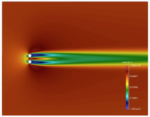


Closed-loop controls for fluids

Jeff Borggaard w/ Serkan Gugercin and Lizette Zietsman

19 February 2020

Virginia Tech

Outline

1. Flow Control Problem
2. Nonlinear Feedback Control
3. Quadratic-Quadratic Regulator
4. Burgers
5. van der Pol oscillators

Acknowledgments

- The National Science Foundation (DMS-1819110)

Flow Control Problem

Objective

Feedback Flow Control

Current Strategy:

- Compute the (unstable) steady-state solution $(\mathbf{v}_{\mathrm{ss}}, p_{\mathrm{ss}})$
- Write $\mathbf{v} = \mathbf{v}_{\mathrm{ss}} + \mathbf{v}'$ and $p = p_{\mathrm{ss}} + p'$
- Linearize the system about this steady-state (Oseen equations)

$$\begin{aligned} \dot{\mathbf{v}}' &= -\mathbf{v}_{\mathrm{ss}} \cdot \nabla \mathbf{v}' - \mathbf{v}' \cdot \nabla \mathbf{v}_{\mathrm{ss}} + \nabla \cdot \tau(\mathbf{v}') - \nabla p' + \mathcal{B}\mathbf{u} \\ 0 &= \nabla \cdot \mathbf{v}' \end{aligned}$$

- Use model reduction to find a smaller surrogate system
- Design the (linear) feedback control law
- Test the performance in the full nonlinear flow equations

Model Reduction for this study (w/ Serkan)

- Discretize the *Oseen equations* and *controlled outputs* (FEM)
- $\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$
- Use model reduction by tangential interpolation:
  Stykel 04; Mehrmann & Stykel 05; Benner & Sokolov 05; . . . ;
  Gugercin, Stykel & Wyatt 13.

$$\begin{bmatrix} \sigma_i \mathbf{E}_{11} - \mathbf{A}_{11} & -\mathbf{A}_{21}^T \\ -\mathbf{A}_{21} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \mathrm{b}_i \\ \mathbf{0} \end{bmatrix},$$

- Apply projection matrices

$$\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}, \quad \mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \mathbf{B}_r = \mathbf{W}^T\mathbf{B}, \quad \text{and} \quad \mathbf{C}_r = \mathbf{C}\mathbf{V}$$

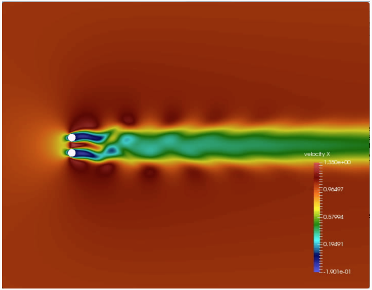
- $\mathbf{G}_r(s) = \mathbf{C}_r(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r$
- Flow simulations not req'd; Input independent; Computational cost is equivalent to several implicit time-steps.

Model Reduction for this study (w/ Serkan)

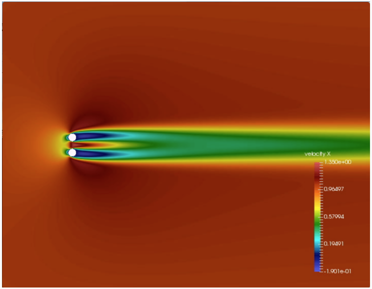
- The full-order model had $n_1 = 111,814$ and $n_2 = 14,336$.
- The reduced model used $r = 142$.
- The relative error in the $\mathcal{H}_\infty$ norm was 1.5154×10^{-5}.
- The reduced model was used to design the control.
- The projection matrices are used to implement the control on the full-order quadratic model.

Twin Cylinder Example

At $Re = 60$: Linear feedback control of the cylinder angular velocities

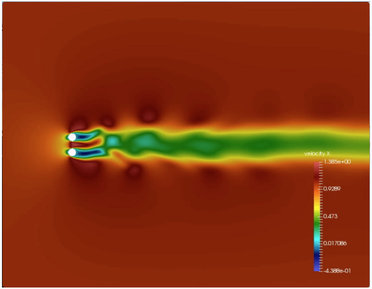


$\Longrightarrow$



Twin Cylinder Example

At $Re = 67$: Linear feedback control of the cylinder angular velocities



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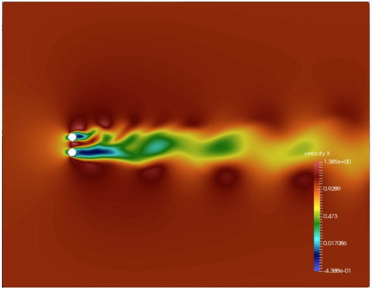
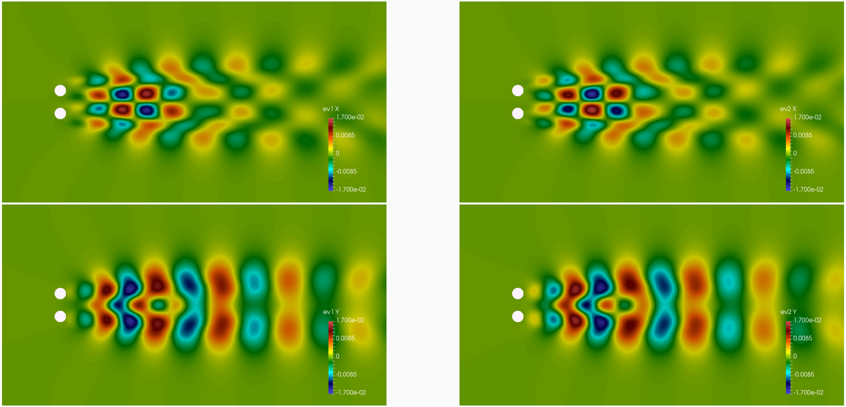
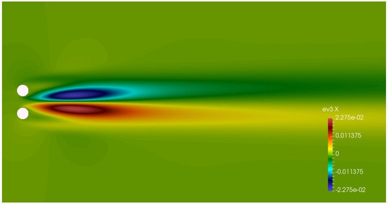


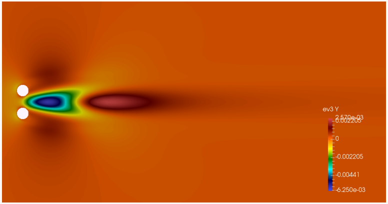
Eigenvectors



$$Re = 60: \;\; \lambda_{1,2} = +0.04019 \pm 0.7468i$$
$$Re = 67: \;\; \lambda_{1,2} = +0.06217 \pm 0.7535i$$

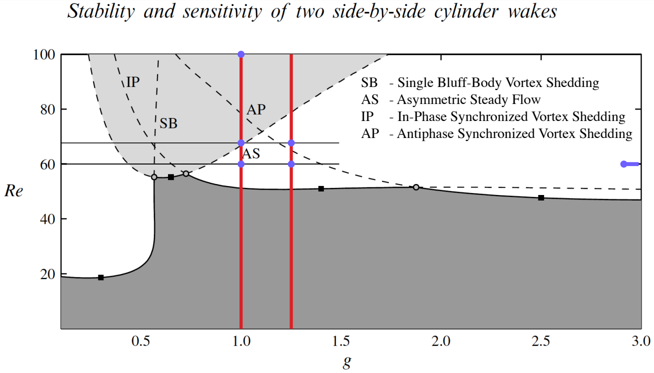
Eigenvectors





$$Re = 60\text{:}\;\; \lambda_3 = -0.006040$$
$$Re = 67\text{:}\;\; \lambda_3 = +0.002863$$

Wake Stabilization by Cylinder Rotation



Linear stability analysis agrees with: M. Carini, F. Giannetti, and F. Auteri, First instability and structural sensitivity of the flow past two side-by-side cylinders, *J. Fluid Mechanics*, 2014.

Nonlinear Feedback Control

Objective

Feedback Flow Control

Linear feedback control can be effective

- It works well to stabilize the steady-state solution, ...
- BUT the stability region may be very small.

Thus, we are motivated to consider nonlinear feedback for these problems.

We need new sets of computational tools:

- model reduction for systems with quadratic nonlinearities, and
- computation of nonlinear feedback laws.

Optimal Control Problem

Find a control $u(\cdot)$ with $u(t) \in \mathbb{R}^m$ that solves

$$\min_u J(z,u) = \int_0^\infty g(z(s),u(s))\ ds$$

subject to

$$\dot{z}(t) = f(z(t),u(t)), \qquad z(0) = z_0 \in \mathbb{R}^n.$$

Let the value function be $v(z_0) = J(z^*(\cdot;z_0),u^*(\cdot))$ and assume the optimal control is given by

$$u(t) = \mathcal{K}(z(t)).$$

For f, g, and v smooth enough, the feedback relation satisfies the Hamilton-Jacobi-Bellman partial differential equations

$$\begin{aligned} 0 &= \frac{\partial v}{\partial z}(z) f(z,\mathcal{K}(z)) + g(z,\mathcal{K}(z)) \\ 0 &= \frac{\partial v}{\partial z}(z) \frac{\partial f}{\partial u}(z,\mathcal{K}(z)) + \frac{\partial g}{\partial u}(z,\mathcal{K}(z)). \end{aligned}$$

Optimal Control Problem

Ideally, we could solve the HJB equations simultaneously for v and $\mathcal{K}$.

The feedback law $u(t) = \mathcal{K}(z(t))$ is the quantity of interest.

The value function $v(z)$ can serve as a Lyapunov function, providing information about the stability region around the steady-state solution.

However, these are notoriously difficult to solve as the HJB equations are nonlinear PDEs to be solved in $\mathbb{R}^n$ (or after model reduction $\mathbb{R}^r$).

Instead, we shall look at constructing polynomial approximations:

$$v(z) \approx v^{[2]}(z) + v^{[3]}(z) + \cdots + v^{[d+1]}(z)$$

and

$$\mathcal{K}(z) \approx k^{[1]}(z) + k^{[2]}(z) + \cdots + k^{[d]}(z).$$

Simple Nonlinear Feedback Example

We seek a control $u(\cdot)$ that minimizes

$$J(z,u) = \int_0^\infty \underbrace{z^2(t) + u^2(t)}_{g(z(t),u(t))} \, dt$$

subject to the dynamics

$$\dot{z}(t) = -0.1z(t) - 4z^2(t) + z^3(t) + u(t), \quad z(0) = z_0 \in \mathbb{R}^1.$$

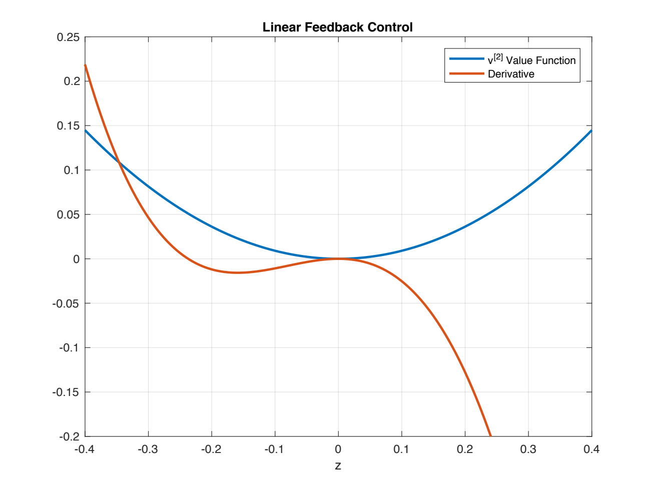
We consider replacing $u(t)$ with either

$$u(t) = \underbrace{k_1 z(t)}_{k^{[1]}(z(t))}, \qquad u(t) = k_1 z(t) + \underbrace{k_2 z^2(t)}_{k^{[2]}(z(t))}$$

or

$$u(t) = k_1 z(t) + k_2 z^2(t) + \underbrace{k_3 z^3(t)}_{k^{[3]}(z(t))}.$$

“Value Function” and its Derivative Along Solutions



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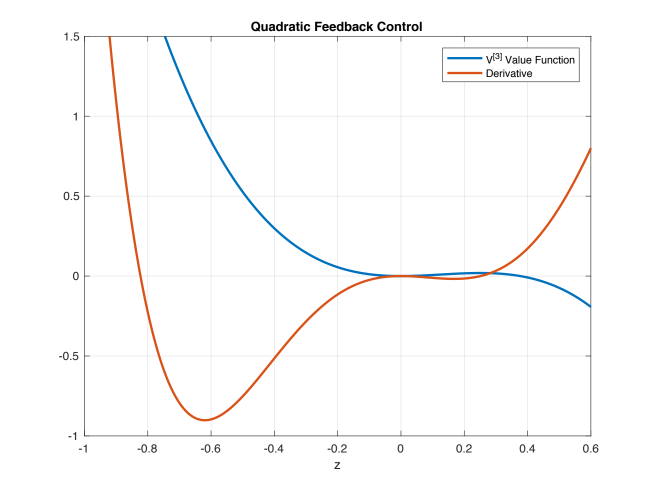

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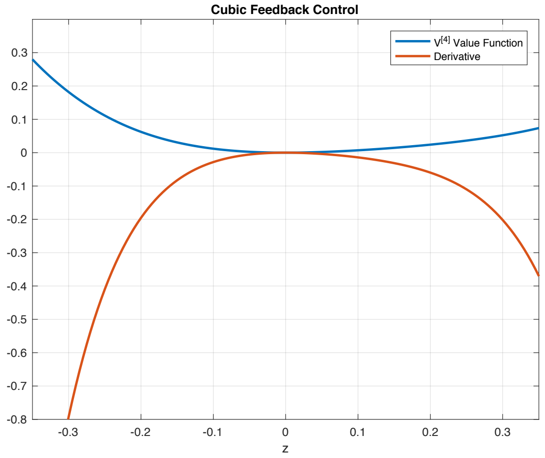
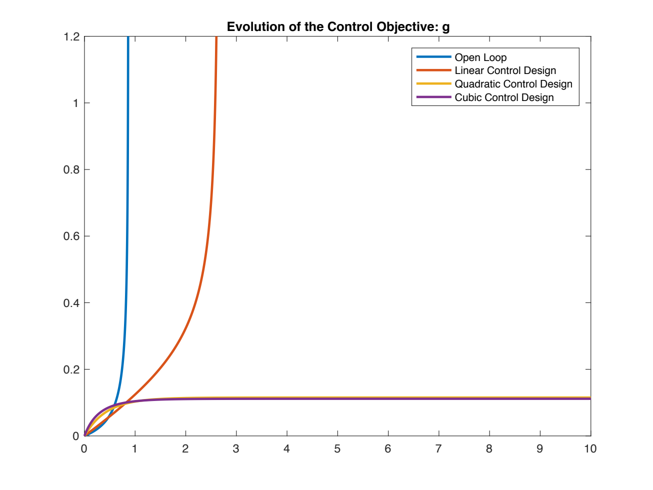
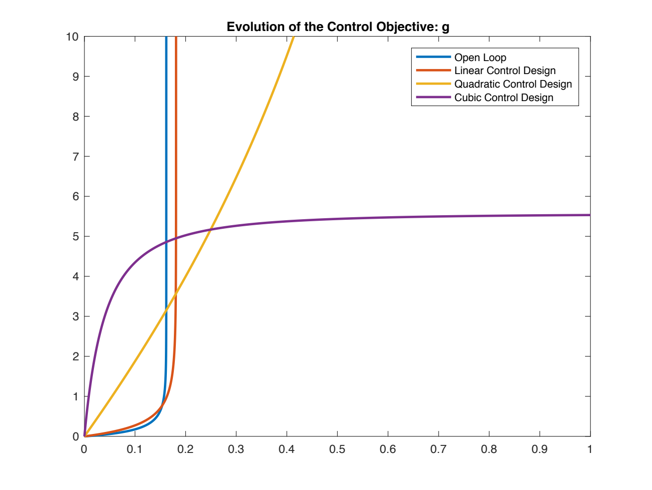


Evolution of System from $z_0 = -0.25$



Evolution of System from $z_0 = -0.95$



Rationale

Stability Region

While not true in general, the stability region often increases with the inclusion of higher degree feedback terms.

Performance

The performance benefit is only guaranteed locally (small z_0)

Goal

Approximate polynomial solutions to the HJB equations to provide these higher degree feedback terms.

It is important to develop efficient computational tools to find these approximating polynomial solutions for more realistic problems.

Outline for the Remainder of the Talk

- Optimal feedback control problem
- A polynomial approximation algorithm
- Simplification: the quadratic-quadratic regulator (QQR)
- Model problem: feedback control for Burgers equation
- Conclusions and future work

Recall the Optimal Control Problem

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$$0 = \frac{\partial v}{\partial z}(z) \frac{\partial f}{\partial u}(z, \mathcal{K}(z)) + \frac{\partial g}{\partial u}(z, \mathcal{K}(z)).$$

Quadratic-Quadratic Regulator

HJB PDEs

ON THE OPTIMAL STABILIZATION OF NONLINEAR SYSTEMS

(OB OPTIMAL'NOI STABILIZATSII NELINEINYKH SISTEM)

PMM Vol.25, No.5, 1961, pp. 836-844

E. G. AL'BREKHT
(Sverdlovsk)

(Received June 26, 1961)

The Nonlinear Systems Toolbox (Krener, 2015) has a routine `hjb.m` to approximate the feedback relation based on an algorithm by Al'brekht (PMM-Journal of Applied Mathematics and Mechanics, **25**:1254-1266, 1961).

$$0 = \frac{\partial v}{\partial z}(z) f(z, \mathcal{K}(z)) + g(z, \mathcal{K}(z)) \tag{1}$$

$$0 = \frac{\partial v}{\partial z}(z) \frac{\partial f}{\partial u}(z, \mathcal{K}(z)) + \frac{\partial g}{\partial u}(z, \mathcal{K}(z)). \tag{2}$$

Specializing Al'brekht's Algorithm for QQR

Assume expansions for v and $\mathcal{K}$ as

$$v(z) = \underbrace{\mathbf{v}_2(z \otimes z)}_{v^{[2]}(z)} + \underbrace{\mathbf{v}_3(z \otimes z \otimes z)}_{v^{[3]}(z)} + \underbrace{\mathbf{v}_4(z \otimes z \otimes z \otimes z)}_{v^{[4]}(z)} + \cdots$$

$$\mathcal{K}(z) = \underbrace{\mathbf{k}_1 z}_{k^{[1]}(z)} + \underbrace{\mathbf{k}_2(z \otimes z)}_{k^{[2]}(z)} + \underbrace{\mathbf{k}_3(z \otimes z \otimes z)}_{k^{[3]}(z)} + \cdots$$

as well as the quadratic expressions for f and g

$$f(z,u) = Az + Bu + N(z \otimes z)$$

$$g(z,u) = \mathbf{q}_2(z \otimes z) + \mathbf{r}_2(u \otimes u)$$

with $\mathbf{q}_2 = \mathrm{vec}(\mathbf{Q}_2)^T$, $\mathbf{Q}_2 = \mathbf{Q}_2^T \geq 0$ and $\mathbf{r}_2 = \mathrm{vec}(\mathbf{R}_2)^T$, $\mathbf{R}_2 = \mathbf{R}_2^T > 0$.

Substitute these expansions into the HJB PDEs (1)-(2), match terms of equal degree.

Al'brekht's Algorithm (cont.)

For example, collecting the degree two from (1) and degree one terms from (2), leads to

$$\mathbf{v}_2\left((\mathbf{A}+\mathbf{B}\mathbf{k}_1)\otimes\mathbf{I}_n+\mathbf{I}_n\otimes(\mathbf{A}+\mathbf{B}\mathbf{k}_1)\right)+\mathbf{q}_2(\mathbf{I}_n\otimes\mathbf{I}_n)+\mathbf{r}_2(\mathbf{k}_1\otimes\mathbf{k}_1)=\mathbf{0}.$$

and

$$\mathbf{v}_2(\mathbf{B}\otimes\mathbf{I}_n+\mathbf{I}_n\otimes\mathbf{B})+\mathbf{r}_2(\mathbf{k}_1\otimes\mathbf{I}_m+\mathbf{I}_m\otimes\mathbf{k}_1)=\mathbf{0}.$$

These can be rearranged as

$$\begin{aligned}(\mathbf{A}+\mathbf{B}\mathbf{k}_1)^T\mathbf{V}_2+\mathbf{V}_2(\mathbf{A}+\mathbf{B}\mathbf{k}_1)+\mathbf{k}_1^T\mathbf{R}_2\mathbf{k}_1+\mathbf{Q}_2&=\mathbf{0}\\ \mathbf{V}_2\mathbf{B}+\mathbf{k}_1^T\mathbf{R}_2&=\mathbf{0}\end{aligned}$$

and upon substitution of $\mathbf{k}_1$ into the first equation,

$$\begin{aligned}\mathbf{A}^T\mathbf{V}_2+\mathbf{V}_2\mathbf{A}-\mathbf{V}_2\mathbf{B}\mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{V}_2+\mathbf{Q}_2&=\mathbf{0}\\ \mathbf{k}_1&=-\mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{V}_2.\end{aligned}$$

Thus $\mathbf{V}_2$ solves the algebraic Riccati equation and $\mathbf{k}_1$ is the familiar solution to the linear-quadratic regulator problem.

Al'brekht's Algorithm (cont.)

Let $\mathbf{A}_c = \mathbf{A} + \mathbf{B}\mathbf{k}_1$. Collecting degree three terms from (1)

$$\mathbf{v}_3\left(\mathbf{A}_c \otimes \mathbf{I}_n \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{A}_c \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{I}_n \otimes \mathbf{A}_c\right)$$
$$= -\mathbf{v}_2\left((\mathbf{N} + \mathbf{B}\mathbf{k}_2) \otimes \mathbf{I}_n + \mathbf{I}_n \otimes (\mathbf{N} + \mathbf{B}\mathbf{k}_2)\right) - \mathbf{r}_2\left(\mathbf{k}_1 \otimes \mathbf{k}_2 + \mathbf{k}_2 \otimes \mathbf{k}_1\right).$$

and the degree two terms from (2)

$$\mathbf{v}_3(\mathbf{B} \otimes \mathbf{I}_{n^2} + \mathbf{I}_{n^2} \otimes \mathbf{B}) + \mathbf{r}_2(\mathbf{k}_2 \otimes \mathbf{I}_m + \mathbf{I}_m \otimes \mathbf{k}_2) = \mathbf{0}.$$

Al'brekht's Algorithm (cont.)

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$$= -\mathbf{v}_2 \left((\mathbf{N} + \mathbf{B}\mathbf{k}_2) \otimes \mathbf{I}_n + \mathbf{I}_n \otimes (\mathbf{N} + \mathbf{B}\mathbf{k}_2) \right) - \mathbf{r}_2 \left(\mathbf{k}_1 \otimes \mathbf{k}_2 + \mathbf{k}_2 \otimes \mathbf{k}_1 \right).$$

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Recall the degree one terms from the previous page:

$$\mathbf{v}_2(\mathbf{B} \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{B}) + \mathbf{r}_2(\mathbf{k}_1 \otimes \mathbf{I}_m + \mathbf{I}_m \otimes \mathbf{k}_1) = \mathbf{0}.$$

and identify all of the $\mathbf{k}_2$ terms in the top equation

Al'brekht's Algorithm (cont.)

Let $\mathbf{A}_c = \mathbf{A} + \mathbf{B}\mathbf{k}_1$. Collecting degree three terms from (1)

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$$-\mathbf{v}_2(\mathbf{B}\mathbf{k}_2 \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{B}\mathbf{k}_2) - \mathbf{r}_2(\mathbf{k}_1 \otimes \mathbf{k}_2 + \mathbf{k}_2 \otimes \mathbf{k}_1)$$
$$= \left(-\mathbf{v}_2(\mathbf{B} \otimes \mathbf{I}_n) - \mathbf{r}_2(\mathbf{I}_m \otimes \mathbf{k}_1)\right)(\mathbf{k}_2 \otimes \mathbf{I}_n) + \cdots$$

Al'brekht's Algorithm (cont.)

Let $\mathbf{A}_c = \mathbf{A} + \mathbf{B}\mathbf{k}_1$. Collecting degree three terms from (1)

$$\mathbf{v}_3 \left(\mathbf{A}_c \otimes \mathbf{I}_n \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{A}_c \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{I}_n \otimes \mathbf{A}_c \right) \\ = -\mathbf{v}_2 \left((\mathbf{N} + \mathbf{B}\mathbf{k}_2) \otimes \mathbf{I}_n + \mathbf{I}_n \otimes (\mathbf{N} + \mathbf{B}\mathbf{k}_2) \right) - \mathbf{r}_2 \left(\mathbf{k}_1 \otimes \mathbf{k}_2 + \mathbf{k}_2 \otimes \mathbf{k}_1 \right).$$

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So **all** of the $\mathbf{k}_2$ terms in the top equation vanish, and eqns decouple.

The first equation can be solved for $\mathbf{v}_3$, then inserted into the second equation to compute $\mathbf{k}_2$. This pattern continues...

Simplified Description of the Al'brekht algorithm

Define the special Kronecker sum as

$$\mathcal{L}_d(\mathbf{X}) \equiv \underbrace{\mathbf{X} \otimes \cdots \otimes \mathbf{I}_n}_{d \text{ terms}} + \cdots + \underbrace{\mathbf{I}_n \otimes \cdots \otimes \mathbf{X}}_{d \text{ terms}}.$$

Then we can write the higher degree terms in the value function as

$$\mathcal{L}_3(\mathbf{A}_c^T)\mathbf{v}_3^T = -\mathcal{L}_2(\mathbf{N}^T)\mathbf{v}_2^T. \tag{3}$$

$$\mathcal{L}_4(\mathbf{A}_c^T)\mathbf{v}_4^T = -\mathcal{L}_3((\mathbf{B}\mathbf{k}_2 + \mathbf{N})^T)\mathbf{v}_3^T - (\mathbf{k}_2^T \otimes \mathbf{k}_2^T)\mathbf{r}_2^T, \tag{4}$$

$$\begin{aligned} \mathcal{L}_5(\mathbf{A}_c^T)\mathbf{v}_5^T = &- \mathcal{L}_4((\mathbf{B}\mathbf{k}_2 + \mathbf{N})^T)\mathbf{v}_4^T - \mathcal{L}_3((\mathbf{B}\mathbf{k}_3)^T)\mathbf{v}_3^T \\ &- (\mathbf{k}_2 \otimes \mathbf{k}_3 + \mathbf{k}_3 \otimes \mathbf{k}_2)^T \mathbf{r}_2^T. \end{aligned} \tag{5}$$

and for all of these...

$$\mathbf{k}_d = -\frac{1}{2}\mathbf{R}_2^{-1}\left(\mathcal{L}_{d+1}(\mathbf{B}^T)\mathbf{v}_{d+1}^T\right).$$

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- $\mathbf{A}_c$ is a stable matrix, by the above, the eigenvalues of $\mathcal{L}_d(\mathbf{A}_c)$ are sums of the eigenvalues of $\mathbf{A}_c$. (these systems are solvable for $\mathbf{v}_d$)
- A block recursive algorithm by Chen and Kressner, which is suitable for more general Kronecker sum systems, can be used here. (the complexity is just $\approx O(n^{d+1})$ work)
- The assembly of the RHS can also be performed efficiently (products of $\mathcal{L}_d \mathbf{v}_d$)
- Software is available on Github (github.com/jborggaard/QQR)

Burgers

Nonlinear Feedback for Burgers Equation

Find $u(\cdot)$ that minimizes

$$J(z,u) = \int_0^\infty \left(\int_0^1 z^2(x,t)\; dx + u^T(t) R u(t) \right)\; dt$$

subject to

$$\dot{z}(x,t) = \epsilon z_{xx}(x,t) - \frac{1}{2}\left(z^2(x,t)\right)_x + \sum_{k=1}^{m} \chi_{[(k-1)/m,k/m]}(x) u_k(t)$$

$$z(\cdot,0) = z_0(\cdot) \in H^1_{\mathrm{per}}(0,1).$$

Discretize with 14 linear FE ($n = 14$), $m = 6$, and take $\epsilon = 0.005$.

Approximate the quadratic-quadratic regulator to compute the control.

Open Loop

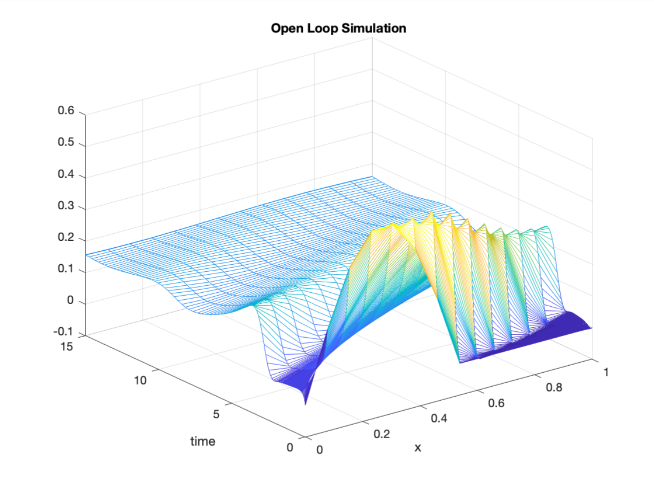
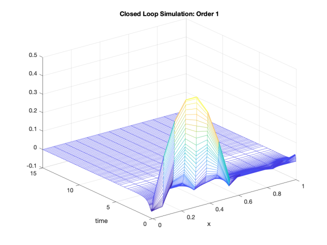
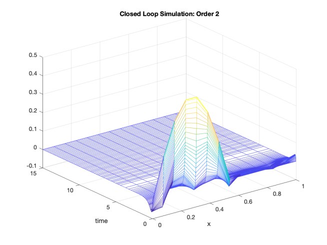


Figure 1: Open Loop Cost to $t = 15$ is 0.215602

Closed Loop Simulation, $d = 1$ and $d = 2$



Closed Loop Simulation: Cost is 3.09792e-03



Closed Loop Simulation: Cost is 3.09443e-03

Similar results for $d = 3$ (Cost is 3.07617e-03).

Recursive Blocked Solvers for Kronecker Sum Systems

For $m = 3$ on this laptop with:

$d = 3$ feedback law for order $n = 64$ computed in 130.2 seconds

$d = 4$ feedback law for order $n = 32$ computed in 182.9 seconds

$d = 5$ feedback law for order $n = 20$ computed in 156.3 seconds

van der Pol oscillators

Cubic-Quadratic Regulator Problems

- As a second test case, we consider controlling a ring of van der Pol oscillators.

$$\ddot{y}_i + (y_i^2 - 1)\dot{y}_i + y_i = y_{i-1} - 2y_i + y_{i+1} + b_i u_i(t),$$

for $i = 1, \ldots, g$ with $y_i(0) = 0.3$ and $\dot{y}_i(0) = 0$.

- We identify $y_{g+1} = y_1$ and $y_g = y_0$ to close the ring.
- The stability of this system was studied in Nana and Woafo 2006 and a related control problem considered in Barron 2016.
- Choosing different values of g and rewriting as a first-order system of differential equations allows us to study the *cubic-quadratic* regulator problem for problems of size $n = 2g$.
- We set b_i as 0 or 1 with $m = \|\mathbf{b}\|_1$.

Convergence of the Value Function with Increasing Degree

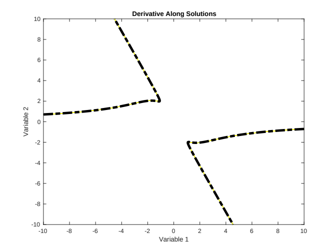
Experiment: $g = 4,\ b_1 = b_2 = 1$.

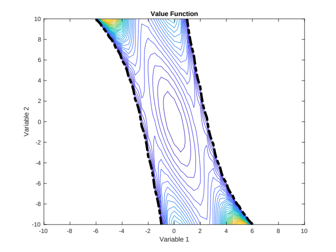
Table 1: van der Pol: Value Function Approx.

d	$\sum_{i=2}^{d+1} v^{[i]}(\mathbf{z}_0)$	$\int_0^{50} g(\mathbf{z}(t), \mathbf{u}(t))dt$
1	4.6380	4.4253
2	4.6380	4.4253
3	4.4125	4.4208
4	4.4125	4.4208
5	4.4246	4.4208
6	4.4246	4.4208
7	4.4242	4.4208

Improving Stability Region

Experiment: $g = 6,\ b_1 = b_2 = 1$.





Left: Derivative of $v^{[2]}$ along solutions ($v^{[2]}$ is positive definite)

Right: The value function $\sum_{d=2}^{5} v^{[d]}$ (the derivative along solutions is negative definite)

Conclusions and Future Work

Conclusions

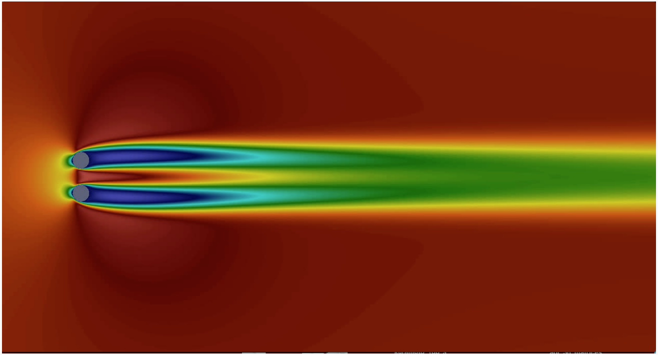
- Reasonably fast computation of low degree feedback for the quadratic-quadratic regulator problem.
- A similar structure appears in the cubic-quadratic regulator problem, etc.
- These allow for higher degree feedback computation with many mathematical models of interest, including flow control problems.

Future Work

- Add a special version for more general costs: $g(z, u)$.
- Develop quadratic model reduction for the flow control problem.
- Test this methodology on flow control problems for ability to expand stability region of closed loop system.

Extra Slides

$Re = 100$, $g = 1$ Case: Closed Loop From $t = 10$



Influence of Controlled Nodes

8 oscillators and 4 controls

Table 2: van der Pol: Value Function Approx.

nodes	linear	cubic	quintic
(1,2,3,4)	77.9977	blow-up	75.7120
(1,2,3,5)	29.9355	29.1139	29.0181
(1,2,3,6)	8.3986	8.3910	8.3910
(1,2,4,5)	29.4803	28.6854	28.5952
(1,2,4,6)	7.7364	7.7293	7.7292
(1,2,4,7)	6.9549	6.9489	6.9489
(1,2,5,6)	8.8505	8.8417	8.8417