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AM Session

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BASIC NOTIONS IN COTANGENT SCHUBERT CALCULUS
INTRO WORKSHOP ON COMBINATORIAL ALGEBARIC GEOMETRY
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PROBLEMS

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These problems use the notations, and refer to notions from my lecture "Basic notions in cotangent Schubert Calculus" at CIRM 2021. Lecture notes are available upon request. The meaning and relevance of the statements made in these problems are also explained in that lecture.

**Problem 1** (*on* $H^*_T(\mathbb{P}^1)$)
Consider the ring homomorphism
\[ \text{Loc} : \mathbb{Z}[t, z_1, z_2] \to \mathbb{Z}[z_1, z_2] \oplus \mathbb{Z}[z_1, z_2] \]
\[ f(t, z_1, z_2) \mapsto (f(z_1, z_1, z_2), f(z_2, z_1, z_2)). \]
Prove the following two characterizations of the image (range) of Loc:
\[ \text{Im}(\text{Loc}) = \{ (f_1(z_1, z_2), f_2(z_1, z_2)) \in \mathbb{Z}[z_1, z_2] \oplus \mathbb{Z}[z_1, z_2] : f_1(u, u) = f_2(u, u) \}, \]
\[ \text{Im}(\text{Loc}) = \{ (f_1(z_1, z_2), f_2(z_1, z_2)) \in \mathbb{Z}[z_1, z_2] \oplus \mathbb{Z}[z_1, z_2] : (z_1 - z_2)|(f_1 - f_2) \}. \]

**Problem 2** (*on* equivariant Schubert classes in $H^*_T(\mathbb{P}^{n-1})$)
Let $j \leq n$ be non-negative integers. Invent (that is, give a formula for) a polynomial $f(t, z_1, z_2, \ldots, z_n)$ such that
\begin{itemize}
  \item $f$ is of homogeneous degree $n - j$ (where $\deg t = \deg z_i = 1$);
  \item $f(z_j, z_1, z_2, \ldots, z_n) = \prod_{i=j+1}^n(z_i - z_j)$;
  \item $f(z_i, z_1, z_2, \ldots, z_n) = 0$ if $j < i \leq n$.
\end{itemize}

**Problem 3** (*on* equivariant CSM classes in $H^*_T(\mathbb{P}^{n-1})$)
Let $j \leq n$ be non-negative integers. Invent a polynomial $f(t, z_1, z_2, \ldots, z_n, h)$ such that
\begin{itemize}
  \item $f$ is of homogeneous degree $n - 1$ (where $\deg t = \deg z_i = \deg h = 1$);
  \item $f(z_j, z_1, z_2, \ldots, z_n) = \prod_{i=1}^{j-1}(z_i - z_j + h) \prod_{i=j+1}^n(z_i - z_j)$;
  \item $f(z_i, z_1, z_2, \ldots, z_n) = 0$ if $j < i \leq n$;
  \item $f(z_i, z_1, z_2, \ldots, z_n)$ is divisible by $h$ for $i < j$;
  \item $f(z_i, z_1, z_2, \ldots, z_n)$ is divisible by $\prod_{s=1}^{j-1}(z_s - z_i + h)$.
\end{itemize}
**Problem 4** (on the MacPherson property of CSM classes)
Consider the polynomial $f$ you defined in Problem 3, and let us call it $f_{j,n}$. Define $F_n = \sum_{j=1}^{n} f_{j,n}$. Show that $F_n|_{t=z_i}$ is a product of linear factors, for all $n$ and $i$.

**Problem 5** (on equivariant Littlewood-Richardson coefficients on $\mathbb{P}^1$)
In the lecture we saw that in $H^*_T(\mathbb{P}^1)$ we have
\begin{align*}
[\Omega_1] &= (z_2 - z_1, 0), \\
[\Omega_2] &= (1, 1).
\end{align*}
Calculate the products $[\Omega_i] \cdot [\Omega_j]$ as $\mathbb{Z}[z_1, z_2]$-linear combinations of $[\Omega_1]$ and $[\Omega_2]$.

**Problem 6** (on CSM versions of equivariant Littlewood-Richardson coefficients on $\mathbb{P}^1$)
In the lecture we saw that in $H^*_T(\mathbb{P}^1)$ we have
\begin{align*}
c^{\text{sm}}(\Omega_1) &= (z_2 - z_1, \quad 0), \\
c^{\text{sm}}(\Omega_2) &= (h, \quad z_1 - z_2 + \hbar).
\end{align*}
Calculate the products $c^{\text{sm}}(\Omega_i) \cdot c^{\text{sm}}(\Omega_j)$ as $\mathbb{Z}[z_1, z_2, \hbar]$-linear combinations of $c^{\text{sm}}(\Omega_1)$ and $c^{\text{sm}}(\Omega_2)$.

**Problem 7** (on the R-matrix property on CSM classes)
In the lecture we claimed that
\begin{equation}
\begin{pmatrix}
c^{\text{sm}}(\Omega_1^\text{opposite}) \\
c^{\text{sm}}(\Omega_2^\text{opposite}) \\
c^{\text{sm}}(\Omega_1^\text{opposite}_2) \\
c^{\text{sm}}(\Omega_2^\text{opposite}_2)
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{z_1 - z_2}{h} & \frac{h}{z_1 - z_2 + \hbar} & 0 \\
0 & \frac{z_1 - z_2 + \hbar}{z_1 - z_2} & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c^{\text{sm}}(\Omega_1^\text{opposite}) \\
c^{\text{sm}}(\Omega_1^\text{opposite}_2) \\
c^{\text{sm}}(\Omega_2^\text{opposite}) \\
c^{\text{sm}}(\Omega_2^\text{opposite}_2)
\end{pmatrix}
\end{equation}
(and that the occurring matrix satisfies the parameterized Yang-Baxter equation). Verify (1). Find the analogous matrix if we replace $c^{\text{sm}}(\Omega_i)$'s with $[\Omega_i]$'s.