# Spinor regular ternary quadratic lattices

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Computational Challenges in the Theory of Lattices
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[1] doo-KANE

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**The Representation Problem**

*Can we determine the set of all integers represented by $f$?*

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Matiyasevich (1970) $\rightarrow$ no general solution exists.

Theorem (Siegel, 1972)

*For $f$ quadratic, there exists a number $C$ depending on $a$ and $f$, such that if $f(x_1, ..., x_n) = a$ has an integer solution, then it must have one with*

$$\max_{i \leq i \leq a} | x_i | \leq C.$$

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tl;dr $\rightarrow$ it's possible, but totally impractical.

Theorem (Hasse, 1920)

*For $f$ quadratic, the equation*

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*has a rational solution if and only if has a solution over $\mathbb{Q}_p$ for every prime $p$, and over $\mathbb{R}$.*

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$\rightarrow$ Local-Global Principle

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*but clearly* $f(x, y) = 3$ *has no integral solution.*

The Big Question:

*To what extent does an integral local-global principle hold? When does it fail? And why? And how badly?*

# The General Setup

A quadratic polynomial $f(\vec{x})$ can be written as

$$f(\vec{x}) = q(\vec{x}) + \ell(\vec{x}) + c$$

where

- $q$ is a homogeneous quadratic.
- $\ell$ is a homogeneous linear.
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For $f(\vec{x}) = q(\vec{x})$ homogeneous (positive definite), define

$$L = (\mathbb{Z}^n, q).$$

Then $L$ is a **quadratic lattice**, and

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For $p$ prime, define the **local lattice** as

$$L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

and $q(L_p)$ accordingly.

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Similarly to $q(L)$, define

- $q(\mathrm{spn}(L))=$ the set of integers represented by $M \in \mathrm{spn}(L)$.
- $q(\mathrm{gen}(L))=$ the set of integers represented by $M \in \mathrm{gen}(L)$.

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$$\text{“}a \in q(\mathrm{gen}(L)) \Longleftrightarrow a \in q(L)\text{”}$$

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$$\text{“}a \in q(\text{gen}(L)) \iff a \in q(L)\text{”}$$

...but that would be incorrect (recall example 1).

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*Here,*

$$\mathrm{gen}(L) = \mathrm{spn}(L) = cls(L)$$

*so clearly*

$$q(\mathrm{gen}(L)) = q(L).$$

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Theorem (Kloosterman, 1926, Tartakowsky, 1929)

*For positive definite $L$ with $rk(L) \geq 4$ then*

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*provided that $a \gg 0$ (and $p^s \nmid a$ for $p$ anisotropic when $n = 4$).*

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- Icaza (1999): Made $C$ effective.

Theorem (Duke, Schulze-Pillot, 1990)

*For positive definite $L$ with $rk(L) = 3$,*

$$a \in^* q(\mathrm{spn}(L)) \iff a \in q(L)$$

*provided that $a \gg 0$.*

What might the genus look like?

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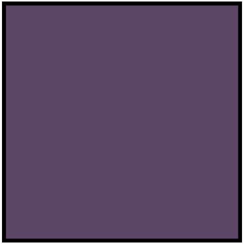
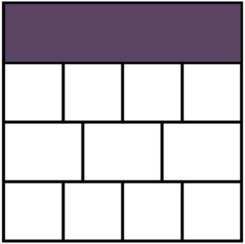
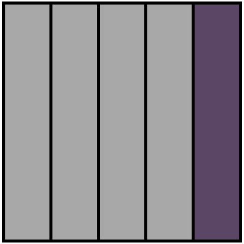
Class Number One



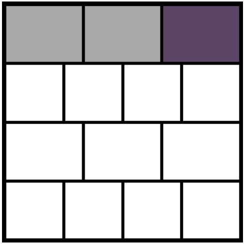
Spinor-Class Number One



Single Spinor Genus



Worst Case Scenario



Theorem (Earnest, Hsia, 1991)

*For a positive-definite lattice $L$ with rank $n \geq 5$,*

$$\mathrm{gen}(L) = cls(L) \Longleftrightarrow \mathrm{spn}(L) = cls(L)$$

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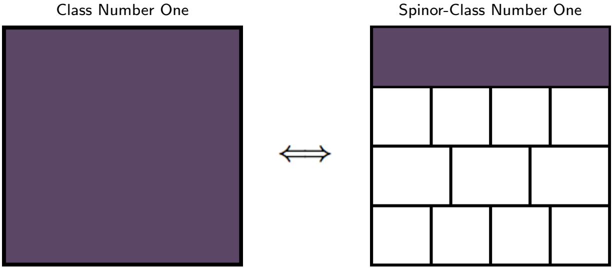

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*and* **spinor regular**, *that is,*

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When $rk(L) \geq 4$, there are infinitely many regular forms.

**Theorem (Jagy, Kaplansky, Schiemann, 1997)**

*There are at most 913 regular ternary lattices, that is, lattices for which*

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- Lemke Oliver, 2015: Confirmed remaining 14 assuming GRH.

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*There are 29 spinor regular ternary lattices which aren't regular, that is, lattices for which*

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*for which* $dL < 575,000$.

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Theorem (Earnest, H-, 2017)

*Jagy's list is complete.*

# The Watson Transformation

For an odd prime $p$ and

$$L_p \cong \langle a, p^{\beta} b, p^{\gamma} c \rangle$$

with $a, b, c \in \mathbb{Z}_p^{\times}$ and $\beta \leq \gamma$, define

$$(\lambda_p(L))_p = \begin{cases} \langle a, b, p^{\gamma-2} c \rangle & \text{if } \beta = 0 \\ \langle b, p^{\beta-1} a, p^{\gamma-1} c \rangle & \text{if } \beta = 1 \\ \langle a, p^{\beta-2} b, p^{\gamma-2} c \rangle & \text{if } \beta \geq 2. \end{cases}$$

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- For $q \neq p$, $\left(\lambda_p(L)\right)_q = L_q^u$ for $u \in \mathbb{Z}_q^{\times}$.

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- $\mathrm{ord}_p(d\lambda_p(L)) = \mathrm{ord}_p(dL) - 1, 2, 4$

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- *If $L$ does not behave well, then $\lambda_p(L)$ is spinor regular.*
- *There exists $L'$ with $ord_p(dL') = ord_p(dL)$ and $L'$ behaves well at all $q \neq p$.*

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$$\begin{aligned} L' \text{ not behaves well at } p_i &\implies \lambda_{p_i}(L') \text{ is spinor regular} \\ &\vdots \\ &\implies \lambda_{p_i}^{\delta}(L') \text{ is regular} \end{aligned}$$

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Therefore,

$$p_i \in \{2, 3, 5, 7, 11, 13, 17, 23\}$$

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and any triple must be of the form

$$2 \cdot p \cdot q$$

with $p \cdot q$ coming from above.

# The "Skip 4" Method

Lemma

*For a prime p with* $t > r_p$ *and* $\gcd(p, m) = 1$*, if*

$$p^t m,\ \ p^{t+1} m,\ \ p^{t+2} m \ \ \textit{and,}\ p^{t+3} m$$

*are not regular or spinor regular discriminants, then*

$$p^{t_0} m$$

*is not a spinor regular discriminant for any* $t_0 > t$*.*

# Discriminant Elimination

Suppose $L$ is spinor regular but not regular with $dL = 2^k \cdot 17^m$, and

$$L_{17} \cong \langle a, 17^{\beta} b, 17^{\gamma} c \rangle.$$

If $\beta + \gamma > 2$ then

$$\left(\lambda_{17}^{\delta}(L)\right)_{17} = \langle a, 17^{\beta'} b, 17^{\gamma'} c \rangle$$

is spinor regular where $\beta' + \gamma' = 1, 2$.

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$\rightarrow$ appeal to JKS list of 913.

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| $3^k \cdot 7^m$ | $7$ | $7^2$ | $7^3$ | $7^4$ | $7^5$ | $7^6$ |
|---|---|---|---|---|---|---|
| $3$ | $d_r$ | $d_r$ | - | - | - | - |
| $3^2$ | $d_r$ | $d_r$ | - | - | - | - |
| $3^3$ | $d_r$ | $d_r$ | - | - | - | * |
| $3^4$ | - | - | - | - | * | * |
| $3^5$ | - | - | - | * | * | * |
| $3^6$ | - | - | - | * | * | * |
| $3^7$ | - | - | * | * | * | * |

$d_r =$ discriminant of a regular form
$* =$ product greater than 575,000
$r_3 = 5$
$r_7 = 2$

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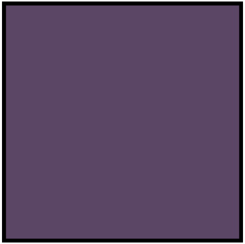
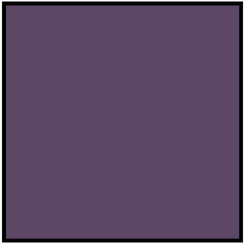


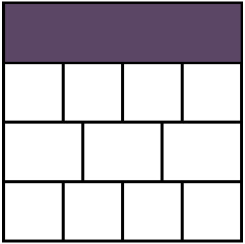


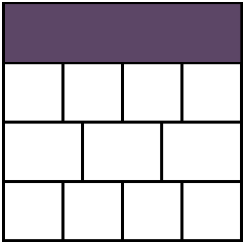
Theorem (Kirshmer, Lorch, 2013)

*An enumeration of all positive definite $L$ with*

$$\text{gen}(L) = \text{spn}(L) = cls(L),$$

*that is, $L$ has class number 1.*



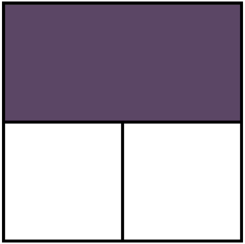


Theorem (Earnest, H-, 2017)

*There are 27 ternary forms, for which*

$$\text{gen}(L) \neq \text{spn}(L) = cls(L),$$

*that is, L has spinor class number 1, but L has class number greater than 1.*



Theorem (Earnest, H-, 2018)

*There is only one quaternary form,*

$$q(x, y, z, w) = x^2 + xy + 7y^2 + 3z^2 + 3zw + 3w^2,$$

*which has spinor class number 1, but class number greater than 1.*

Algorithm (Earnest, Nipp, 1991)

*INPUT:* *A prime $p$ and a lattice discriminant $D$.*

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3. *Reduce the set of all $P^t AP$ up to isometry.*

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- Use Algorithm with Nipp quaternary tables as input.
- Explicit computation of genus and spinor genus using Magma.

An Open Question:

*Can all of the above classifications be extended to primitive representations? That is, when does*

$$a \in^* q(\mathrm{gen}(L)) \Longleftrightarrow a \in^* q(\mathrm{spn}(L)) \Longleftrightarrow a \in^* q(L)$$

*hold, and when does it fail? And why...and how badly?*

# The Inhomogeneous Case

For $f(\vec{x}) = q(\vec{x}) + \ell(\vec{x})$ inhomogeneous,

$$f(x_1, ..., x_n) = a$$

has a solution, if and only if

$$a \in q(v + L)$$

where $v + L$ is a **lattice coset** for $v \in \mathbb{Q}L$.

**Theorem (Chan, Ricci, 2015)**

*Under certain arithmetic conditions, there are only finitely many equivalence classes of $v + L$ for which*

$$a \in q(\mathrm{gen}(v + L)) \iff a \in q(v + L)$$

An Open Question:

*Under what conditions does*

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*fail, and why, and how badly?*

Thank You!