a,\ -a,$$

we obtain a constant multiple of a vector $(0, a+bp, a+bp)^\top$. Coordinates of this vector are $\leq 1/b$ in absolute value, so it $\to \mathbf{0}$ as $b \to \infty$, i.e. $\Lambda(\mathcal{F})$ is not discrete.

Summary of our results

Table: There exists an (n, k) real unit ETF $\mathcal{F}$ with:

(n,k)	$c(\mathcal{F})$	**Eutactic?**	**Perfect?**	$\delta(\Lambda(\mathcal{F}))$	$\delta_{\max}$
$(k+1,k)$	$1/k$	strongly	only $(3,2)$	–	–
$\mathbf{(6,3)}$	$\mathbf{1/\sqrt{5}}$	**n/a**	**n/a**	**n/a**	**n/a**
$(10,5)$	$1/3$	strongly	no	0.3701...	0.4652...
$(16,6)$	$1/3$	strongly	no	0.2725...	0.3729...
$\mathbf{(14,7)}$	$\mathbf{1/\sqrt{13}}$	**n/a**	**n/a**	**n/a**	**n/a**
$(28,7)$	$1/3$	strongly	yes	0.2157...	0.2953...
$\mathbf{(18,9)}$	$\mathbf{1/\sqrt{17}}$	**n/a**	**n/a**	**n/a**	**n/a**
$(26,13)$	$1/5$	strongly	no	0.0024...	0.0320...
$(276,23)$	$1/5$	yes	yes	?	0.0019...
$(50,25)$	$1/7$	?	no	–	–

$\delta_{\max}$ = max packing density known in $\mathbb{R}^k$.

Further results on ETF lattices

Theorem 5 (Böttcher, F., Garcia, Maharaj, Needell (2016))

1. *Lattices* $\Lambda(\mathcal{F})$ *from the* $(k+1, k)$, $(10, 5)$, $(16, 6)$, $(28, 7)$, $(26, 13)$ *entries of Table 1 all have bases of minimal vectors.*
2. *There are infinitely many* k *for which there exist* $(2k, k)$*-ETFs* $\mathcal{F}$ *such that* $\Lambda(\mathcal{F})$ *is a full-rank lattice, e.g.* $(10, 5)$, $(26, 13)$.
3. *Lattice* $\Lambda(\mathcal{F})$ *from the* $(276, 23)$ *entry of Table 1 is generated by minimal vectors.*

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Remark 2

1. There are often multiple ETFs with the same parameters (n, k): we exhibit two lattices from $(10, 5)$-ETFs, three lattices from $(26, 13)$-ETFs, and ten lattices from $(50, 25)$-ETFs.

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1. There are often multiple ETFs with the same parameters (n,k): we exhibit two lattices from $(10,5)$-ETFs, three lattices from $(26,13)$-ETFs, and ten lattices from $(50,25)$-ETFs.
2. Perfection of the lattice from $(28,7)$-ETF (constructed differently) was established in 2015 by R. Bacher.

More remarks

Minimal vectors of ETF lattices are often precisely the $\pm$ frame vectors (this is the case with all our examples). The corresponding symmetric matrices are known to be linearly independent, and so there are

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In case of equality, we likely *always* get a perfect strongly eutactic (hence extremal) lattice with the minimal required number of minimal vectors for the perfection condition. This being said, the only known cases of equality in the Gerzon bound are $(3,2)$, $(6,3)$, $(28,7)$ and $(276,23)$; the $(3,2)$ case is precisely the planar hexagonal lattice.

More generally

Let $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$ be a spanning set, and let $\Lambda(\mathcal{F}) = \operatorname{span}_{\mathbb{Z}} \mathcal{F}$. Define the associated **norm-form of the frame** to be

$$Q_{\mathcal{F}}(\boldsymbol{a}) = \left\| \sum_{i=1}^{n} a_i \boldsymbol{f}_i \right\|^2$$

for each $\boldsymbol{a} \in \mathbb{Z}^n$. We say that $Q_{\mathcal{F}}$ **has separated values** if

$$\inf\{|Q_{\mathcal{F}}(\boldsymbol{a}) - Q_{\mathcal{F}}(\boldsymbol{b})| : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n, Q_{\mathcal{F}}(\boldsymbol{a}) \neq Q_{\mathcal{F}}(\boldsymbol{b})\} > 0.$$

We call the frame $\mathcal{F}$ **rational** if the inner products $(\boldsymbol{f}_i, \boldsymbol{f}_j)$ are rational numbers for all $1 \leq i, j \leq n$. This is equivalent to saying that the $n \times n$ Gram matrix of $Q_{\mathcal{F}}$ is rational.

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Proposition 6 (Böttcher & F., 2017)

If $\mathcal{F}$ is rational, then $\Lambda(\mathcal{F})$ is a lattice and the values of $Q_{\mathcal{F}}$ are separated.

Proof

Let B be the Gram matrix of $Q_{\mathcal{F}}$, i.e.

$$Q_{\mathcal{F}}(\boldsymbol{a}) = \boldsymbol{a}^{\top} B \boldsymbol{a}$$

for any $\boldsymbol{a} \in \mathbb{R}^n$. Since $Q_{\mathcal{F}}$ is rational, there is an integer $d > 0$ such that all entries of dB are integers. Hence

$$dQ_{\mathcal{F}}(\boldsymbol{a}) = d(\boldsymbol{a}^{\top} B \boldsymbol{a}) = \boldsymbol{a}^{\top} (dB) \boldsymbol{a}$$

assumes values in $\{0, 1, 2, \ldots\}$ for $\boldsymbol{a} \in \mathbb{Z}^n$, which shows that $Q_{\mathcal{F}}$ has separated values and does not take values in $(0, 1/d)$.

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In particular, this means that if $\mathbf{0} \neq \boldsymbol{x} \in \Lambda(\mathcal{F})$, then for some $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{Z}^n$,

$$\|\boldsymbol{x}\|^2 = Q_{\mathcal{F}}(\boldsymbol{a}) > 1/d.$$

Hence 0 is not an accumulation point of the norm values on $\Lambda(\mathcal{F})$. Since $\Lambda(\mathcal{F})$ is an additive group, this means it is discrete, and hence a lattice.

Irrational?

If $\mathcal{F}$ is irrational so that $Q_{\mathcal{F}}$ is not a constant multiple of a rational quadratic form, the situation is less clear.

For instance, positive definite irrational quadratic forms in $n \geq 5$ variables do not have separated values (this was Lewis-Davenport conjecture, resolved by F. Götze in 2004 for $n \geq 5$).

On the other hand, this does not prevent corresponding lattices from being discrete.

Our form $Q_{\mathcal{F}}$ is positive *semidefinite*, to which these results do not apply directly.

Tight frames

Suppose now $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$ is an (n, k) tight frame, i.e. $\exists\ \gamma \in \mathbb{R}$ such that for every $\boldsymbol{x} \in \mathbb{R}^k$,

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Theorem 7 (Böttcher & F., 2017)

Let $\mathcal{F} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\}$ be an $(n, 2)$ or an $(n, 3)$ tight frame containing at least one unit vector. Then the following are equivalent:

(i) $\mathcal{F}$ is rational,

(ii) $\Lambda(\mathcal{F})$ is a lattice,

(iii) $Q_{\mathcal{F}}$ has separated values.

Some open questions: ETFs

We demonstrated a full-rank lattice construction $\Lambda(\mathcal{F}) \subseteq \mathbb{R}^k$ from any (n, k) unit ETF $\mathcal{F}$ with rational angle. Several questions arise:

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3. A necessary condition for $\Lambda(\mathcal{F})$ to be perfect is that $\mathcal{F}$ is of maximal possible cardinality $n = \frac{k(k+1)}{2}$. Are there any other examples of that besides dimensions $k = 2, 3, 7, 23$? It is known that for $k > 2$, $k + 2$ has to be an odd perfect square, but finding other such examples is a well known open problem.

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4. Do these lattices always have bases of minimal vectors? If (1) above is true, they are at least generated by minimal vectors.

Some open questions: general tight frames

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1. Are there any irrational tight frames generating lattices?
2. Can the Davenport-Lewis conjecture be extended to at least some semidefinite irrational quadratic forms? Namely, if Q is an irrational quadratic form in n variables (where n is sufficiently big) with $Q(\boldsymbol{x}) \geq 0 \ \forall \boldsymbol{x} \in \mathbb{R}^n$ and

$$\dim_{\mathbb{R}} \{\boldsymbol{x} \in \mathbb{R}^n : Q(\boldsymbol{x}) = 0\} > 0,$$

is it necessarily true that

$$\inf\{|Q(\boldsymbol{a}) - Q(\boldsymbol{b})| : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n, Q(\boldsymbol{a}) \neq Q(\boldsymbol{b})\} = 0?$$

What if Q comes from a tight frame as above?

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1. A. Böttcher, L. Fukshansky, S. R. Garcia, H. Maharaj, D. Needell, *Lattices from tight equiangular frames*, Linear Algebra and its Applications, vol. 510 (2016), pg. 395–420
2. A. Böttcher, L. Fukshansky, *Addendum to: Lattices from tight equiangular frames*, Linear Algebra and its Applications, vol. 531 (2017), pg. 592–601

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Thank you!