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EXERCISES FOR THE ICERM CLUSTER ALGEBRA CLASS.

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(1) Mutate the following quiver at vertex 1. Alternatively, mutate the quiver at vertex 2.

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(2) Start with the following labelled seed and perform the following sequence of mutations: \( \mu_1, \mu_3, \mu_2, \mu_1, \mu_3, \mu_2, \mu_1, \mu_3, \mu_2, \mu_1, \mu_3, \mu_2, \). Compute the cluster variables you get at each step and make sure that they are Laurent polynomials in \( \{x_1, x_2, x_3\} \) with positive coefficients.

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(3) Verify that for any quiver \( Q \) and vertex \( k \), \( \mu_k^2(Q) = Q \).

(4) If \( T \) is a triangulation and \( T' \) is obtained by flipping at diagonal \( d \), then \( Q'_T = \mu_d(Q_T) \). (Try verifying in some examples, then prove it.)

(5) Prove that for any \( A \in Gr_{2,n} \) and for any \( i < j < k < \ell \),
\[ p_{ik}(A)p_{j\ell}(A) = p_{ij}(A)p_{k\ell}(A) + p_{i\ell}(A)p_{jk}(A). \]

(6) Draw the flip graph of the triangulations of a hexagon.

(7) (To do after the second lecture) Show that the rectangles seed gives a cluster structure on \( \mathbb{C}[Gr_{k,n}] \). More specifically:
- Show that if one mutates at any mutable cluster variable, the new cluster variable is a regular function which is coprime to the old cluster variable (so that one can apply the Starfish Lemma).
- Show that one can obtain any Plücker coordinate from the rectangles seed by an appropriate sequence of mutations.

(8) (To do after the second lecture) Although the equation
\[ P_{135}P_{246} - P_{134}P_{256} - P_{136}P_{245} - P_{123}P_{456} = 0 \]
does not lie in the ideal generated by the exchange relations, show that we can multiply it by a monomial in the Plücker coordinates so that the result lies in the exchange ideal.