Complex Pisot Numbers and Newman Representatives

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Introduction

How small can roots get?

- **Theorem.** (Kronecker, 1857). Let $f(z) \in \mathbb{Z}[z]$ be irreducible and monic, with roots $\theta_1, \ldots, \theta_n$. If $|\theta_i| \leq 1$ for all i, then the θ_i are all cyclotomic factors.

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- **Def.** Let $f(z)$ be irreducible and monic, with roots θ_i. Then its Mahler measure is defined as $M(f) = \prod_{i=1}^{n} \max\{1, |\theta_i|\}$.
- **Conj.** There exists a $c > 1$ such that for all $f \in \mathbb{Z}[z]$, $M(f) < c$ implies $M(f) = 1$.

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- **Conj.** σ exists and is close to $\sqrt{2}$.

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III. A family of CPNs that we found

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IV. A computational approach to the Newman-division conjecture

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Note. $M(f) = \beta$ for f irreducible with Pisot root β.

Pisot Numbers

Theorem. (Hare & Mossinghoff, 2014): If β is a Pisot number with $\beta < \tau$, then there exists a Newman polynomial that has $-\beta$ as a root.

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- Let $S^{(1)}$ denote the set of limit points of S. For each $k \in \mathbb{N}$, let $S^{(k)}$ denote the set of limit points of $S^{(k-1)}$.
- **Theorem.** (Dufresnoy & Pisot, 1953): For all $k \in \mathbb{N}$, $S^{(k)}$ is nonempty.

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Solution. David Boyd (1978) designed an algorithm that can find all the Pisot numbers with arbitrary closeness to 2, i.e., for every $\delta > 0$, Boyd's algorithm enumerates all of $S \cap [1, 2 - \delta]$ in a finite amount of time. (The central ideas are due to Dufresnoy and Pisot.)

Complex Pisot Numbers

Analogizing to the complex realm

- **Def.** An algebraic integer β with modulus greater than 1 is a *complex Pisot number* if all its algebraic conjugates aside from $\overline{\beta}$ have modulus less than 1.

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Enumerating All the Complex Pisot Numbers

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- Looked up to degree 25 and found no complex Pisots above degree 16.

Dufresnoy and Pisot's fundamental theorem (1955).

Figure : Charles Pisot

Source: Classora, http://bit.ly/1qYBC1T

Enumerating All the Complex Pisot Numbers

This theorem can be adapted nicely to the complex realm:

Theorem. (Chamfy, 1958) If $f(z)$ is as in Dufresnoy and Pisot's theorem, except that $P(z)$ is complex Pisot, then for each coefficient sequence there exists an n_0 such that Dufresnoy and Pisot's theorem holds (modulo small details) for each $n > n_0$.

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- Our goal: Determine all complex Pisots of modulus less than $\sqrt{\tau} \approx 1.272$.
- Our strategy: Take advantage of hefty computing power to get around the algorithm's limitations.

Complex Pisot Families

It is well known that every Pisot number less than τ is a root of one of the following polynomials:

$$p_{2n}(z) = z^{2n+1} - z^{2n-1} - z^{2n-2} - \cdots - z - 1$$

$$q_{2n+1}(z) = z^{2n+1} - z^{2n} - z^{2n-2} - \cdots - z^2 - 1$$

$$r_n(z) = z^n(z^2 - z - 1) + z^2 - 1$$

$$g(z) = z^6 - 2z^5 + z^4 - z^2 + z - 1$$

Complex Pisot Families

The following families capture small negative Pisot numbers:

$$P'_n(z) = z^n(z^2 + z - 1) + 1,$$

$$Q'_n(z) = z^n(z^2 + z - 1) - 1,$$

$$R'_n(z) = z^n(z^2 + z - 1) + z^2 - 1,$$

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To create four families of complex Pisot numbers, we apply the following construction:

$$F_n(z) = F'_{\frac{n}{2}}(z^2).$$

Complex Pisot Families

In doing this, we obtain the following families:

$$P_n(z) = 1 - z^n + z^{n+2} + z^{n+4},$$

$$Q_n(z) = -1 - z^n + z^{n+2} + z^{n+4},$$

$$R_n(z) = -1 + z^4 - z^n + z^{n+2} + z^{n+4},$$

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Question. Are these families complex Pisot for all n?

Complex Pisot Families

Theorem. (Rouché's Theorem): Suppose that $f(z)$ and $g(z)$ are meromorphic functions defined in the simply connected domain D, that C is a simply closed contour in D, and that $f(z)$ and $g(z)$ have no zeros or poles for $z \in C$. If the strict inequality

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

holds for all $z \in C$, then $Z_f - P_f = Z_g - P_g$.

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- Consider $P_n(z) = 1 - z^n + z^{n+2} + z^{n+4}$, and $P_n^*(z) = 1 + z^2 - z^4 + z^{n+4}$, the reciprocal of $P_n(z)$.

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- To show this, we will utilize Rouché's theorem over a contour containing the unit circle.
- First, we must check that $P_n(z)$ has no roots on the unit circle other than $z = -1$ for $n \geq 3$. This result will follow if we can show that $P_n(z)$ has no reciprocal factors.

Complex Pisot Families

Suppose that $R(z) \neq z+1$ is a reciprocal factor of $P_n(z)$. Then,

$$R(z) \mid P_n^*(z) \text{ and } R(z) \mid P_n^*(z) - P_n(z).$$

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$R(z) \mid 1 + z^{n-2}$ for all $n \geq 3$.
Further, we assert that $R(z) \mid P_n^*(z) - z^6(1+z^{n-2})$, so $R(z) \mid (1+z^2)(1-z^4)$. Similarly, we can discard the $(1-z^4)$ factor. So we must have $R(z) \mid (1+z^2)$.

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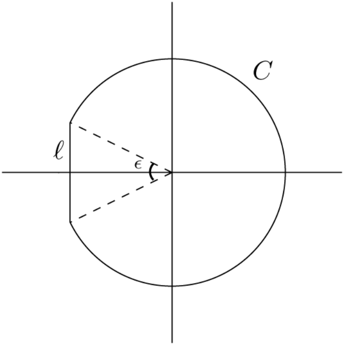
However, since $\pm i$ are not roots of $P_n(z)$ we have a contradiction.



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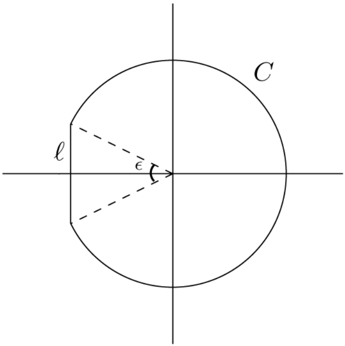
Complex Pisot Families



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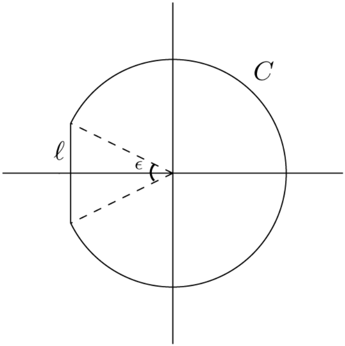
- Let $f(z) = -1 - z^2 + z^4$. Notice that $f(z)$ has exactly two roots inside of the contour.



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- Let $f(z) = -1 - z^2 + z^4$. Notice that $f(z)$ has exactly two roots inside of the contour.
- For z on the arc C we have that

$$|f(z) + P_n^*(z)| < |f(z)| + |P_n^*(z)|.$$

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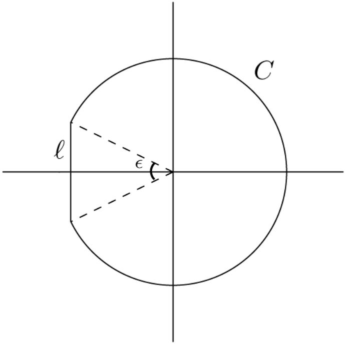
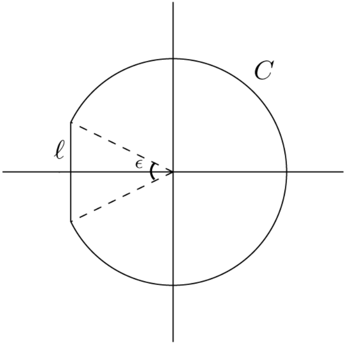


- Now, we must consider the chord ℓ for small $\epsilon > 0$.



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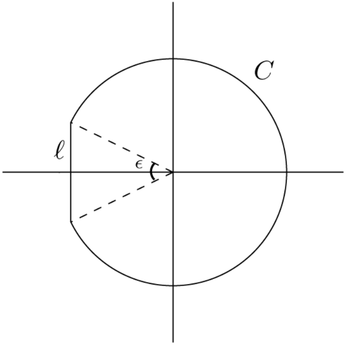

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- For values of z on ℓ, we have that $|f(z) + P_n^*(z)| = |z^{n+4}| < 1$
- By parameterizing ℓ, we are able to see that $|f(z)| > 1$.

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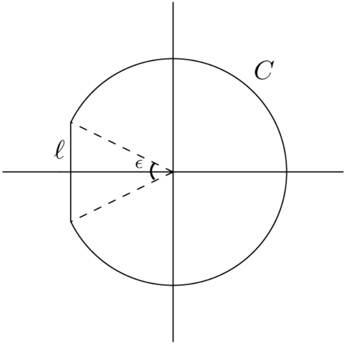

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  Hence, $P_n^*(z)$ has two roots inside of the unit circle, as desired.

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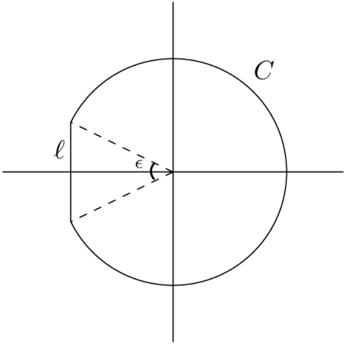

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  Hence, $P_n^*(z)$ has two roots inside of the unit circle, as desired.
- To show that these roots are non real, we use Descartes' rules of signs.

Theorem. The Mahler measure of $P_n(z)$, $Q_n(z)$, $R_n(z)$ and $S_n(z)$ is greater than τ for $n \geq 5$.

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To show this, we can utilize Rouché's theorem over the contour
$$\mathcal{C} = \left\{ z \in \mathbb{C} : |z| = \frac{1}{\sqrt{\tau}} \right\}.$$

Checking for Newman Divisibility

Testing a root β (*Hare & Mossinghoff*, 2014)

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- Idea: Construct only the polynomials that when evaluated at the algebraic integer β for $|\beta| > 1$ lie within a closed disk

$$I(\beta) = \left\{ z \in \mathbb{C} : |z| \leq \frac{|\beta|}{|\beta| - 1} \right\},$$

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$$I(\beta) = \left\{ z \in \mathbb{C} : |z| \leq \frac{|\beta|}{|\beta| - 1} \right\},$$

- If there is a Newman polynomial F and $F(\beta) \notin I(\beta)$, then:
  - $\beta F(\beta) \notin I(\beta)$ and
  - $\beta F(\beta) + 1 \notin I(\beta)$.

Newman Representatives

$$\mathcal{N}(\beta, d) = \left\{1,\ \beta,\ \beta + 1, \ldots,\ \beta^d + 1, \ldots,\ \beta^d + \cdots + 1\right\}$$

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(Hare & Mossinghoff, 2014; Garsia, 1962): In an analogous case for a *real Pisot number* $\beta < -1$

$$I(\beta) = \left[\frac{-1}{\beta^2-1}, \frac{-\beta}{\beta^2-1}\right], \text{ and}$$

the algorithm must terminate if:

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- or if, for some d, $\mathcal{N}(\beta, d) = \mathcal{N}(\beta, d+1)$.

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But if β is complex, does that happen?

Experimental evidence suggests so.

Newman Representatives

$f(z)$	Measure	Representable
$z^6 - z^5 - z^3 + z^2 + 1$	1.55601	No
$z^9 + z^5 - z^4 + z^3 - z^2 + 1$	1.55491	Yes
$z^9 - z^8 + z^7 - z^6 + z^4 - z + 1$	1.55111	Yes
$z^6 - z^5 + z^4 - z + 1$	1.48638	Yes
$z^7 - z^6 + z^5 - z + 1$	1.48415	Yes
$z^5 - z^4 + z^3 - z + 1$	1.34972	Yes
$z^3 - z^2 + 1$	1.33033	Yes

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- Polynomial of smallest complex Pisot number found without a Newman representative has measure 1.55601.
  This same result is noted by Hare & Mossinghoff (2014).

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- Polynomial of smallest complex Pisot number found without a Newman representative has measure 1.55601.
  This same result is noted by Hare & Mossinghoff (2014).
- Could this suggest a value σ such that if $M(f) < \sigma$ then $f \mid F$ for F a Newman polynomial?

Newman Representatives

Theorem. (Bloch & Pólya, 1932; Pathiaux, 1973). If $M(f) < 2$, then it has a height-one multiple.

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Conjecture. There exists $\sigma > 1$ such that for all polynomials f, if $M(f) < \sigma$, then $f \mid F$ for some $F \in \mathcal{N}$, where $\mathcal{N}$ denotes the set of Newman polynomials.

Newman Representatives

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We examined this conjecture experimentally in the following two ways:

- Exhaustive search of polynomials with Mahler measure less than 1.625 and degree at most 12
- Search of height one polynomials with Mahler measure less than 1.625 and degree at most 25

Exhaustive Degree 12 Search

$f(z)$	Measure
$1 - z^3 - z^4 - z^5 + z^7 + z^8 + z^9$	1.436
$1 - z^3 - z^4 - z^5 + z^8 + z^9 + z^{10}$	1.475
$1 - z^2 - z^3 + z^{10} + z^{11}$	1.477
$1 - z^2 - z^3 + z^5 - z^7 + z^9 + z^{10}$	1.481
$1 - z^2 - z^3 + z^8 + z^9$	1.483
$1 + z^1 - z^3 - z^4 - z^5 - z^6 - z^7 + z^9 + z^{10} + z^{11} + z^{12}$	1.504
$1 - z^2 - z^5 - z^6 + z^8 + z^{11} + z^{12}$	1.505
$1 + z^1 - z^3 - z^4 - z^5 - z^6 + z^9 + z^{10} + z^{11}$	1.509
$1 + z^1 - z^4 - 2z^5 - z^6 - z^7 + z^9 + z^{10} + z^{11} + z^{12}$	1.514
$1 - z^3 - z^5 - z^7 + z^{10} + z^{11} + z^{12}$	1.515

Exhaustive Degree 12 Search

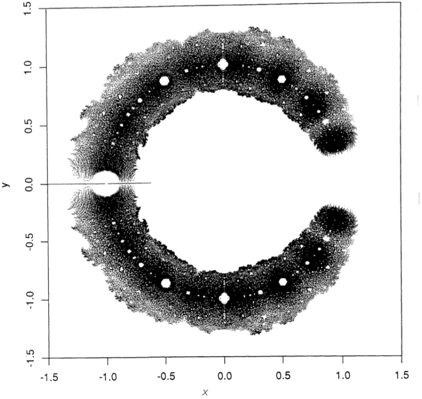
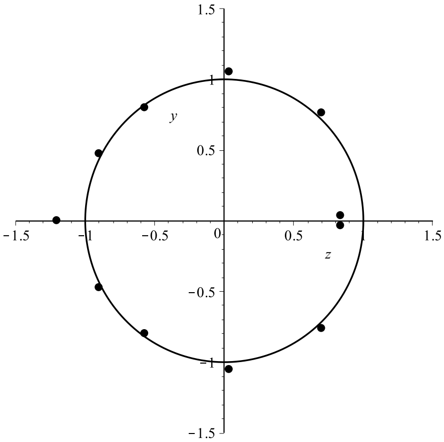


Figure : The roots of all Newman polynomials of degree at most 16 (Odlyzko & Poonen, 1993).

Exhaustive Degree 12 Search

Figure : Roots of $1 - z^2 - z^3 + z^{10} + z^{11}$



Degree 25 Search

Smallest Measures	
1.37905134385879	1.37943718419369
1.38079214221058	1.38080769188561
1.38081779528952	1.38088307116138
1.3808922392546	1.380902191116
1.38093742926323	1.3809897218323
1.38106730194978	1.38243483486449
1.3825870929398	1.38263986768736
1.382693053076	1.3827358846431
1.38286109073846	1.41940463238822
1.43287732589793	1.43663226063485

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Sanya

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Our dear colleagues

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