Computational complexity of lattice problems and cyclic lattices

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Euclidean lattices

A **lattice** in Euclidean space $\mathbb{R}^n$ is a nonzero discrete subgroup. If $\Lambda \subset \mathbb{R}^n$ is a lattice, then there exist $\mathbb{R}$-linearly independent vectors

$$\mathbf{a}_1, \ldots, \mathbf{a}_k \in \Lambda, \ 1 \leq k \leq n,$$

called a **basis** for Λ, such that

$$\Lambda = \left\{ \sum_{i=1}^{k} m_i \mathbf{a}_i : m_i \in \mathbb{Z} \right\} = A\mathbb{Z}^k,$$

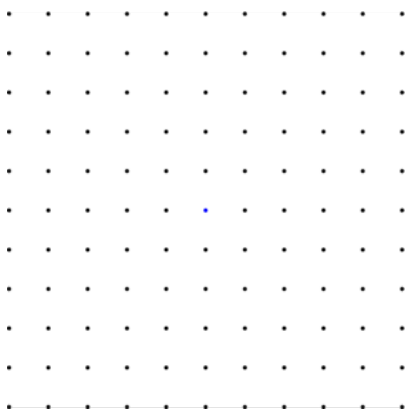
where

$$A = (\mathbf{a}_1 \ldots \mathbf{a}_k)$$

is the corresponding $n \times k$ **basis matrix**. Then k is called the **rank** of Λ, and $k = n$ if and only if the quotient group $\mathbb{R}^n/\Lambda$ is compact.

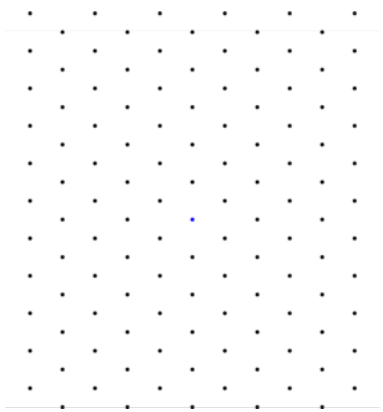
Examples of lattices in the plane

Square lattice



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2$$

Hexagonal lattice



$$\begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix} \mathbb{Z}^2$$

Determinant of a lattice

Determinant or **covolume** of a lattice $\Lambda = A\mathbb{Z}^k \subset \mathbb{R}^n$ is

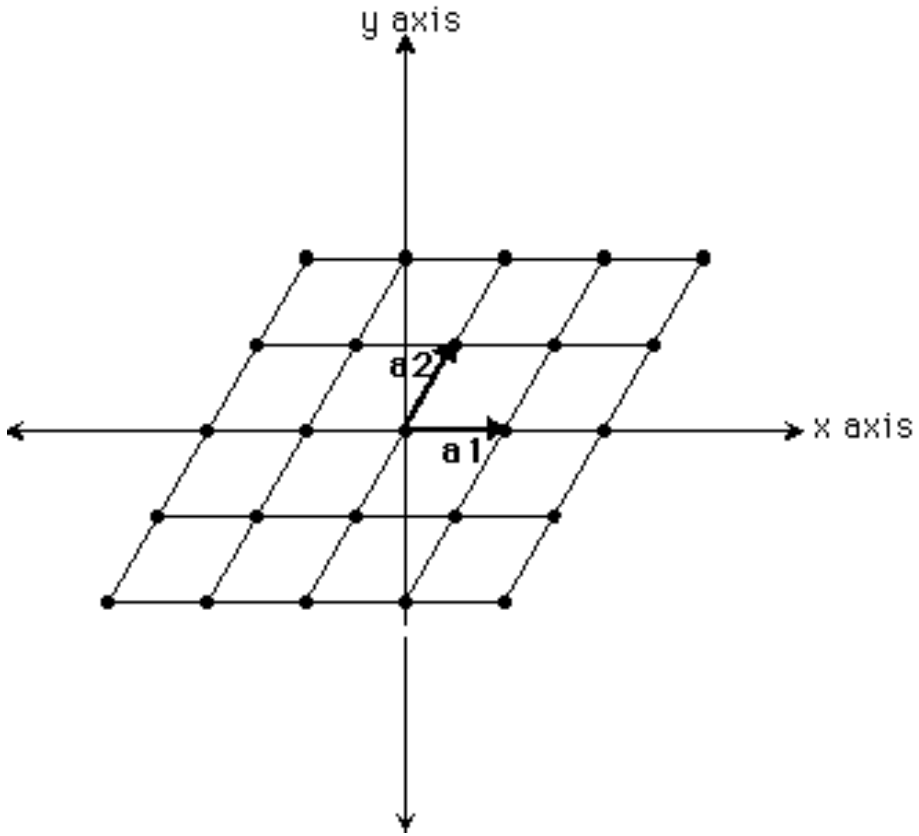
$$\sqrt{\det(A^t A)}.$$

This is equal to the volume of the compact quotient V/Λ, where

$$V = \operatorname{span}_{\mathbb{R}} \Lambda$$

is a k-dimensional subspace of $\mathbb{R}^n$.

Example of a fundamental domain



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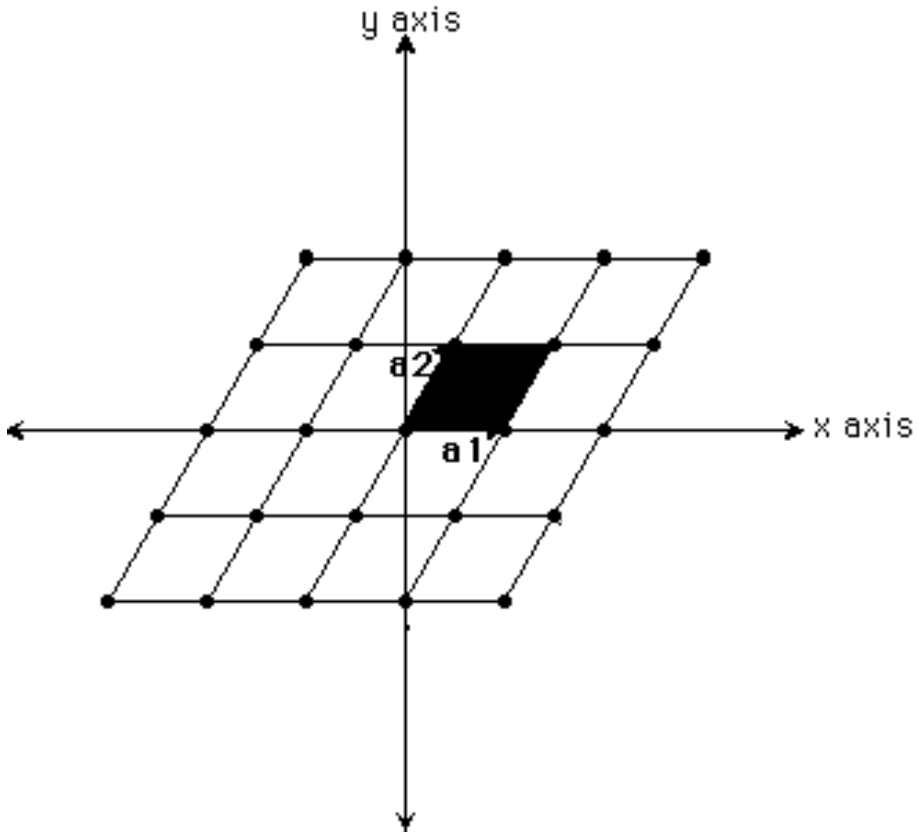

$$\text{Volume} = \det \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$$

Successive minima

Let $\mathbb{B}_n$ be a unit ball centered at the origin in $\mathbb{R}^n$. If $\Lambda \subset \mathbb{R}^n$ is a lattice of rank k, then its **successive minima**

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$$

are real numbers such that

$$\lambda_i \mathbb{B}_n \cap \Lambda$$

contains at least i linearly independent vectors for each $1 \leq i \leq k$ – we call these the **vectors corresponding to successive minima**. They are not necessarily unique, but there are finitely many of them.

Important remark

Vectors corresponding to successive minima do not necessarily form a basis for the lattice. For instance, the 5-dimensional lattice

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 \end{pmatrix} \mathbb{Z}^5$$

contains the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_5$, and hence

$$\lambda_1 = \cdots = \lambda_5 = 1,$$

however these vectors do not span Λ over $\mathbb{Z}$.

Lattice problems

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Definition 1 (Shortest Vector Problem – SVP)

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Output: A shortest nonzero vector in Λ, i.e. $\mathbf{x} \in \Lambda$ such that

$$\|\mathbf{x}\| = \min\{\|\mathbf{y}\| : \mathbf{y} \in \Lambda \setminus \{\mathbf{0}\}\},$$

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Remark 1

This is precisely a vector corresponding to λ_1, the first successive minimum.

Lattice problems

Definition 2 (Shortest Independent Vector Problem – SIVP)

Input: An $n \times n$ basis matrix A for a lattice $\Lambda = A\mathbb{Z}^n \subset \mathbb{R}^n$.
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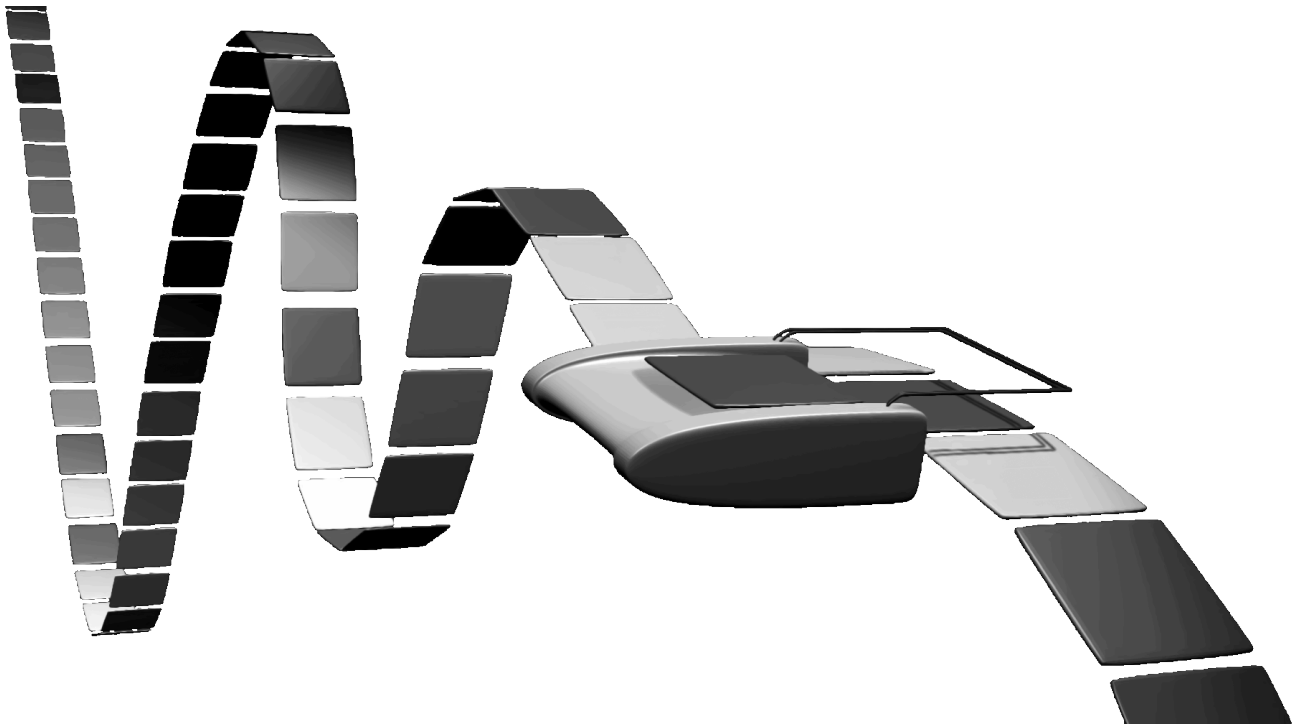
How much harder?

To answer this question, we need to explain how we measure "hardness".

Turing machine

Device with a **head** and an infinite **tape** going through it:



Elementary operations: read 1 cell, write 1 cell, move tape left 1 cell, move tape right 1 cell.

Example: a modern computer

Complexity classes: **P** and **NP**

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Definition 4

A problem is called **non-deterministic polynomial** if the number of elementary operations required to verify a potential answer for it on a Turing machine is polynomial in the size of the input. If this is the case, we say that the problem can be **verified in polynomial time**. The class of all such problems is denoted by **NP**.

More complexity: **NP**-hard and **NP**-complete

It is clear that every problem which can be solved in polynomial time, can be verified in polynomial time, and so

$$\mathbf{P} \subseteq \mathbf{NP}.$$

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A problem is called **NP-complete** if it is in **NP** and is **NP**-hard.

P vs **NP**: a million dollar problem

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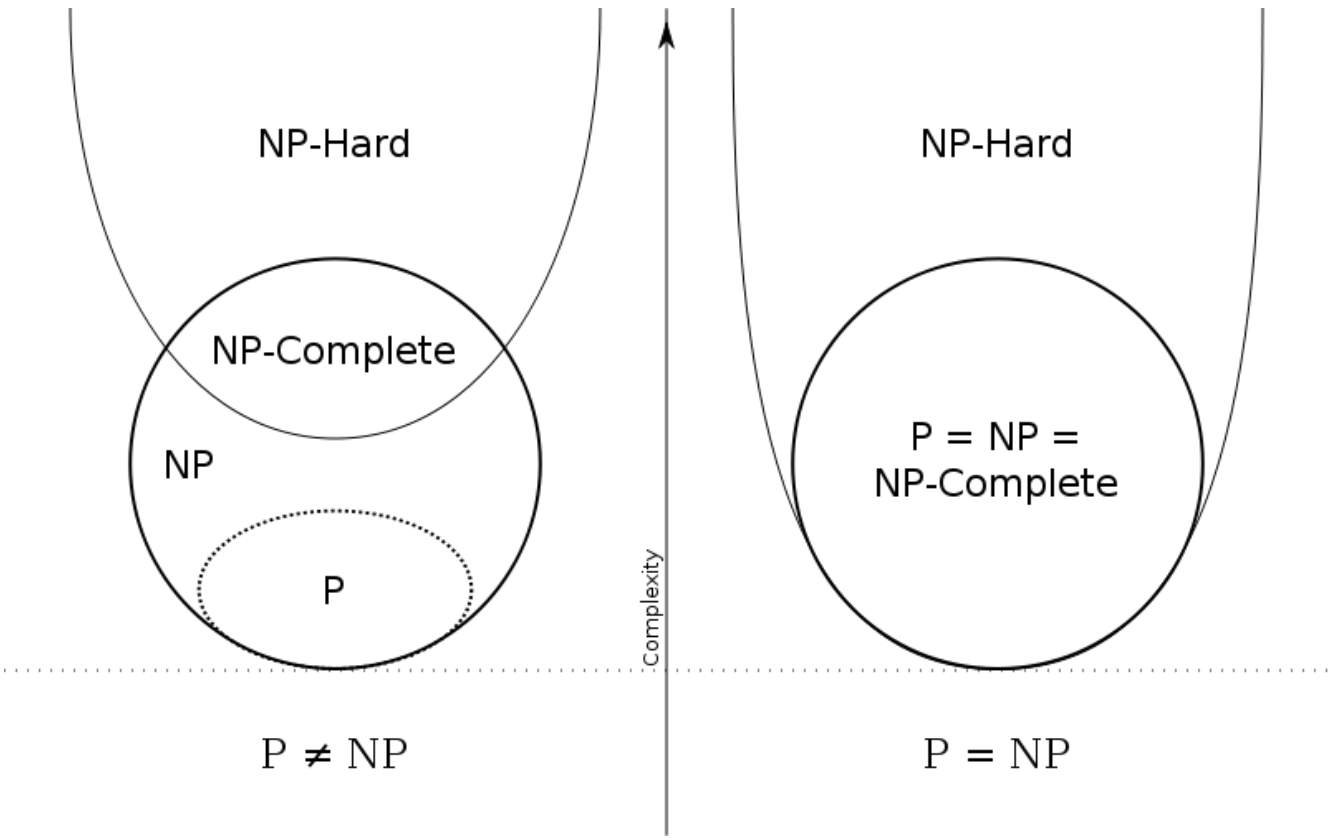

Complexity of lattice problems

SVP and SIVP are both known to be **NP**-hard. In fact, even the problem of finding the first successive minimum λ_1 (respectively, all successive minima $\lambda_1, \ldots, \lambda_n$) of a given lattice is **NP**-hard: it is as hard as SVP (respectively, SIVP).

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Theorem 1 (SIVP to SVP reduction)

For lattices of rank n, there exists a polynomial time reduction algorithm that, given an oracle for SVP, produces an approximate solution to SIVP within an approximation factor of $\sqrt{n}$ – that is, a collection of linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \Lambda$ with

$$\|\mathbf{a}_1\| \leq \|\mathbf{a}_2\| \leq \cdots \leq \|\mathbf{a}_n\| \leq \sqrt{n}\lambda_n.$$

Hard is good: cryptography connection

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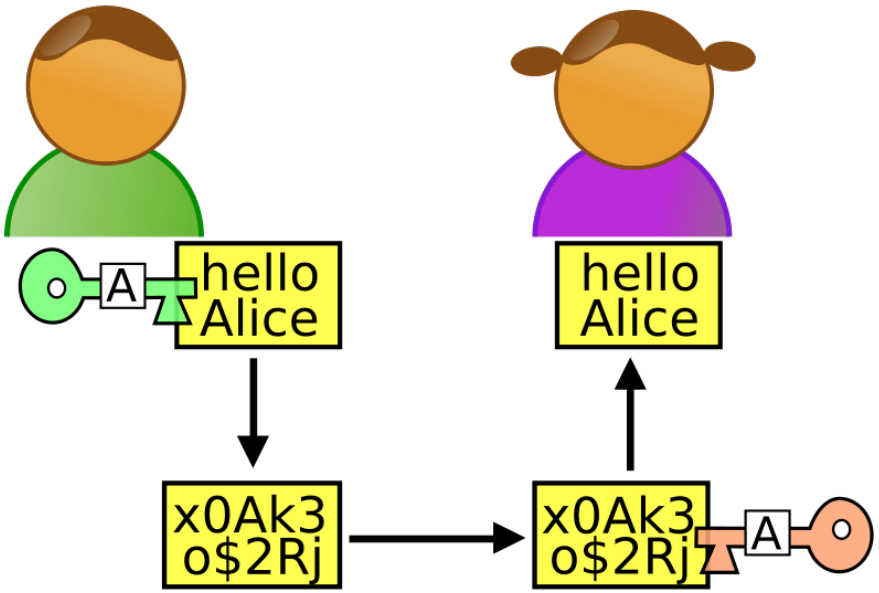

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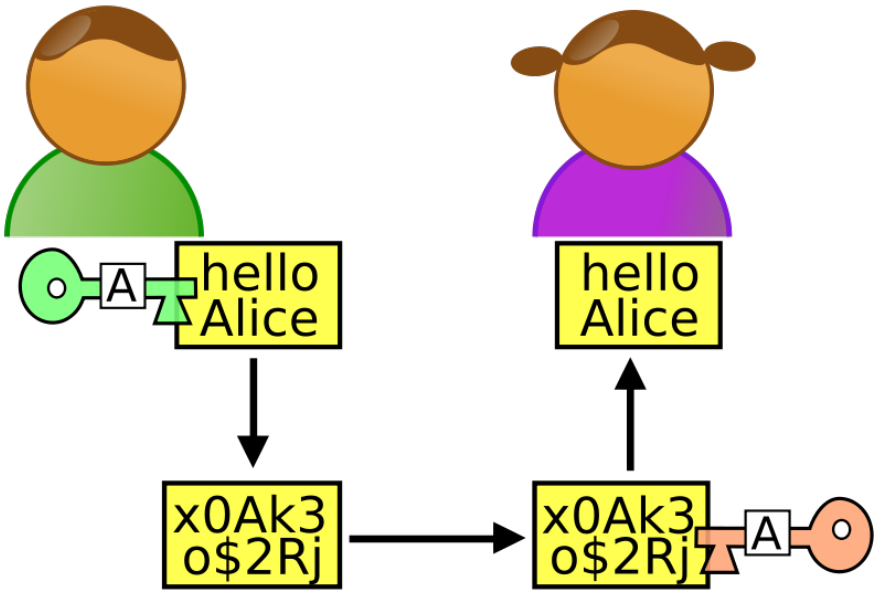

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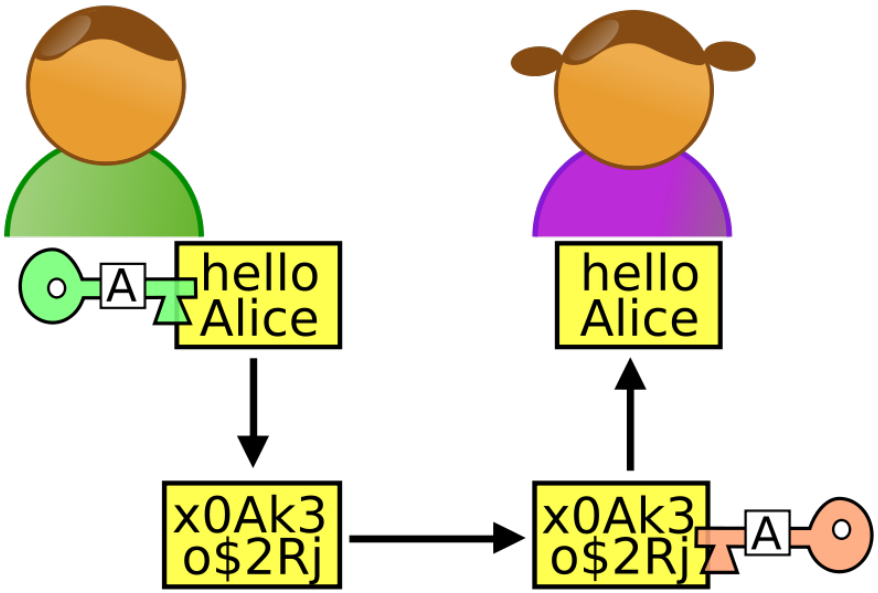

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Some possible choices: SVP, SIVP.

Encryption challenge

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Question 2

Are there lattices which can be described by the input data of size less than n^2?

Cyclic lattices: definition

Define the **rotational shift operator** on $\mathbb{R}^n$, $n \geq 2$, by

$$\mathrm{rot}(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1})$$

for every $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n$. We will write rot^k for iterated application of rot k times for each $k \in \mathbb{Z}_{>0}$ (then rot^0 is just the identity map, and $\mathrm{rot}^k = \mathrm{rot}^{n+k}$). It is also easy to see that rot (and hence each iteration rot^k) is a linear operator. A sublattice Γ of $\mathbb{Z}^n$ is called **cyclic** if $\mathrm{rot}(\Gamma) = \Gamma$, i.e. if for every $\mathbf{x} \in \Gamma$, $\mathrm{rot}(\mathbf{x}) \in \Gamma$. Clearly, $\mathbb{Z}^n$ itself is a cyclic lattice.

Cyclic lattices from ideals in $\mathbb{Z}[x]/(x^n - 1)$

Let

$$p(x) = \sum_{k=0}^{n-1} a_k x^k \in \mathbb{Z}[x]/(x^n - 1).$$

Define a map $\rho : \mathbb{Z}[x]/(x^n - 1) \to \mathbb{Z}^n$ by

$$\rho(p(x)) = (a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n,$$

then for any ideal $I \subseteq \mathbb{Z}[x]/(x^n - 1)$, $\rho(I)$ is a sublattice of $\mathbb{Z}^n$ of full rank. Notice that for every $p(x) \in I$,

$$xp(x) = a_{n-1} + a_0 x + a_1 x^2 + \cdots + a_{n-2} x^{n-1} \in I,$$

and so

$$\rho(xp(x)) = (a_{n-1}, a_0, a_1, \ldots, a_{n-2}) = \mathrm{rot}(\rho(p(x))) \in \rho(I).$$

In other words, $\Gamma \subseteq \mathbb{Z}^n$ is a cyclic lattice if and only if $\Gamma = \rho(I)$ for some ideal $I \subseteq \mathbb{Z}[x]/(x^n - 1)$.

Cyclic lattices in cryptosystems

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This motivates studying cyclic lattices more in depth.

Cyclic lattices: basic properties - 1

Definition 7

For a vector $\mathbf{a} \in \mathbb{Z}^n$, define

$$\Lambda(\mathbf{a}) = \text{span}_{\mathbb{Z}} \left\{ \mathbf{a}, \text{rot}(\mathbf{a}), \ldots, \text{rot}^{n-1}(\mathbf{a}) \right\}.$$

This is always a cyclic lattice.

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Lemma 2

Let $\mathbf{a} \in \mathbb{Z}^n$ and let $p_{\mathbf{a}}(x) \in \mathbb{Z}[x]/(x^n - 1)$ be a polynomial with coefficient vector $\mathbf{a}$. Then $\mathbf{a}, \text{rot}(\mathbf{a}), \ldots, \text{rot}^{n-1}(\mathbf{a})$ are linearly dependent if and only if $p_{\mathbf{a}}(x)$ is divisible by some cyclotomic polynomial divisor of $x^n - 1$.

Cyclic lattices: basic properties - 2

Let

$$C_R^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| := \max\{|x_1|, \ldots, |x_n|\} \leq R\}$$

for every $R \in \mathbb{R}_{>0}$, i.e. C_R^n is a cube of side-length $2R$ centered at the origin in $\mathbb{R}^n$.

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Lemma 3

Let $R > \frac{n-1}{2}$, *then*

$$Prob_{\infty,R}\left(\mathrm{rk}(\Lambda(\mathbf{a})) = n\right) \geq 1 - \frac{n}{2R+1},$$

where probability $\mathrm{Prob}_{\infty,R}(\cdot)$ *is with respect to the uniform distribution among all points* $\mathbf{a}$ *in the set* $C_R^n \cap \mathbb{Z}^n$.

Cyclic lattices: cryptographic use

Hence if we pick $\mathbf{a} \in \mathbb{Z}^n$ with large $|\mathbf{a}|$, the probability that

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is high, and the size of the input data necessary to describe this lattice is only n. This observation makes cyclic lattices very attractive for cryptographic purposes.

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But are cyclic lattices hard enough? In other words, are SVP, SIVP still **NP***-hard on cyclic lattices?*

This is an open question, but many people believe that the answer is **yes**, at least in the worst case.

SIVP to SVP on cyclic lattices

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Theorem 4 (Peikert, Rosen (2005))

Let n be a **prime** *and let $\Lambda \subset \mathbb{R}^n$ be a lattice of rank n. There exists a polynomial time algorithm that, given an oracle for SVP, produces an approximate solution to SIVP on Λ within an approximation factor of 2. In other words, given $\mathbf{a}_1 \in \Lambda$ with $\|\mathbf{a}_1\| = \lambda_1$ we can find a collection of linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \Lambda$ with*

$$\|\mathbf{a}_1\| \leq \|\mathbf{a}_2\| \leq \cdots \leq \|\mathbf{a}_n\| \leq 2\lambda_n$$

polynomial time. Moreover, only one call to the oracle is necessary.

Well-rounded lattices

More generally, we can show that for **every** n, SIVP is **equivalent** to SVP on a positive proportion of cyclic lattices. To explain what this means, we need more notation.

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A lattice $\Gamma \subset \mathbb{R}^n$ of rank n is called **well-rounded** (abbreviated WR) if

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Notice that for a WR lattice, finding λ_1 is equivalent to finding all successive minima.

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Theorem 5 (F., Sun (2013))

For each dimension $n \geq 2$, there exist real constants

$$0 < \alpha_n \leq \beta_n \leq 1,$$

depending only on n, such that

$$\alpha_n \leq \frac{\#\{\Gamma \in \mathcal{C}_n : \lambda_n(\Gamma) \leq R,\ \Gamma \text{ is WR}\}}{\#\{\Gamma \in \mathcal{C}_n : \lambda_n(\Gamma) \leq R\}} \leq \beta_n \text{ as } R \to \infty. \quad (1)$$

For instance, one can take $\alpha_2 = 0.261386...$ and $\beta_2 = 0.348652...$, meaning that between 26% and 35% of full rank cyclic sublattices of $\mathbb{Z}^2$ are WR.

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Corollary 6 (F., Sun (2013))

Let $R \in \mathbb{R}_{>0}$, *then*

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Corollary 7 (F., Sun (2013))

Let $k_1, \ldots, k_{n-1} \in \mathbb{Z}$ *be nonzero integers,* $m = \mathrm{lcm}(k_1, \ldots, k_{n-1})$, *and*

$$\mathbf{a} = \left(m, \frac{m}{k_1}, \ldots, \frac{m}{k_{n-1}}\right)^t \in \mathbb{Z}^n.$$

There exists an integer l, *depending only on* n, *such that whenever* $|k_1|, \ldots, |k_{n-1}| \geq l$, $SVP \equiv SIVP$ *on* $\Lambda(\mathbf{a})$.

Some of my work with students on WR lattices

WR lattices are important in discrete optimization, algebraic number theory, coding theory, cohomology computations of arithmetic groups, etc. Some of my additional recent work with graduate and undergraduate students on WR lattices includes:

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Thank you!