Equidecomposability and Period Collapse

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Overview

The Setting

- Motivation: counting integer lattice points in (rational) polytopes (discrete volume).
- Connections to representation theory, number theory, and toric geometry.
- We study polygons using linear recurrences, graph theory, and plane geometry.

The Natural Symmetries

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- The action of this group preserves discrete volume.
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- We consider the combined action of these two groups into $G = GL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2$.
  - If $g = U \rtimes v \in G$, then $gx := Ux + v$.
  - If P and Q are in the same G-orbit, they are said to be G-equivalent.

Theorem (Ehrhart)

Let P be a rational polygon of denominator d. The expression $\mathrm{ehr}_P(t) = |tP \cap \mathbb{Z}^2|$ is a quasi-polynomial of period d.

- Denominator d indicates the vertices are in $\frac{1}{d}\mathbb{Z} \times \frac{1}{d}\mathbb{Z}$. Suppose P is denominator 3.

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$$\mathrm{ehr}_P(t) = \begin{cases} f_1(t) & : t \equiv 1 \mod 3 \\ f_2(t) & : t \equiv 2 \mod 3 \\ f_3(t) & : t \equiv 3 \mod 3 \end{cases}$$

- The f_i are known as the *constituents* of the Ehrhart quasi-polynomial.

Period Collapse

- *Period collapse* occurs when $\mathrm{ehr}_P(t)$ has minimal period smaller than the denominator of P.

Theorem (McAllister—Woods)

Morally, P has period collapse 1 iff P satisfies Pick's formula:

$$A = i + \frac{b}{2} - 1.$$

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- In many examples, rational polygons with period collapse may be *cut and pasted* into integer polygons.

Equidecomposability: Definitions

Definition (Equidecomposability)

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1. $\mathcal{F}$ sends open faces (vertices, edges, facets, respectively) of $\mathcal{T}_1$ *bijectively* to open faces (vertices, edges, facets, respectively) of $\mathcal{T}_2$.
2. The restriction of $\mathcal{F}$ to a face of $\mathcal{T}_1$ is a G-map.

A Consequence and a Question

Remark

If P and Q are equidecomposable, then $\mathrm{ehr}_P(t) = \mathrm{ehr}_Q(t)$.

- Question (McAllister, Kantor): Is the converse true?

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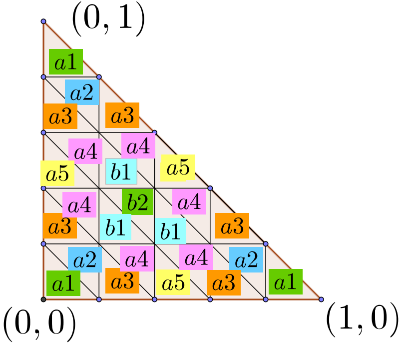
If P and Q are equidecomposable, then $\mathrm{ehr}_P(t) = \mathrm{ehr}_Q(t)$.

- Question (McAllister, Kantor): Is the converse true?
  - Answer: No, if we assume rational "cuts".
  - There exist denominator 5 polygons with the same Ehrhart quasi-polynomial that are *not* equidecomposable.

Classifying Minimal Triangles in $\frac{1}{d}\mathbb{Z} \times \frac{1}{d}\mathbb{Z}$: Definitions

Definition (*d-minimal* triangles)

We say that a denominator d triangle T is *d-minimal* if the only points of $\frac{1}{d}\mathbb{Z} \times \frac{1}{d}\mathbb{Z}$ contained in T occur at the vertices of T. In other words, it is a triangle in $\frac{1}{d}\mathbb{Z} \times \frac{1}{d}\mathbb{Z}$ and has area $\frac{1}{2d^2}$.

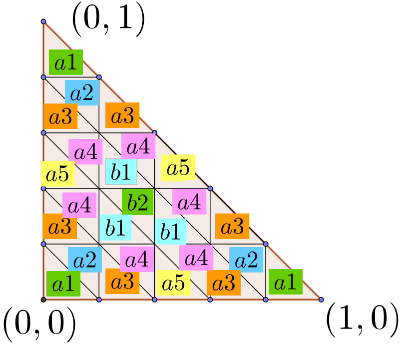


Classification under the actions of G

1. The first proposition we show is that any *d-minimal* triangle can be sent to a right triangle occurring in the unit square $[0,1] \times [0,1]$.
2. Next by observing the possible ways of transforming one right triangles to another, we obtain six matrices, and they form the dihedral group on 3 elements.

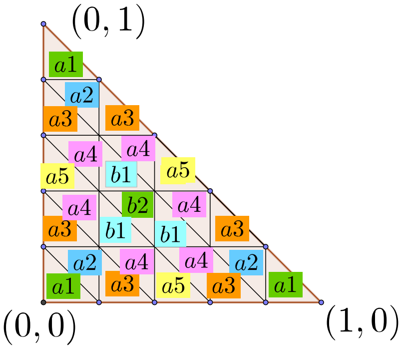
Classification under the actions of G

- Now we can analyze the distribution of *d-minimal* triangles in unit square in terms of actions by D_3.
- We obtain explicit formula of numbers of orbits of *d-minimal* triangles under G.



Invariants: Part i

- Can also define the *weight* of a *d*-minimal triangle.



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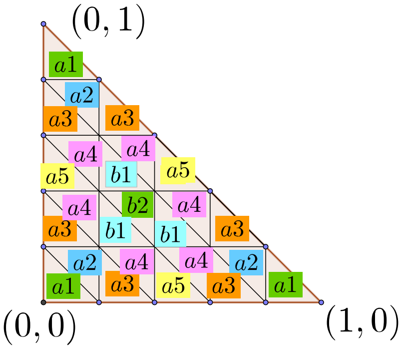
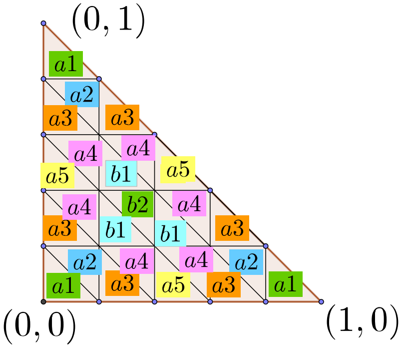


- Note: $a2$ and $a4$ have the same Ehrhart quasi-polynomial.

Invariants: Part i

Lemma

Two d-minimal triangles are G-equivalent iff they have the same weight.

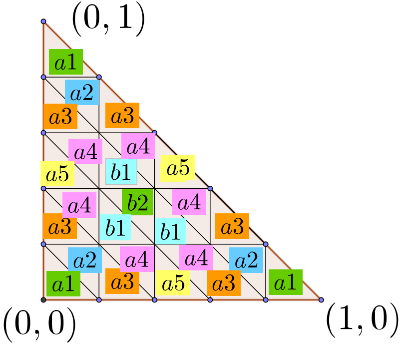


Invariants: Part i

Theorem

Two d-minimal triangles are equidecomposable iff they have the same weight (iff they are G-equivalent).

Proof idea: count the number of signed/unsigned occurences of a weighted edge in a triangulation $\mathcal{T}$ of T.



A Counterexample

Theorem

Two d-minimal triangles are equidecomposable iff they have the same weight (iff they are G-equivalent).

Corollary

Ehrhart equivalence does not imply (rational) equidecomposability.

- Triangles $a2$ and $a4$ have the same Ehrhart quasi-polynomial, do not have the same weight. Therefore they are not rationally equidecomposable.

Invariants: Part ii

- Can construct necessary and sufficient conditions for equidecomposability:
  1. An infinite family of labeled graphs g_d^P for each $d \in \mathbb{N}$ (d-FACES)
  2. An edge weighting system as before, but with an extra piece of information (EDGES)
  3. The Erhart quasi-polynomial (VERTICES)

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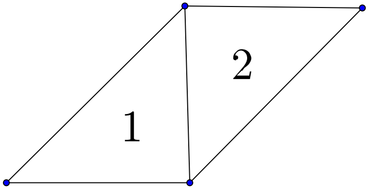
  3. The Erhart quasi-polynomial (VERTICES)
- If P and Q have the same d-FACE data (for some d), EDGE data, and VERTEX data, then P and Q are equidecomposable.

Example: Constructing g_6^P

Concept: flippable pairs.



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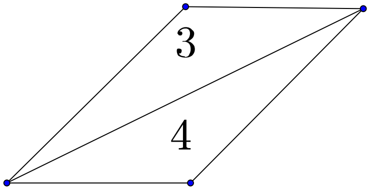
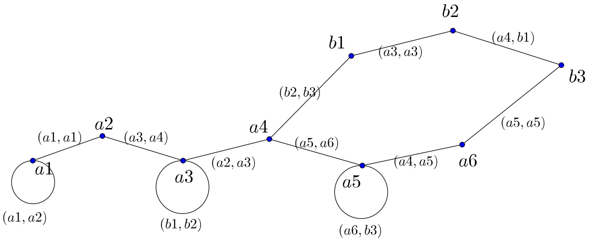


- This means the pair $(1, 2)$ flips to $(3, 4)$.

Example: Constructing g_6^P

- The following graph is a dictionary on how flippable pairs change ($d = 6$). This graph is used to construct g_d^P for any P. This is *not* the graph g_d^P.
  - Vertices: G-equivalence classes of d-minimal triangles.
  - Edges connect flippable pairs.
  - An edge's label tells the result of flipping the triangles represented by its endpoints.



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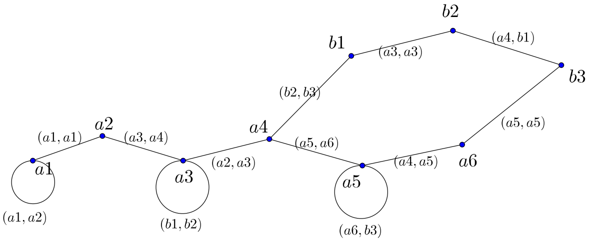
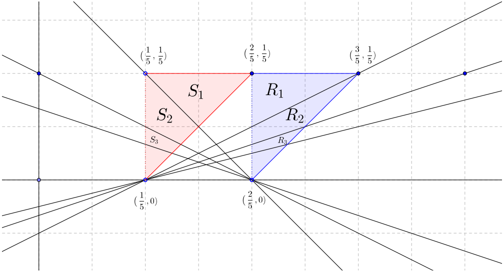


Infinite construction

- In lattice $\frac{1}{5}\mathbb{Z} \times \frac{1}{5}\mathbb{Z}$ we observe triangles $a2$ and $a4$ have the same Ehrhart quasi-polynomial but they are not equidecomposable. Surprisingly, there does however exist an infinite equidecomposability relation between these two triangles if we delete an edge from each triangle.

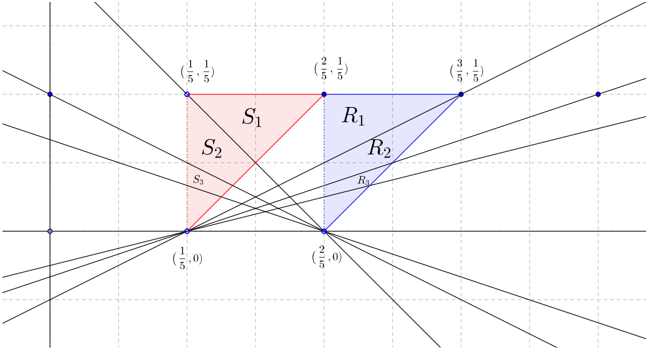
Infinite construction

1. Fix a point $(\frac{1}{5}, 0)$, connecting all $\frac{1}{5}\mathbb{Z} \times \frac{1}{5}\mathbb{Z}$ lattice points on $y = \frac{1}{5}$, cutting the blue triangles into set of pieces $\{R_i\}$, labeled as in the diagram. Do the same thing with $(\frac{2}{5}, 0)$ and cut the red triangle into set of pieces $\{S_i\}$.



Infinite construction

1. Let $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then the i^{th} power of U sends each S_i to R_i.



Another direction...

Now we switch to another direction. In this section we focus on some results we found from the algebraic side of the Ehrhart quasi-polynomial, and then we give a geometric interpretation. The goal is to use these observations to understand the phenomenon of period collapse.

"Explicit" formula for the Ehrhart quasi-polynomial

Theorem

Given a denominator D polygon P with area A, define i_s and b_s to be the number of lattice points in the interior and the boundary of s^{th} dilate of P. Then n^{th} constituent of the Ehrhart quasi-polynomial of P is given by

$$\mathrm{ehr}_P^{(n)}(t) = At^2 + \frac{-2\,DAn + AD^2 - i_{D-n} + i_n + b_n}{D}t + \frac{DAn^2 - nAD^2 + ni_{D-n} - ni_n - nb_n + Di_n + Db_n}{D}$$

simple proof

1. We first start with

$$\mathrm{ehr}_P^{(n)}(n) = An^2 + B_n n + C_n = i_n + b_n$$

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2. Next by reciprocity law, we have

$$\mathrm{ehr}_P^{(n)}(-(D-n)) = A(n-D)^2 + B_n(n-D) + C_n = i_{D-n}$$

3. Then we solve for B_n and C_n in terms of A, D, i_n, b_n and i_{k-n}.

Three linear recurrence relations

Theorem

Given a rational polygon P with area A, define i_s, b_s to be the number of lattice points in the interior and the boundary of s^{th} dilate of P, whose Ehrhart quasi-polynomial is given by $\mathrm{ehr}_P(t)$. Then P has period k if only if i_s, b_s, and $\mathrm{ehr}_P(t)$ satisfy the following three linear recurrence relation.

$$i)\mathrm{ehr}_P(t+2k) = 2\mathrm{ehr}_P(t+k) - \mathrm{ehr}_P(t) + 2Ak^2$$
$$ii)i_{t+2k} = 2i_{t+k} - i_t + 2Ak^2$$
$$iii)b_{t+2k} = 2b_{t+k} - b_t$$

some corollaries

Corollary

$2Ak^2$ is an integer.

Given the area and the denominator of the polygon, this corollary allows us to have some restraint for the choices of k.

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Corollary

d-minimal triangles do not experience period collapse for any d.

Geometric interpretation for the linear recurrence relation

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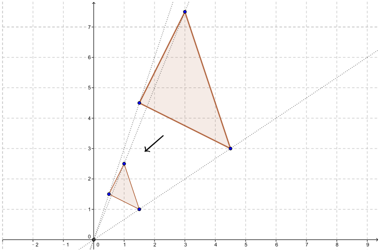
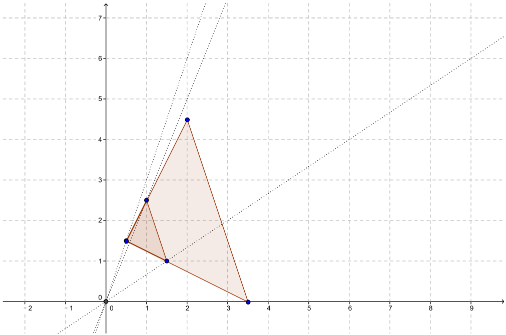
1. If the original triangle's vertices are A, B, C, with coordinate $\frac{1}{D}(A_x, A_y)$, $\frac{1}{D}(B_x, B_y)$, $\frac{1}{D}(C_x, C_y)$ respectively, A_i, B_i, C_i are integers. Then by a dilation of $(D+1)$, the point A will be sent to A', with coordinate $\frac{1}{D}(A_x, A_y) + (A_x, A_y)$.



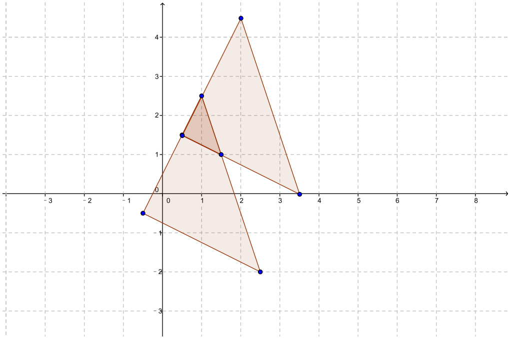
Geometric interpretation for the linear recurrence relation

- In the graph we have the original triangle and its $(1+D)^{th}$ dilate.

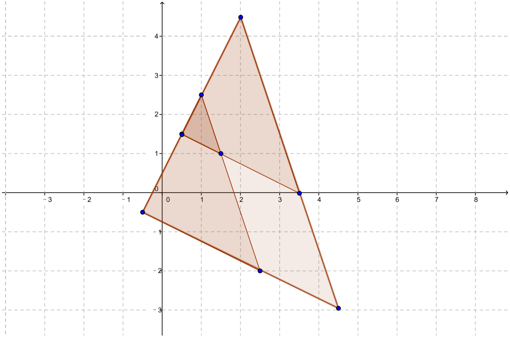
2. Since (A_x, A_y) is integer vector, we are able to move the bigger triangle so that the bigger one contains the small one and A' covers A.



3. Similarly, we construct another $(1+D)^{th}$ dilate that contains the original triangle with one sharing vertex.

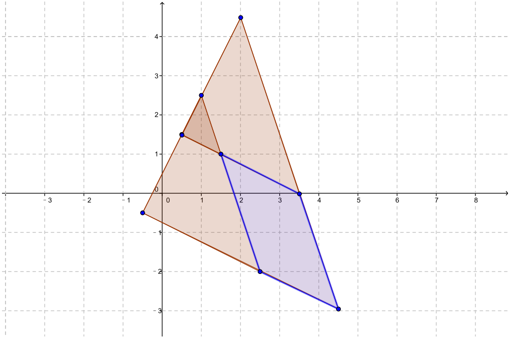


3. What is the $(1+2D)^{th}$ dilate? By an integer translation, we obtain the following picture.



4. We can prove that the parallelogram with half-open boundary resulted from this construction has lattice points $2AD^2$. This completes the geometric proof of the linear recurrence formula.

$$\mathrm{ehr}_P(t+2D) = 2\mathrm{ehr}_P(t+D) - \mathrm{ehr}_P(t) + 2AD^2$$



Some remarks and questions

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The geometric construction gives an alternative proof of Ehrhart theory in dimension 2.

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- Question: The crucial point of the geometric construction is that we can move the bigger one to contain the small one by an integer translation. When a denominator D triangle has period collapse k, can still we do this construction, namely, can we still move the bigger one to a ideal position?

Further results

- The answer to that question is no in general unless we have some condition for the triangles.

1. When a denominator D triangle has period collapse k, two of its vertices can be written as $\frac{1}{k}(r, s)$, $\frac{1}{k}(g, h)$, r, s, g, h being integers, then we can obtain the nice picture as before.
2. The result of this, is that from a special D parallelogram, we can obtain a class of D triangles having property 1, with period collapse k, $k|D$.

Remaining Questions

- Are there "better" necessary and sufficient conditions for equidecomposability?
- What can we say in general about the behavior of the graph g_d^P? Given equidecomposable polygons P and Q, what is the smallest d such that $g_d^P = g_d^Q$?
- Can we find some geometric interpretation for the case when we cannot move the bigger triangle to a nice position?

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[Ehr77] [MW05] [Haa08] [BR09]