Almost Optimal Distance Oracles for Planar Graphs

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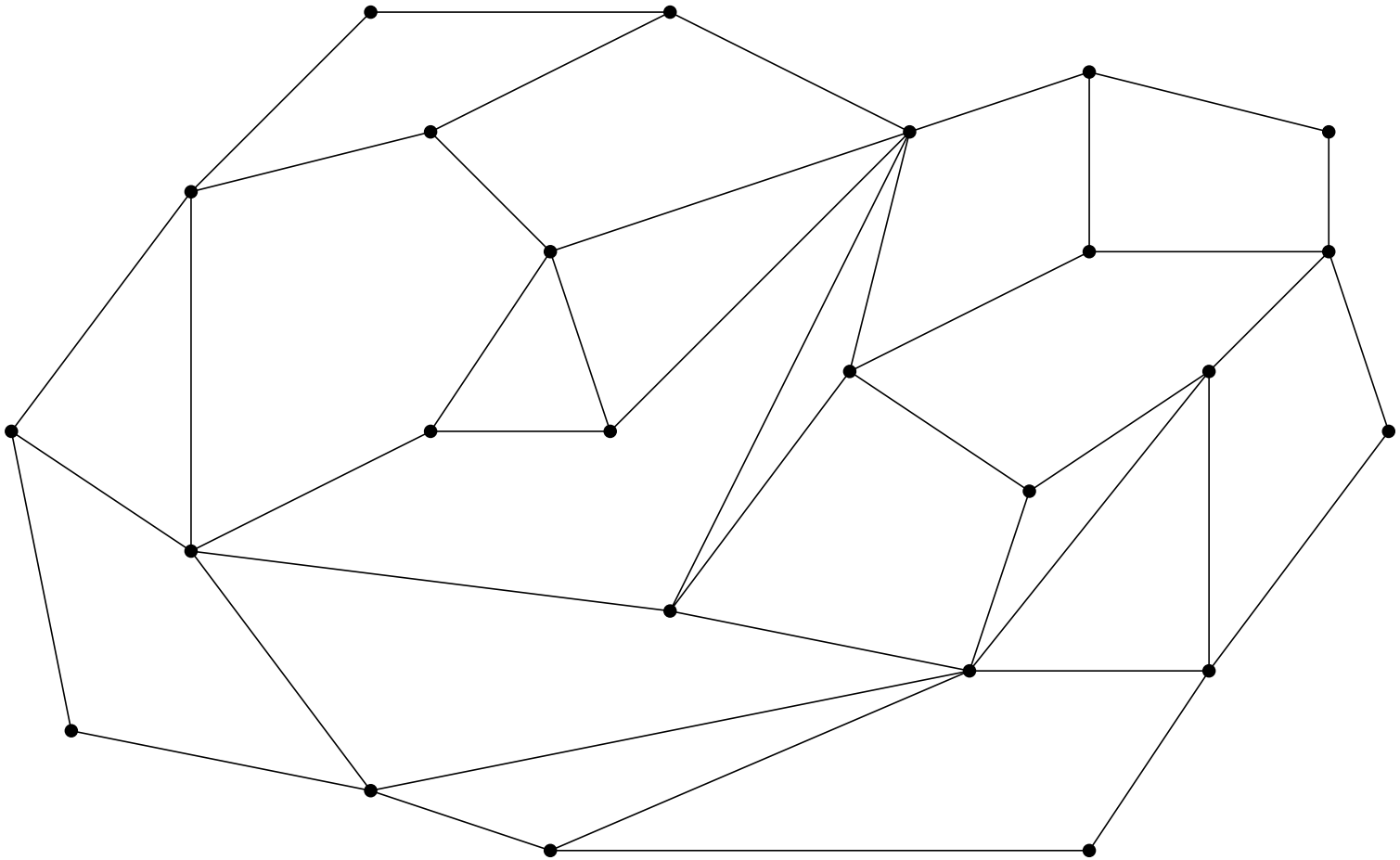
May 15, 2019

Problem definition



Preprocess an n-vertex planar graph $G = (V, E)$ with nonnegative arc lengths, so that given any $u, v \in V$ we can compute $d(u, v)$ efficiently.

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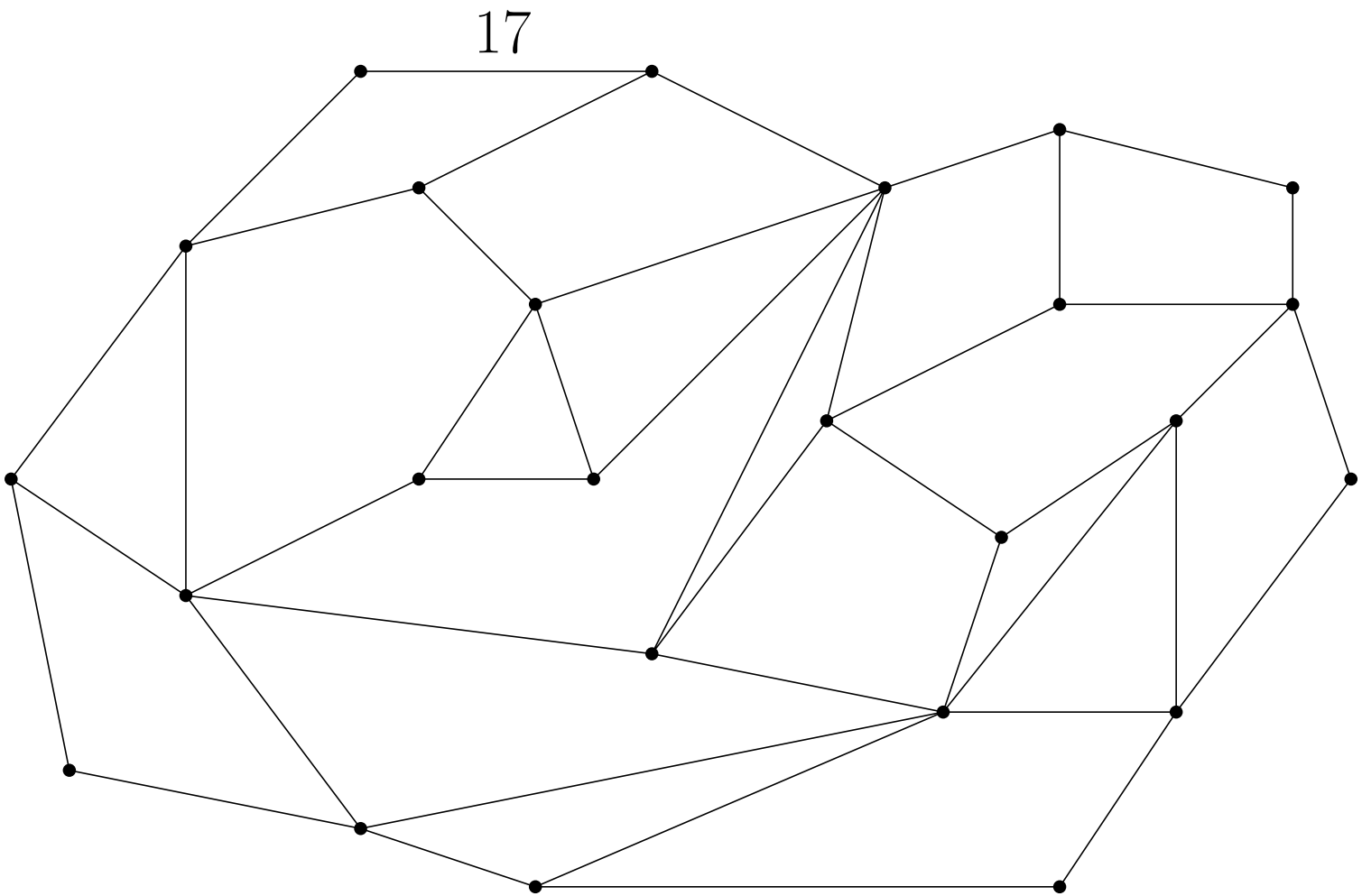

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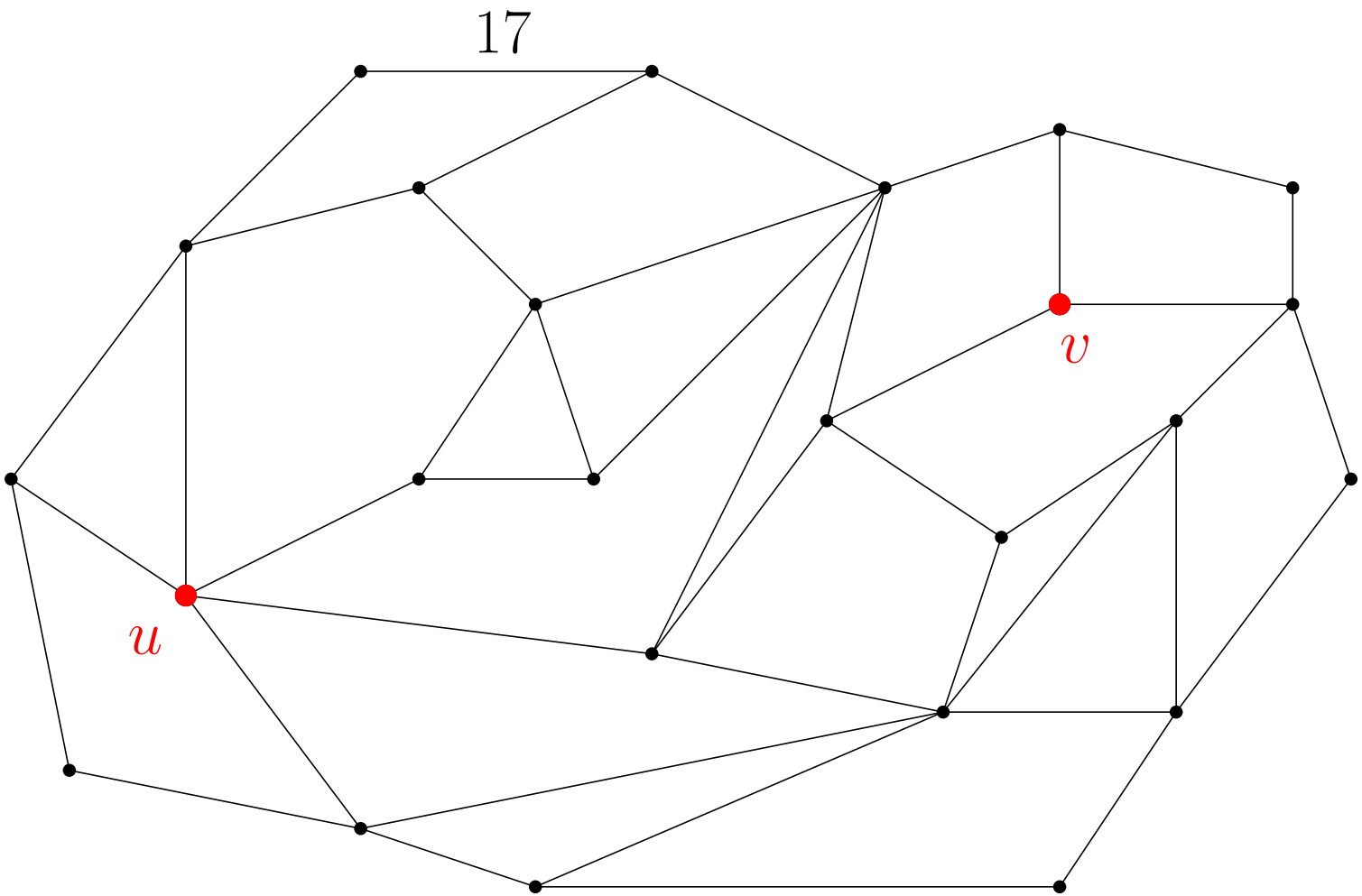

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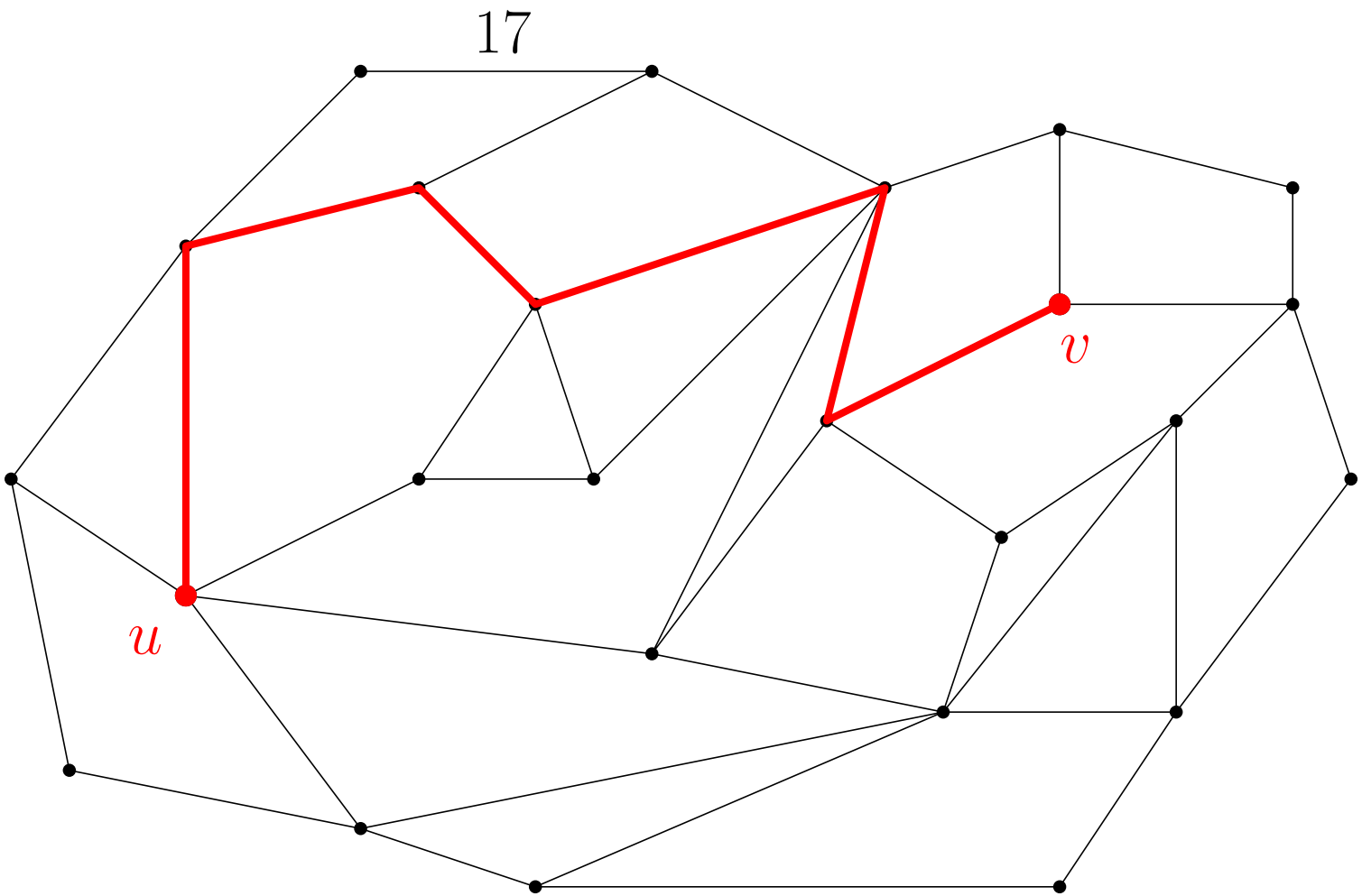

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Goals

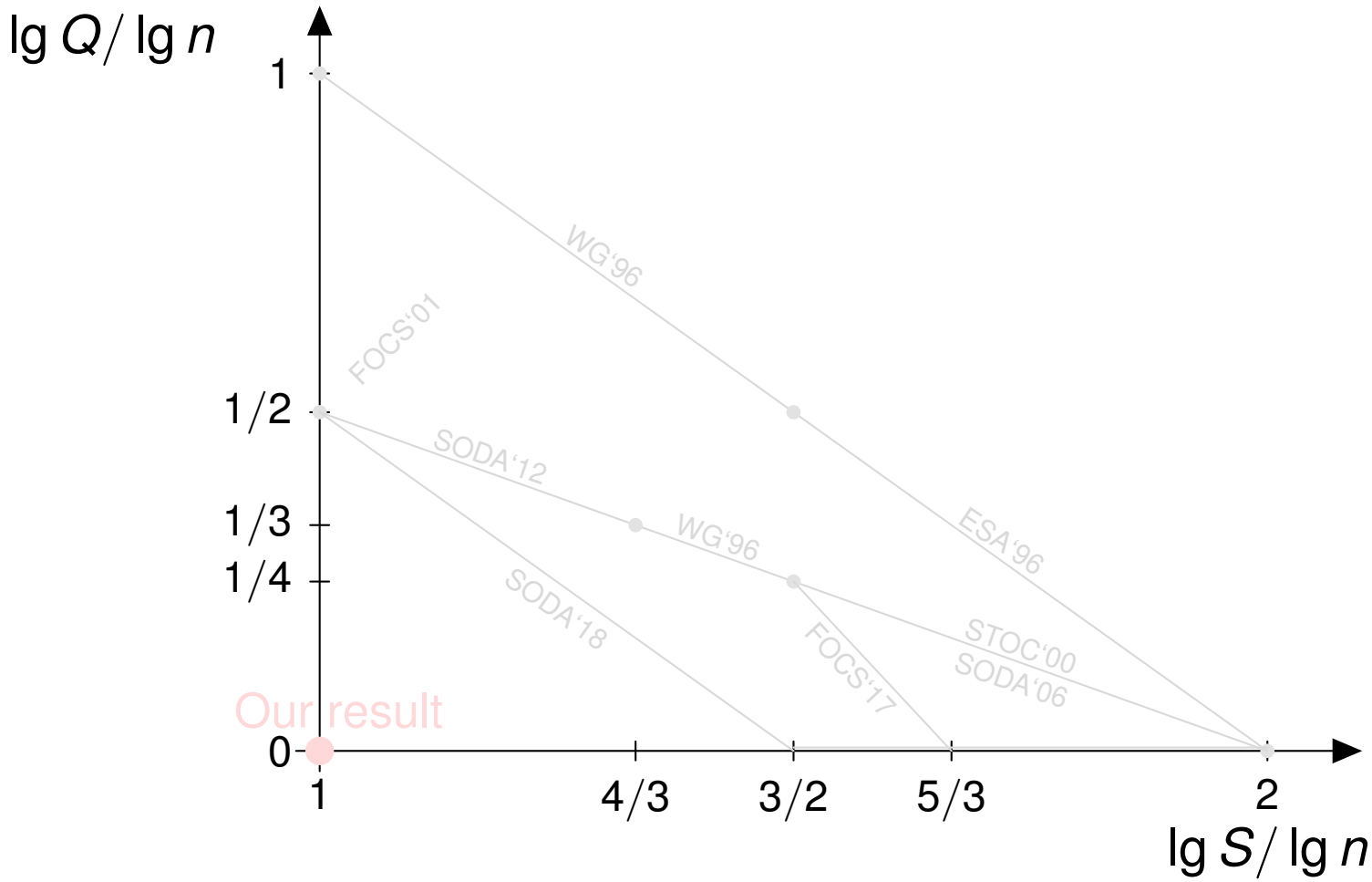
Ideally:

- Fast queries, ideally $Q = O(1)$.
- Small size, ideally $S = O(n)$.
- Fast construction, ideally $T = O(n)$.

The most important tradeoff is between query-time Q and size S.

Previous work

The tradeoff between the query-time Q and the size S of the structure:



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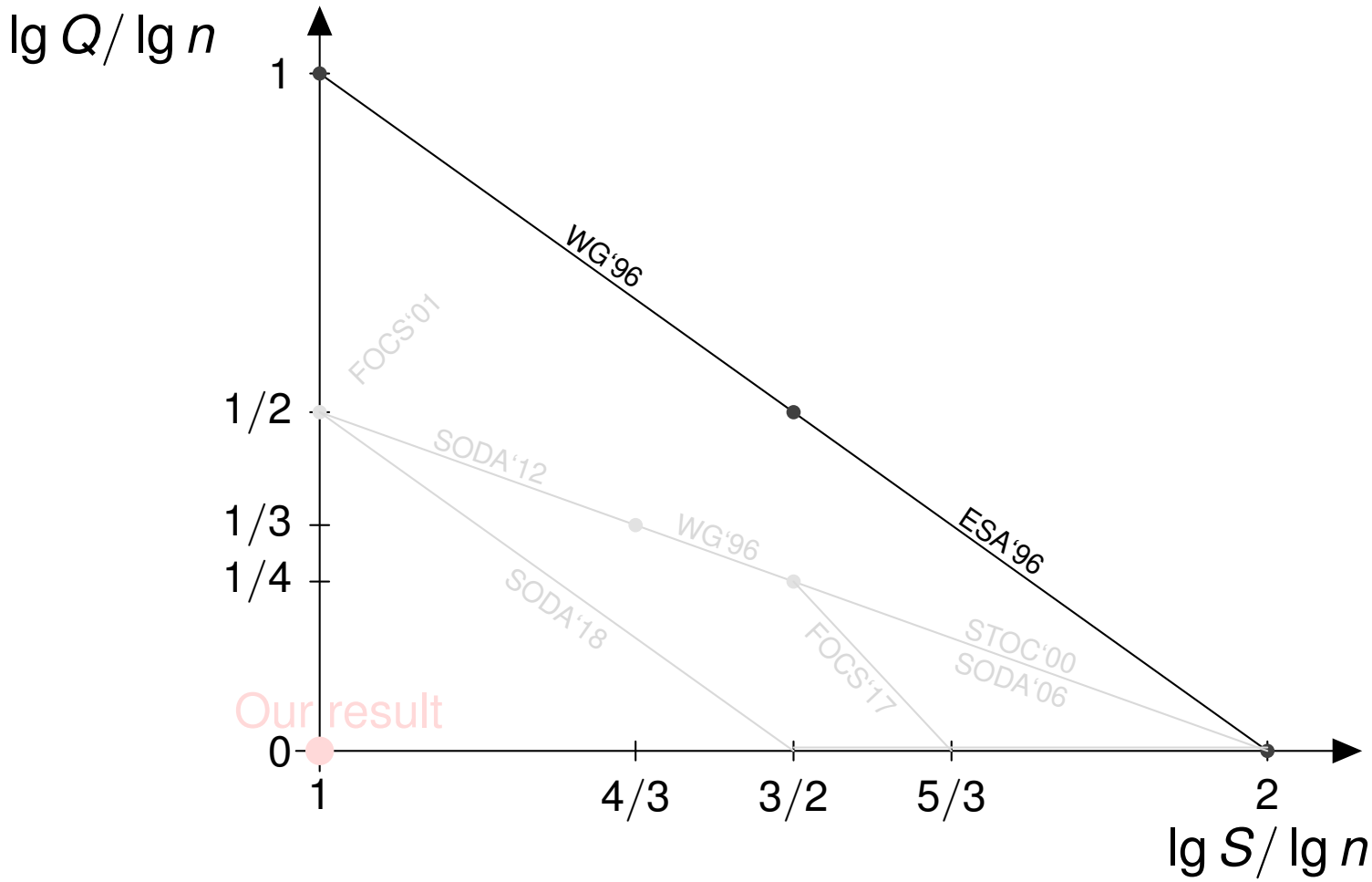

Djidjev and Arikati et al. achieved $Q = O(n^2/S^2)$.

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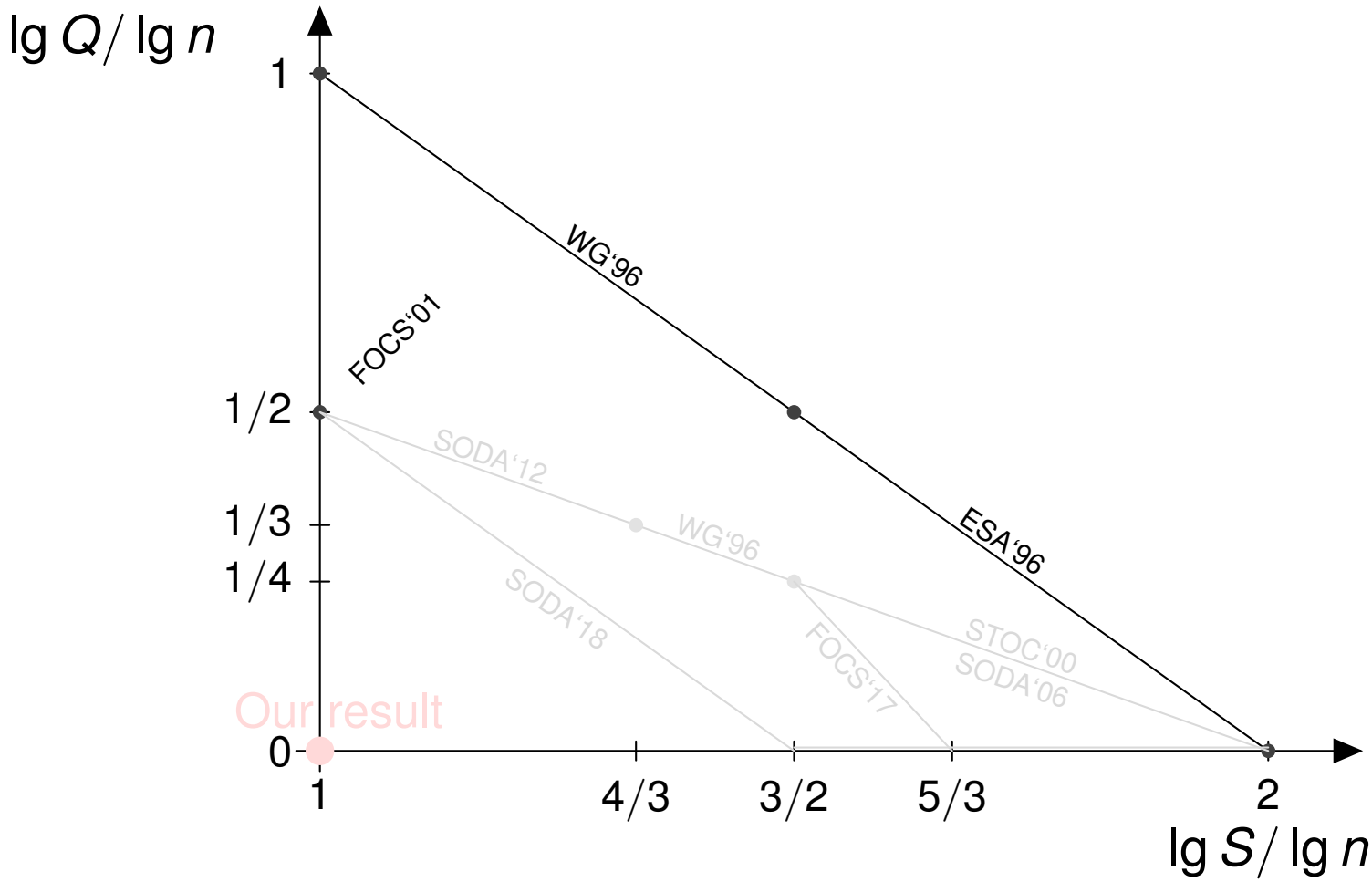
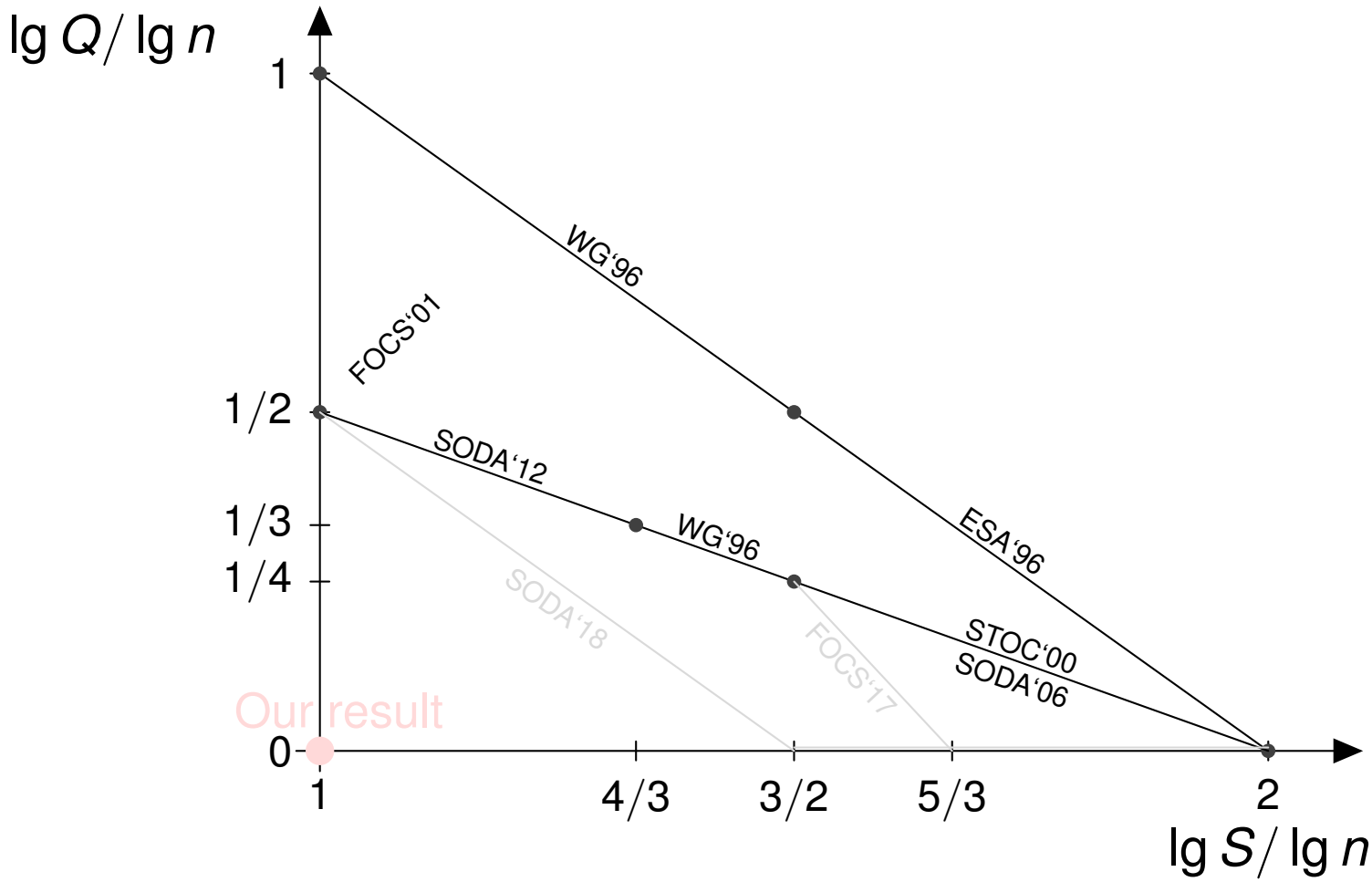


Fakcharoenphol and Rao show that $S = \tilde{O}(n)$ and $Q = \tilde{O}(\sqrt{n})$ is possible.

Previous work

The tradeoff between the query-time Q and the size S of the structure:



This has been extended to $Q = \tilde{O}(n/\sqrt{S})$ for essentially the whole range of S in a series of papers.

Previous work

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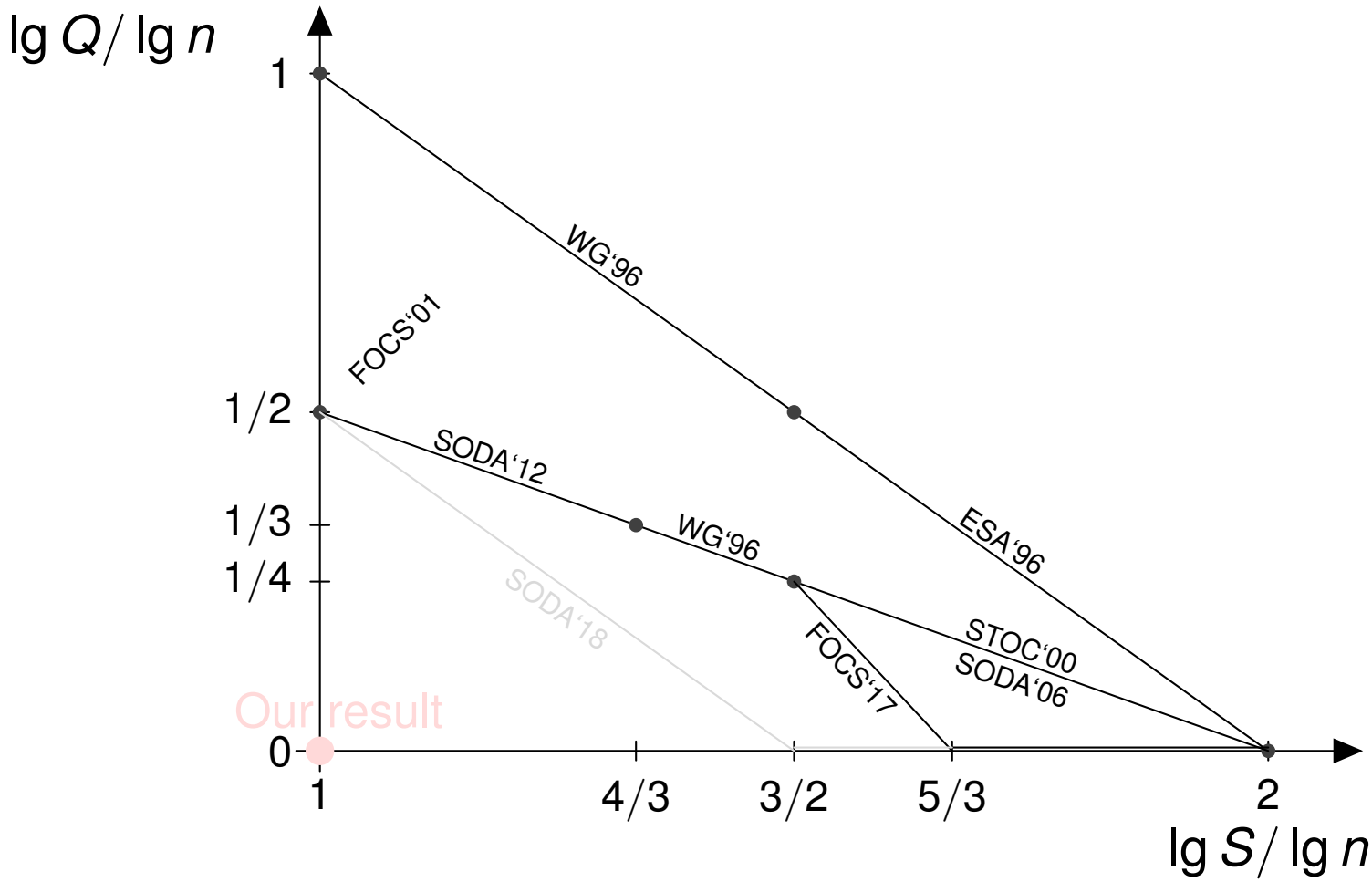

In 2017, Cohen-Addad, Dahlgaard, and Wulff-Nilsen showed that this is not optimal, and $S = O(n^{5/3})$ with $Q = O(\log n)$ is possible.

Previous work

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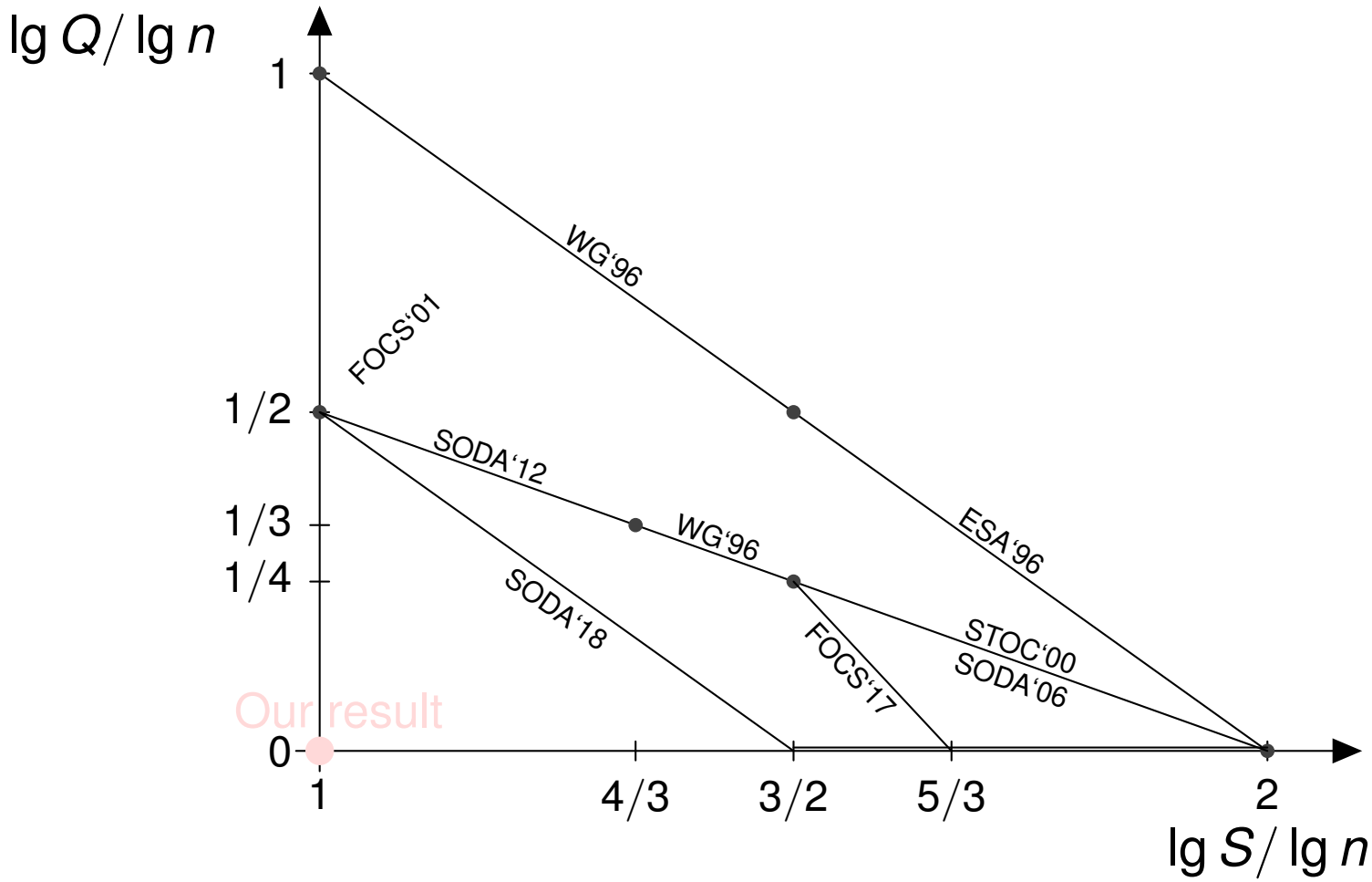
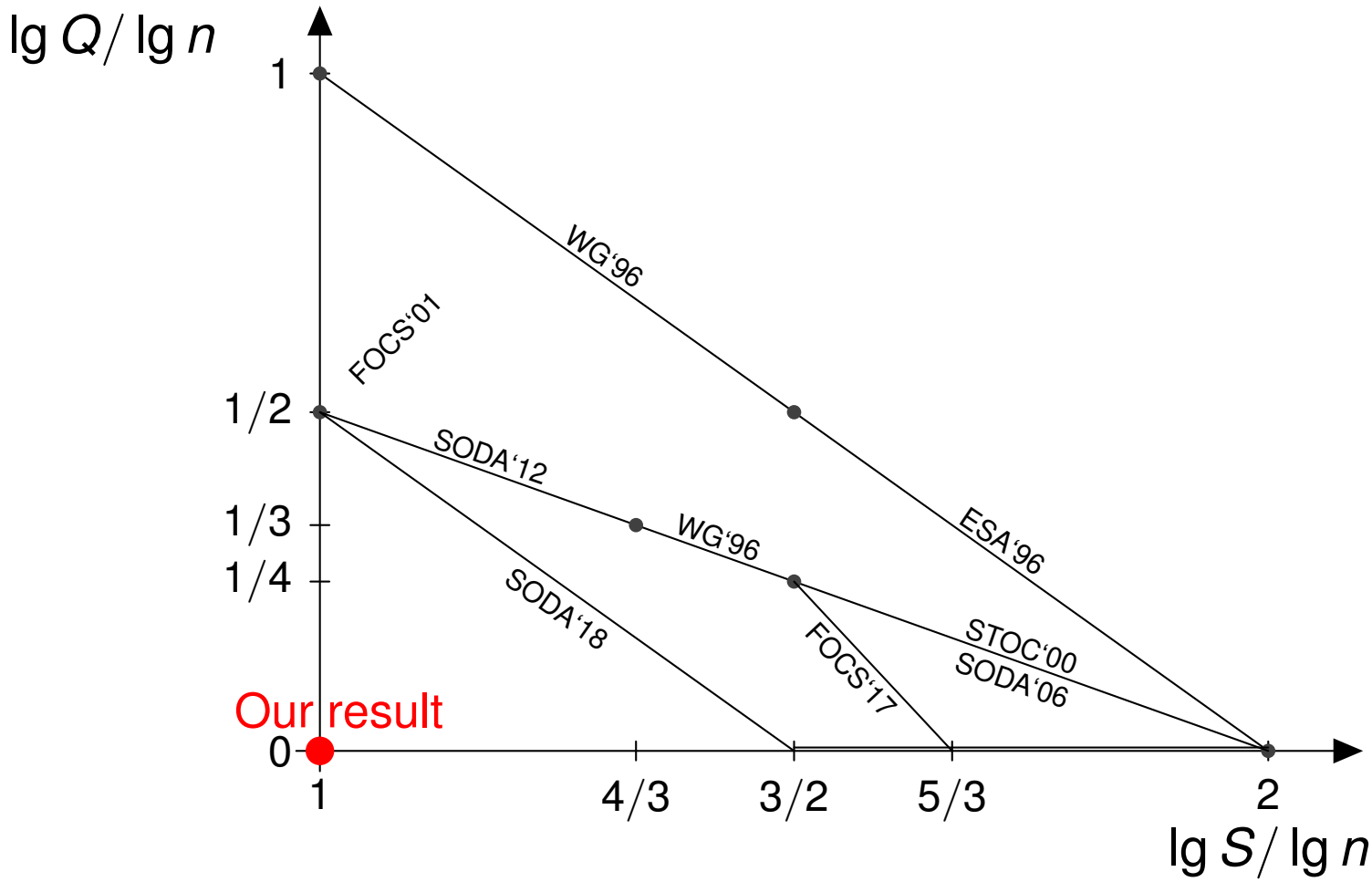


In 2018, Gawrychowski et al. improved this to $S = O(n^{1.5})$ and $Q = O(\log n)$.

Previous work

The tradeoff between the query-time Q and the size S of the structure:



We improve this to $S = O(n^{1+\epsilon})$ and $Q = \tilde{O}(1)$ for any $\epsilon > 0$.

Main result

We show the following tradeoffs for $\langle S, Q \rangle$:

1. $\langle \tilde{O}(n^{1+\epsilon}), O(\log^{1/\epsilon} n) \rangle$, for any constant $1/2 \geq \epsilon > 0$;
2. $\langle O(n \log^{2+1/\epsilon} n), \tilde{O}(n^{\epsilon}) \rangle$, for any constant $\epsilon > 0$;
3. $\langle n^{1+o(1)}, n^{o(1)} \rangle$.

The oracle is based on a recursive view of the point location mechanism for Voronoi diagrams of Gawrychowski et al. [SODA'18].

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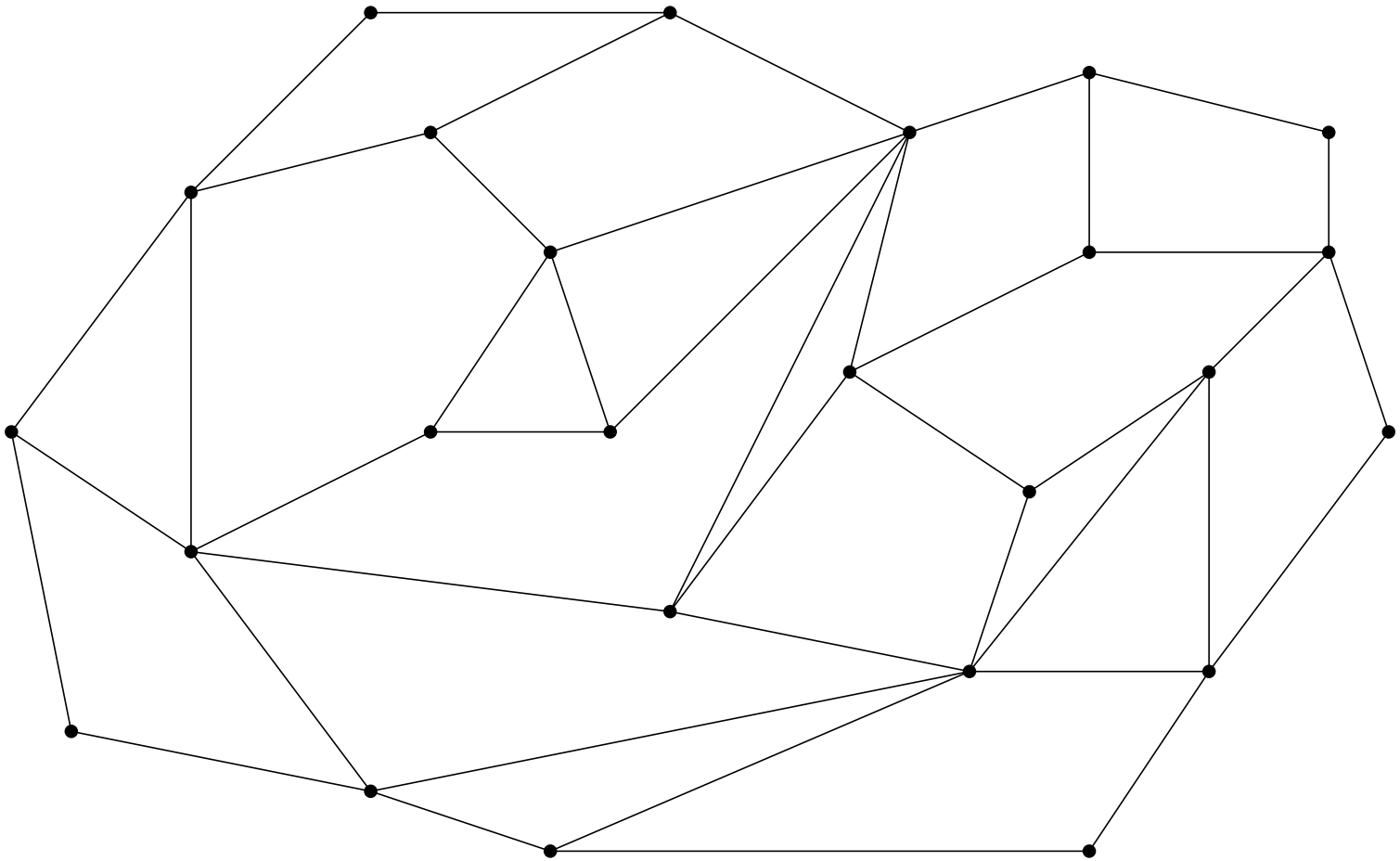
Cycle Separator

Miller [JCSS'86]

There always exists a Jordan curve separator of size $O(\sqrt{n})$ such that there are at most $\frac{2}{3}n$ nodes on its inside/outside.



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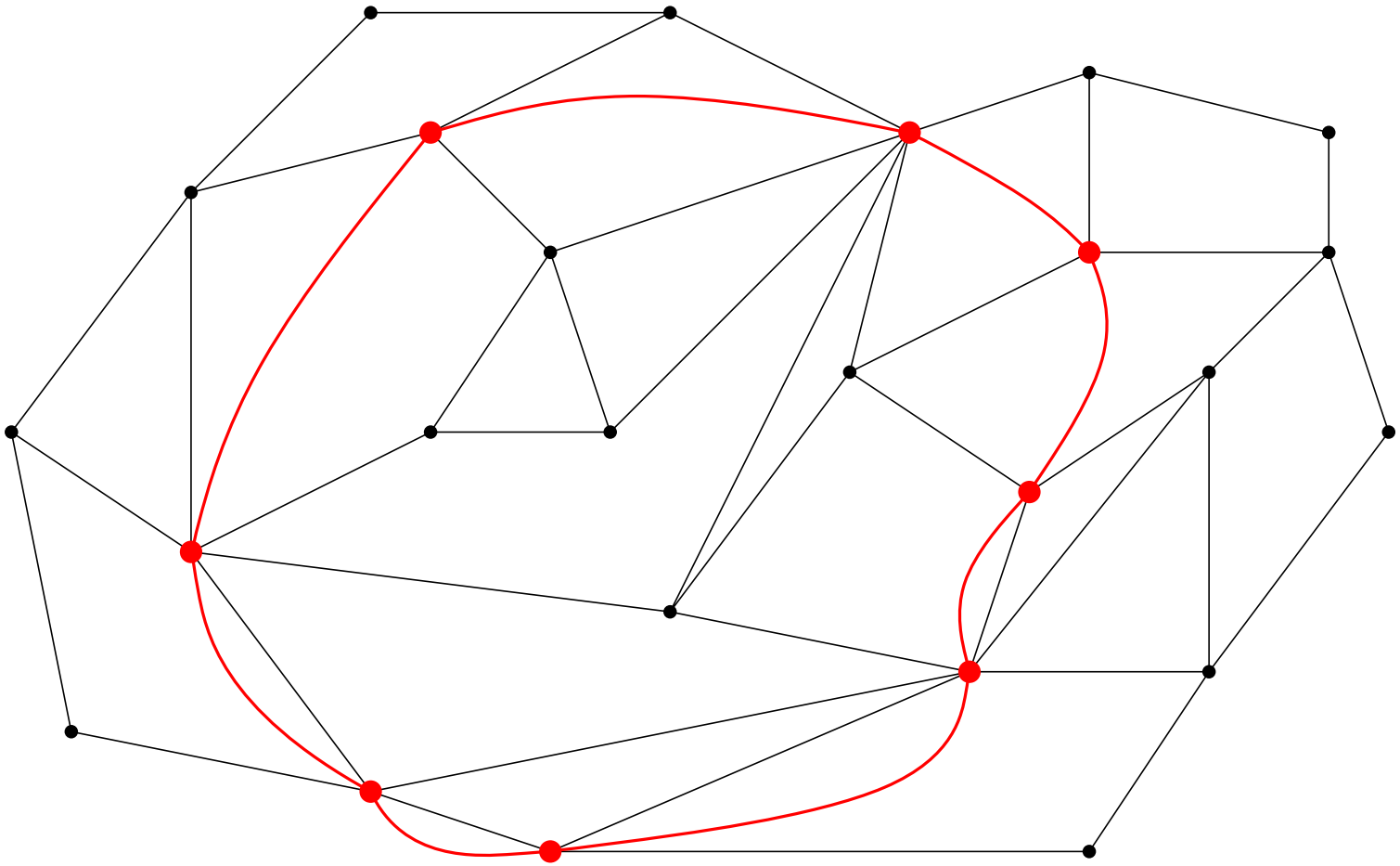
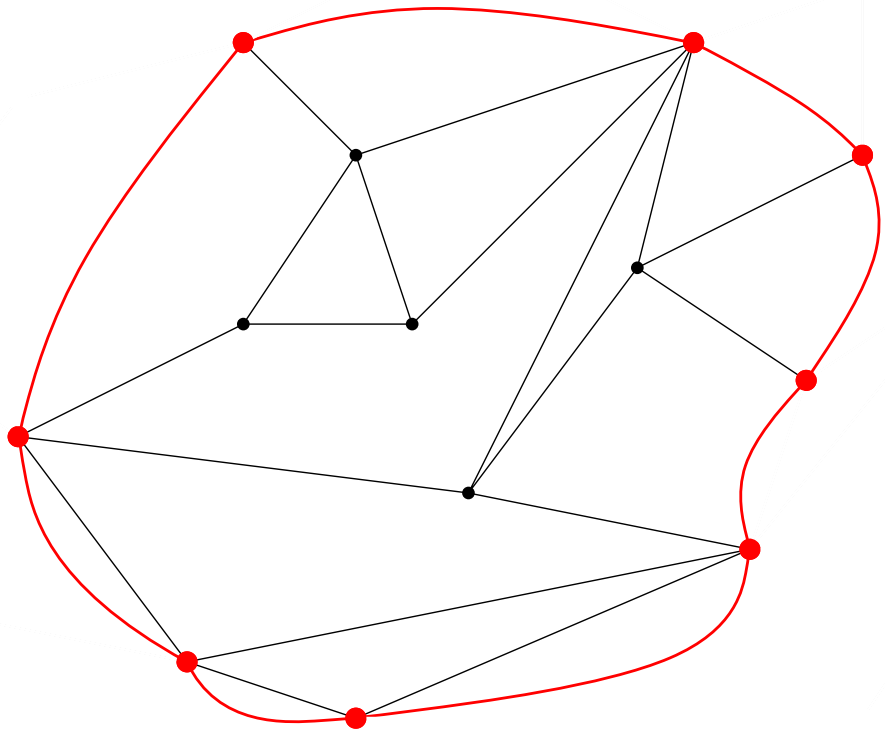


Multiple Source Shortest Paths (MSSP)

Klein [SODA'05]

There exists a data structure requiring $O(n \log n)$ space that can report in $O(\log n)$ time the distance between any node on the infinite face (boundary node) and any node in the graph.



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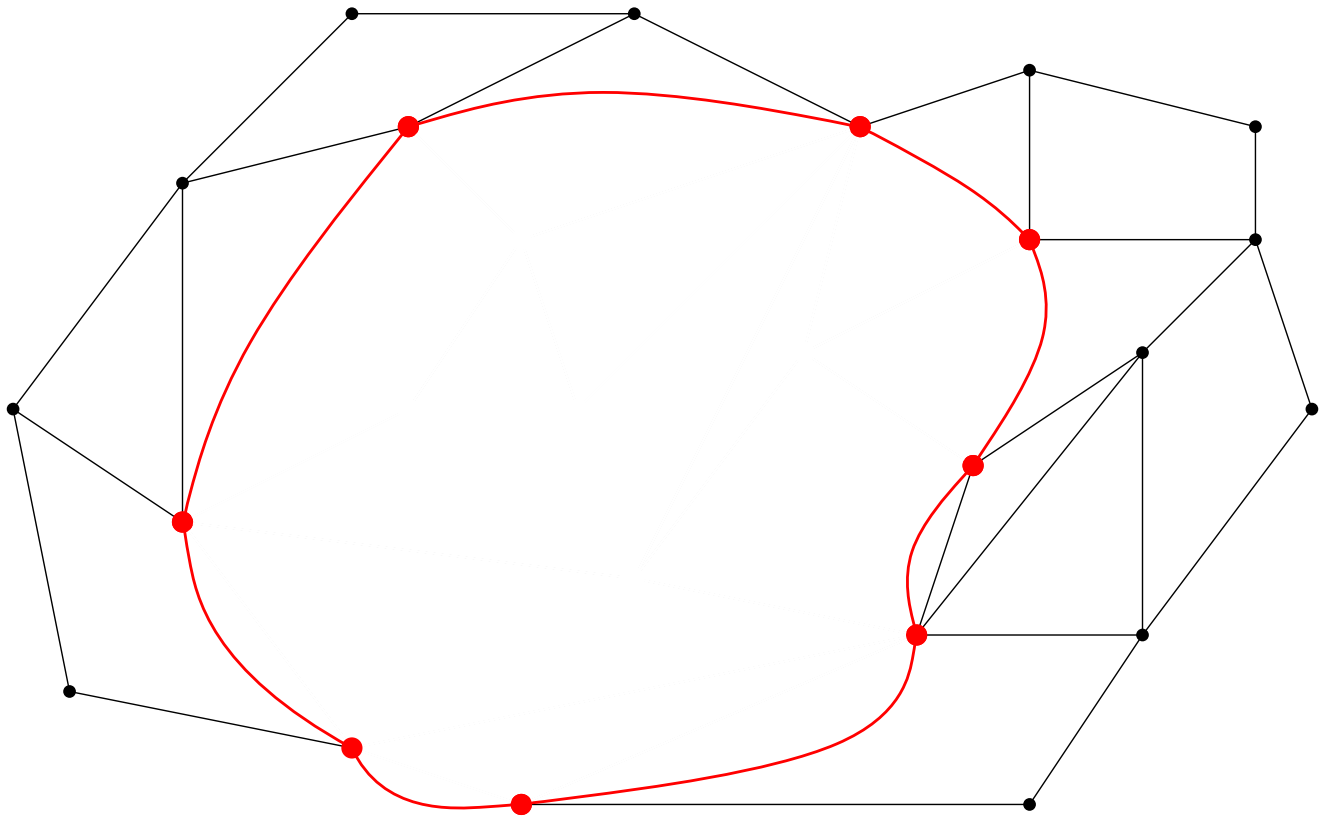
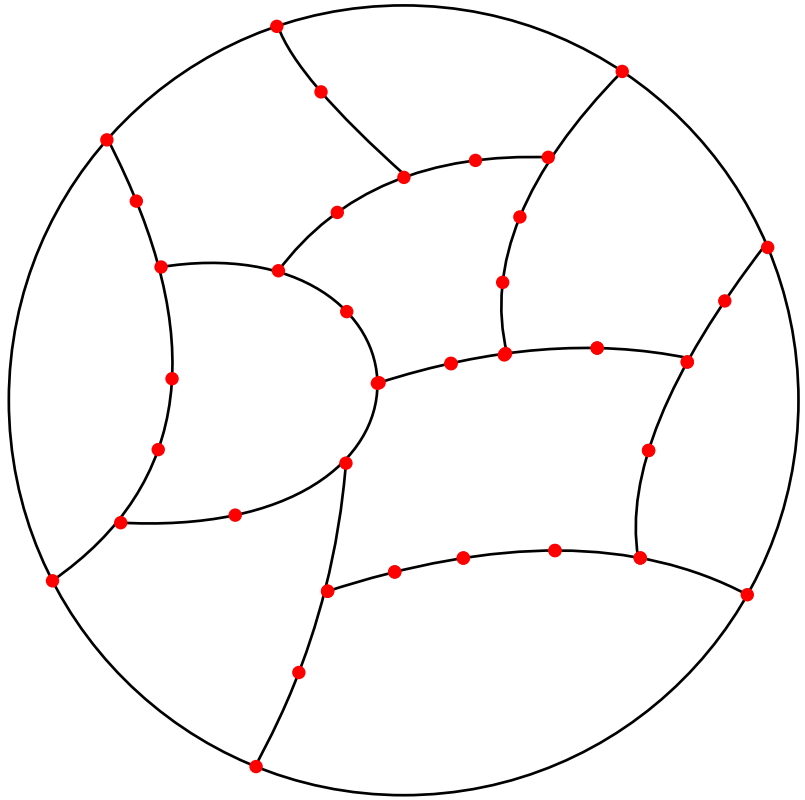


r-divisions

For $r \in [1, n]$, a decomposition of the graph into:

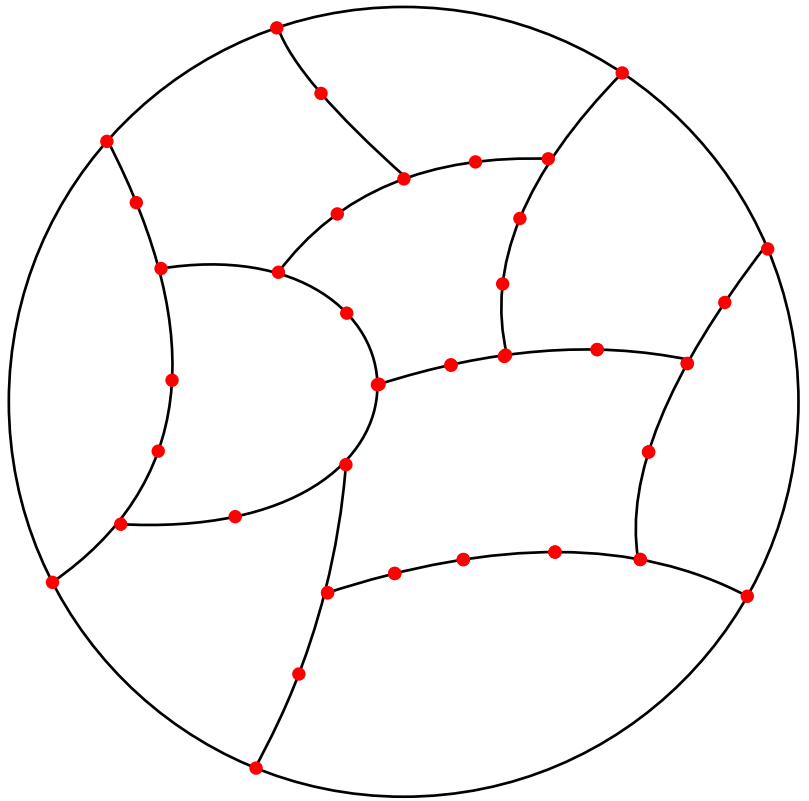
- $O(n/r)$ pieces
- each piece has $O(r)$ vertices.
- each piece has $O(\sqrt{r})$ boundary vertices, (vertices incident to edges in other pieces)



We denote the boundary of a piece P by ∂P and assume that all such nodes lie on a single face of P.

Warm-up: Space $\tilde{O}(n^{4/3})$, Query-time $\tilde{O}(n^{1/3})$

- Compute an r-division.
- For each piece P, for each node $u \in P$, store $d_G(u, p)$ for $p \in \partial P$. Space $O(n \cdot \sqrt{r})$.
- For each piece P, store an MSSP data structure for the outside of P with sources ∂P. Space $\tilde{O}(n/r \cdot n)$.



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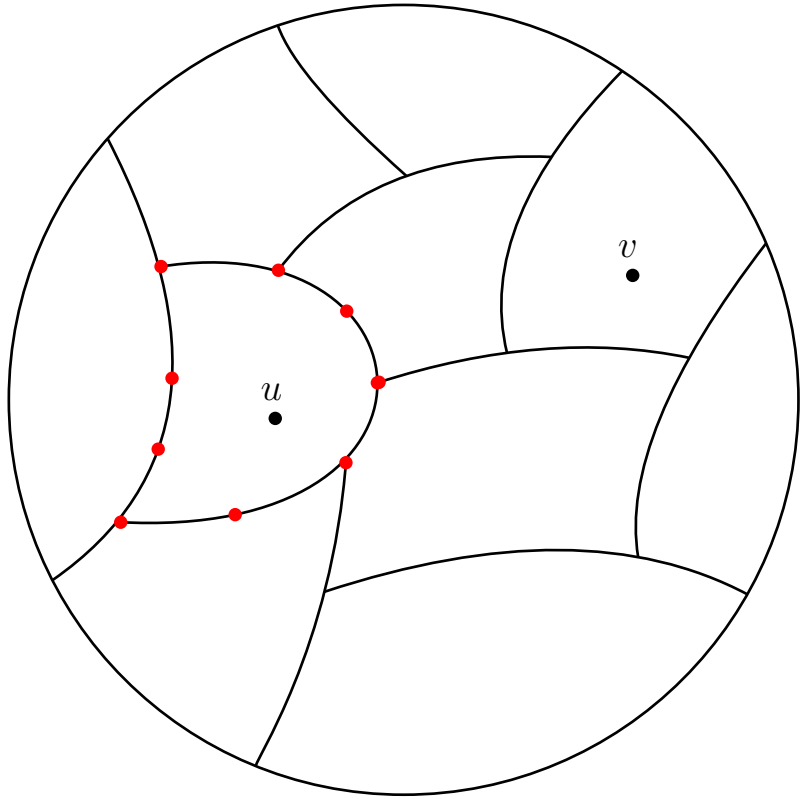

Interesting case.

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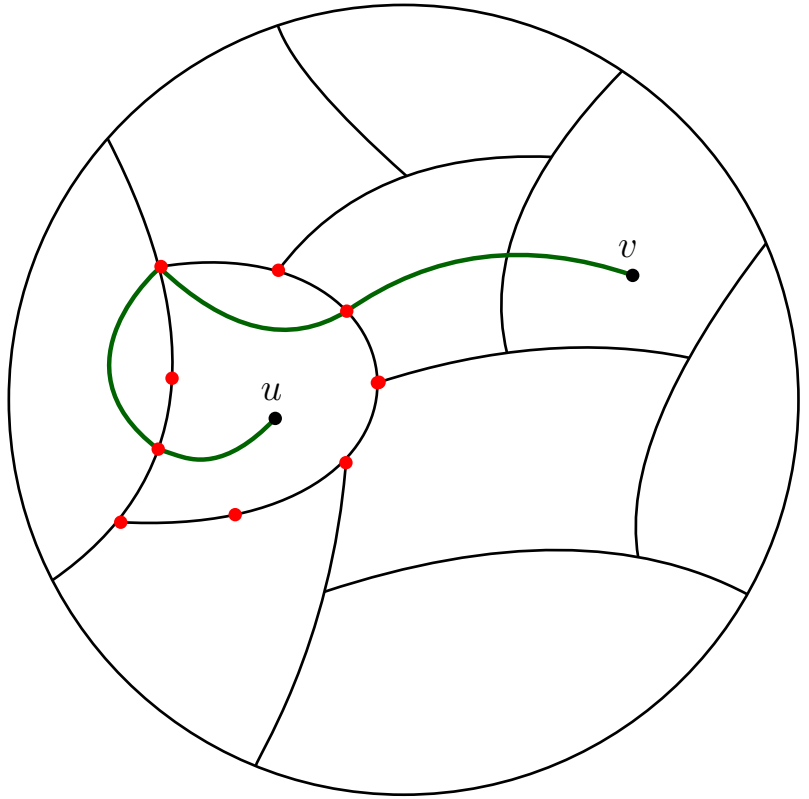

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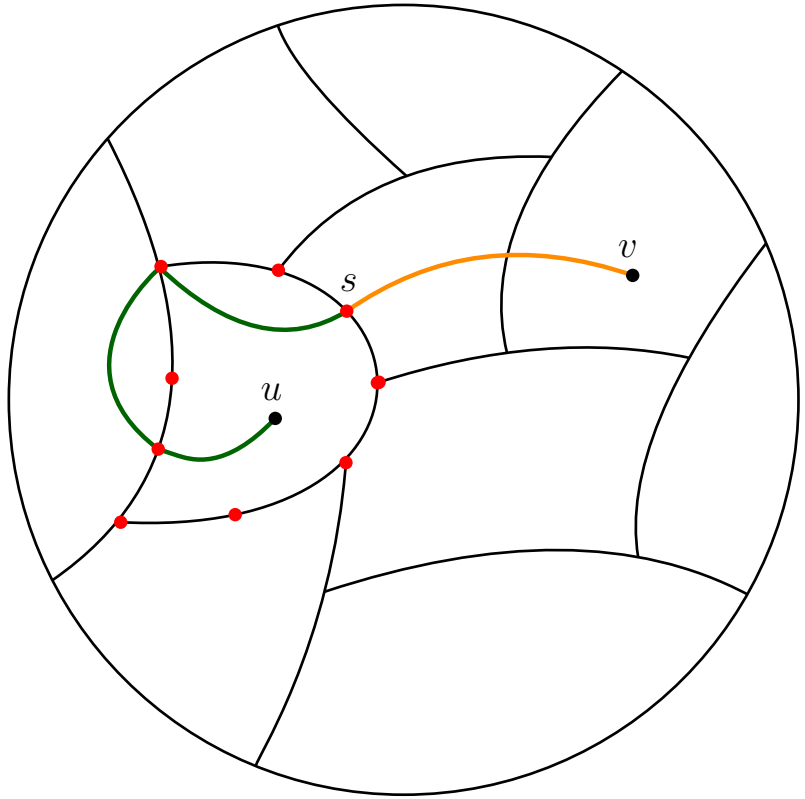

We decompose the path on the last boundary node it visits.

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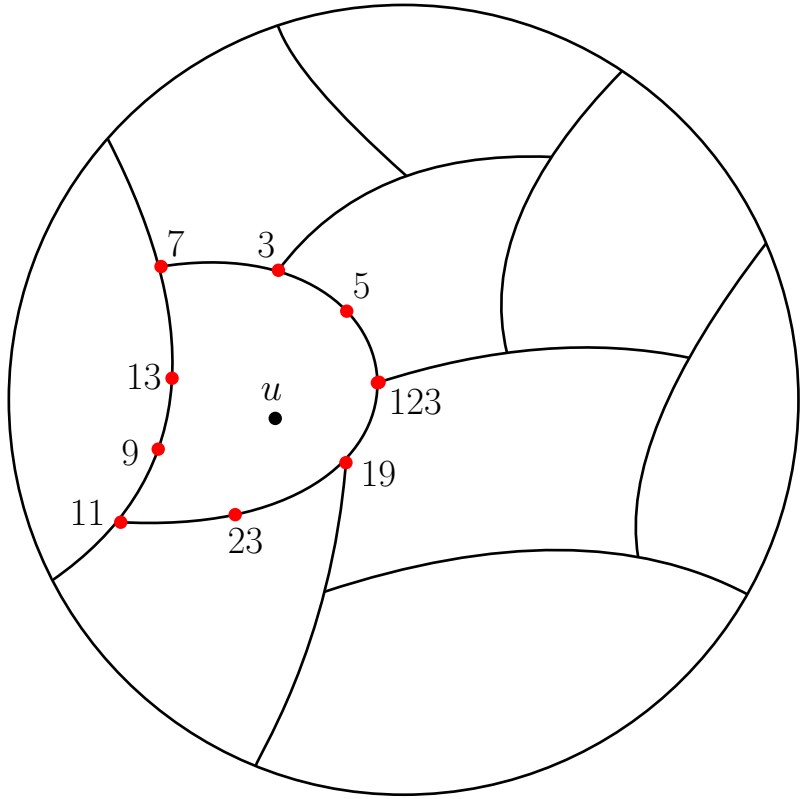

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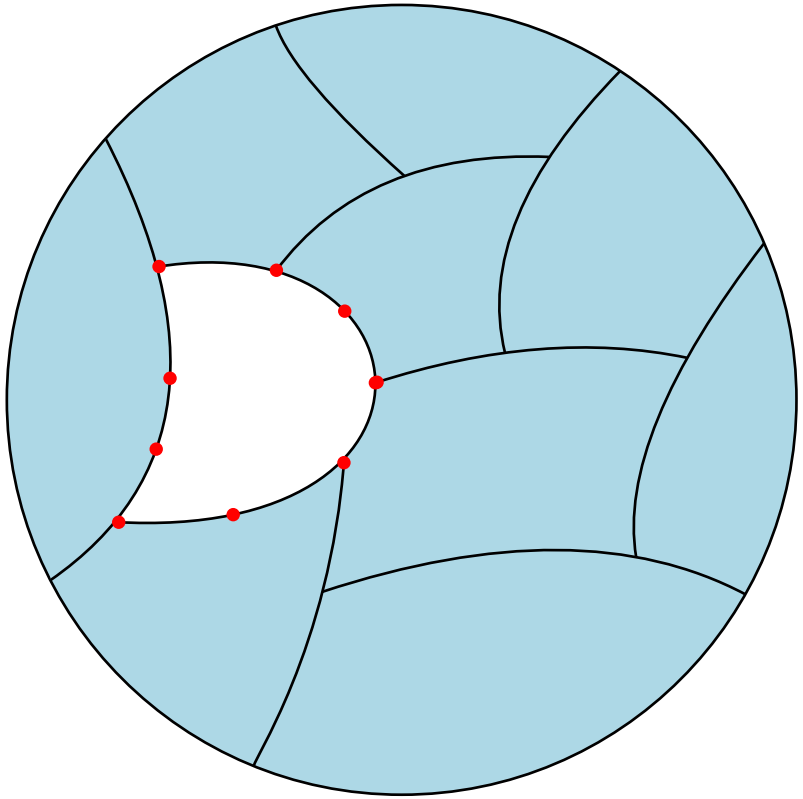

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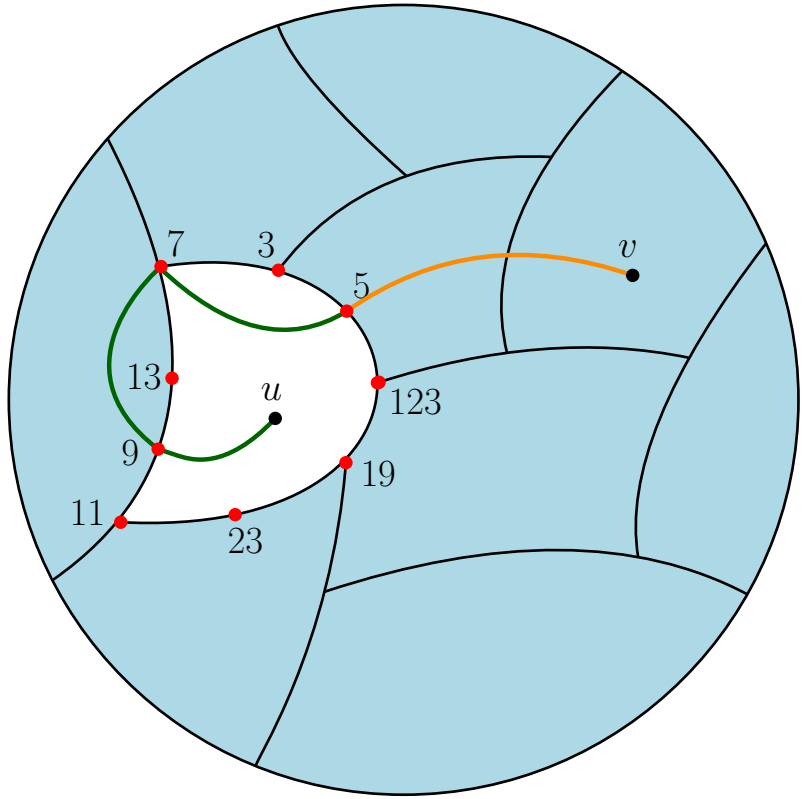

At query, try all possible $O(\sqrt{r})$ candidate boundary nodes.

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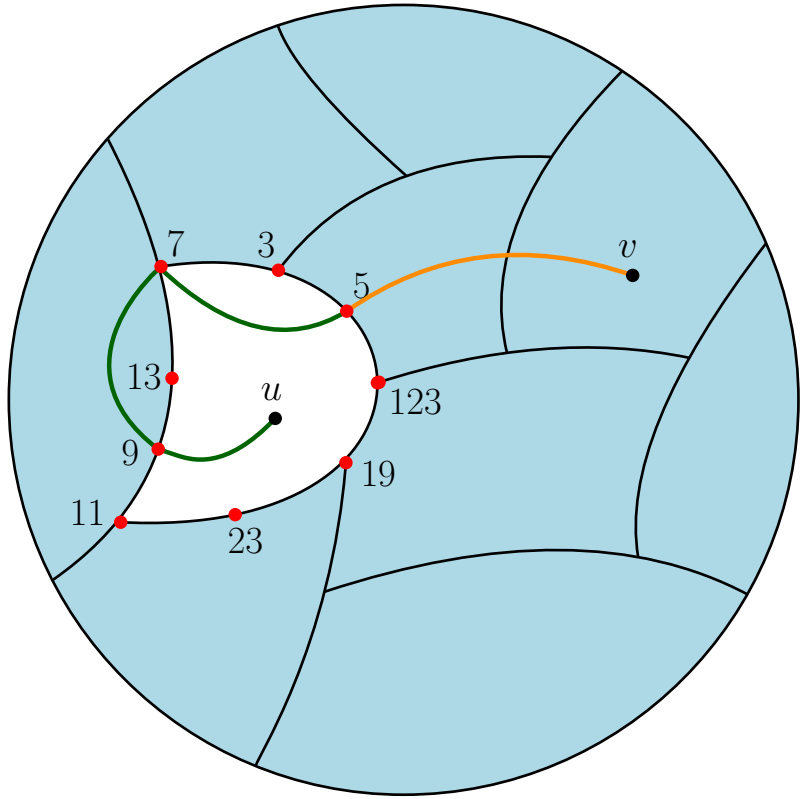

At query, try all possible $O(\sqrt{r})$ candidate boundary nodes.

The balance is at $r = n^{2/3}$.

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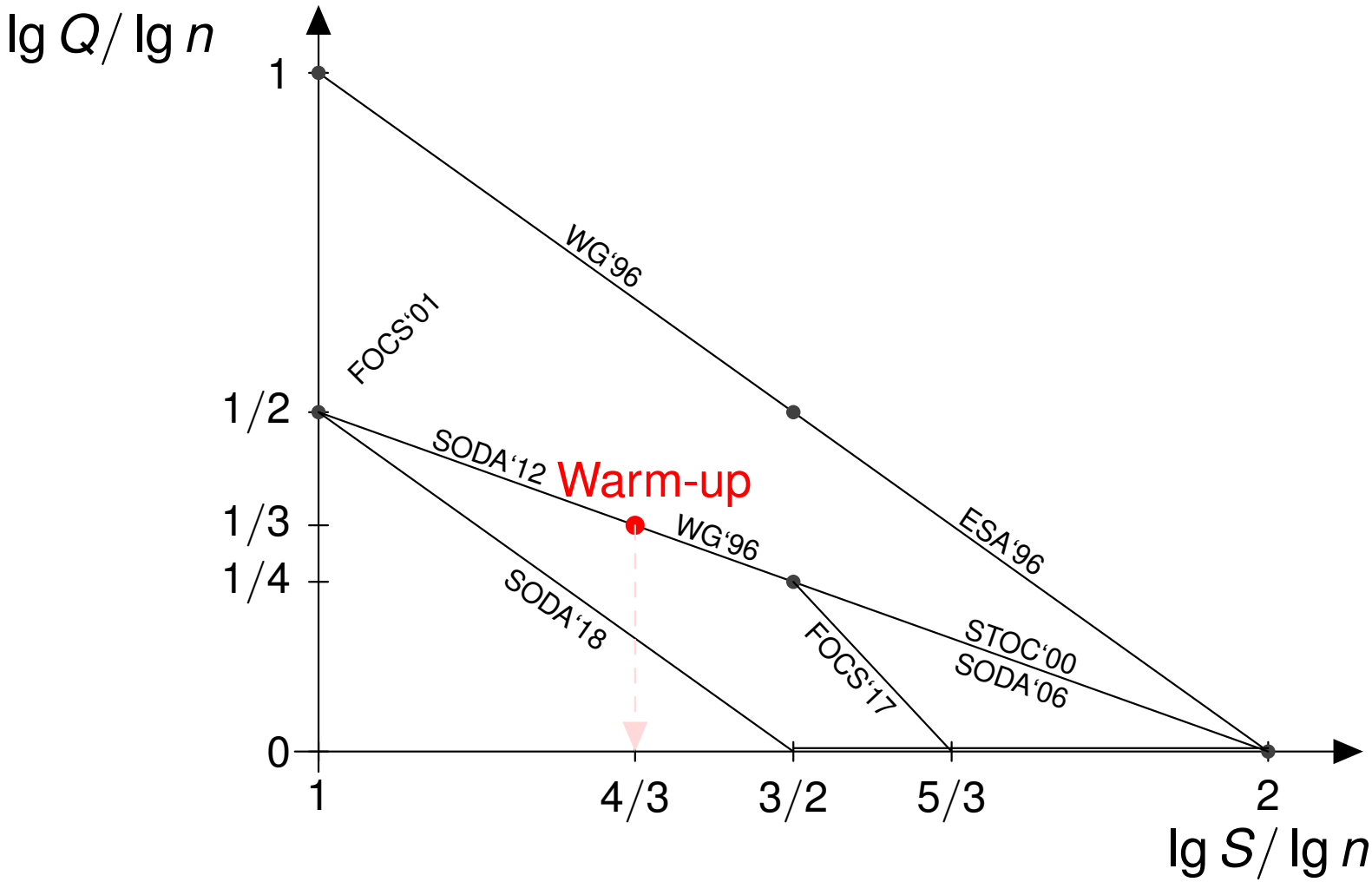

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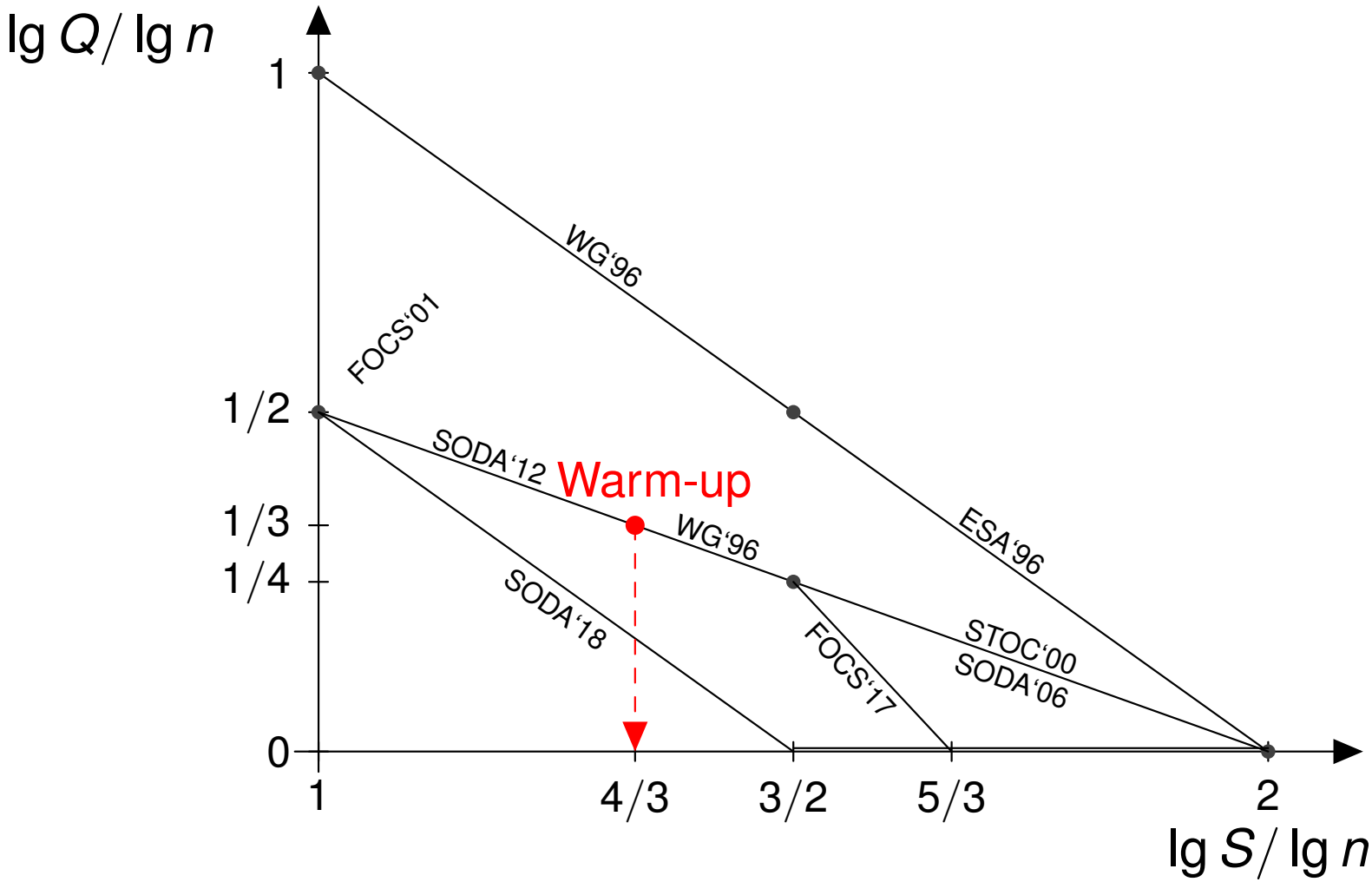
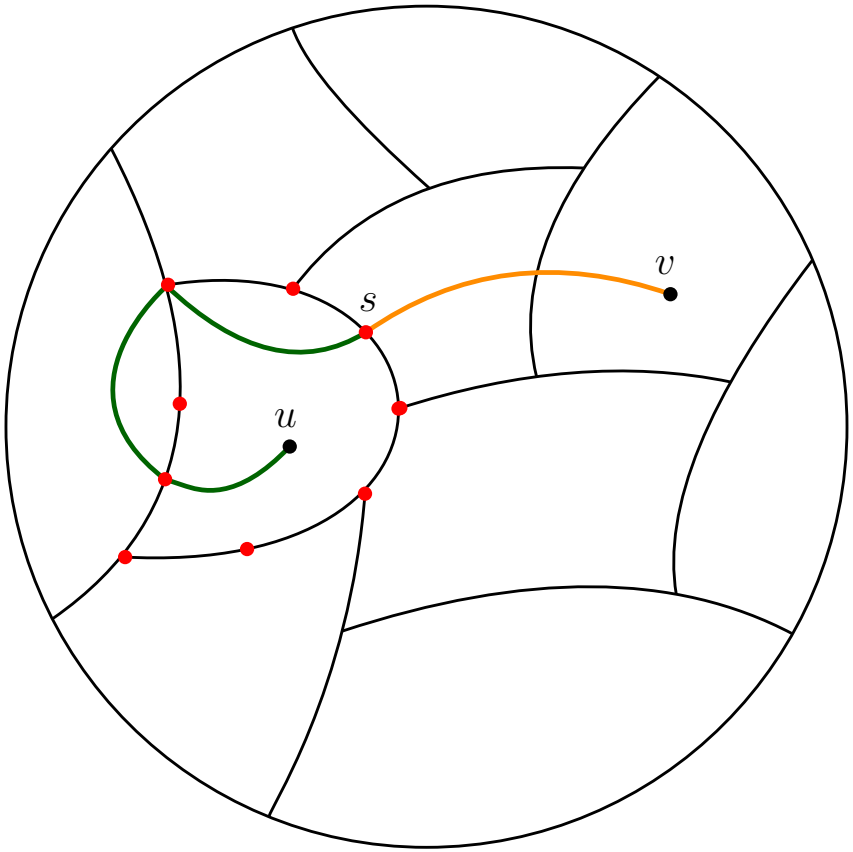


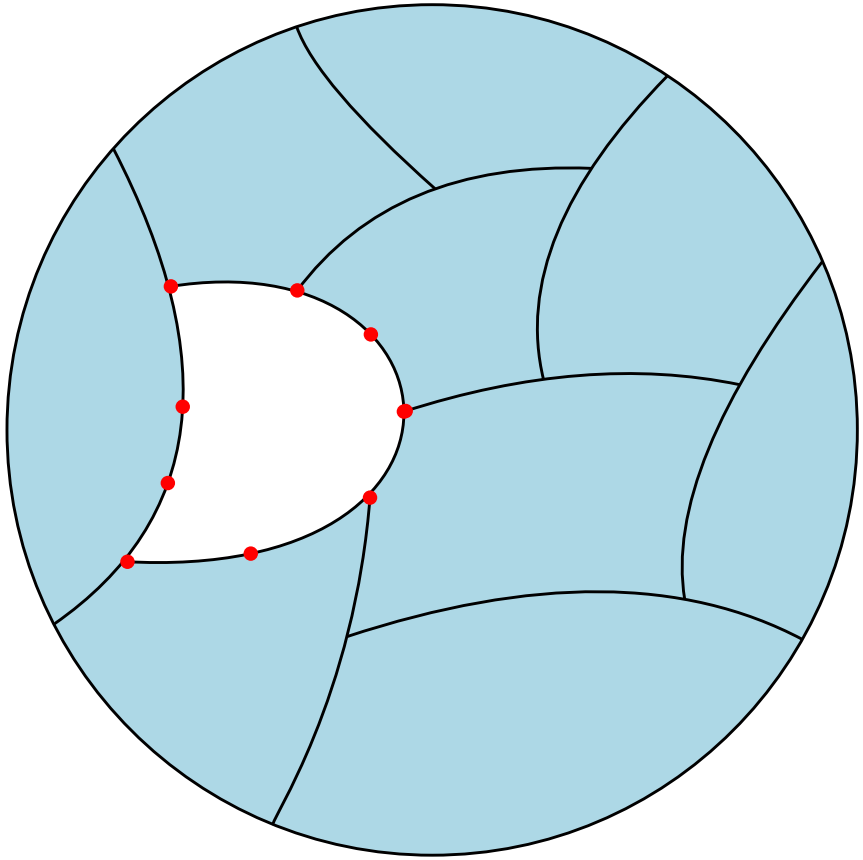
First goal



Instead of trying all possible $O(\sqrt{r}) = O(n^{1/3})$ candidate boundary nodes, we want to compute the last boundary node s visited by the shortest path in $\tilde{O}(1)$ time.

Point location

- Piece P.
- Weights $w_u(s) \geq 0$ for $s \in \partial P$.



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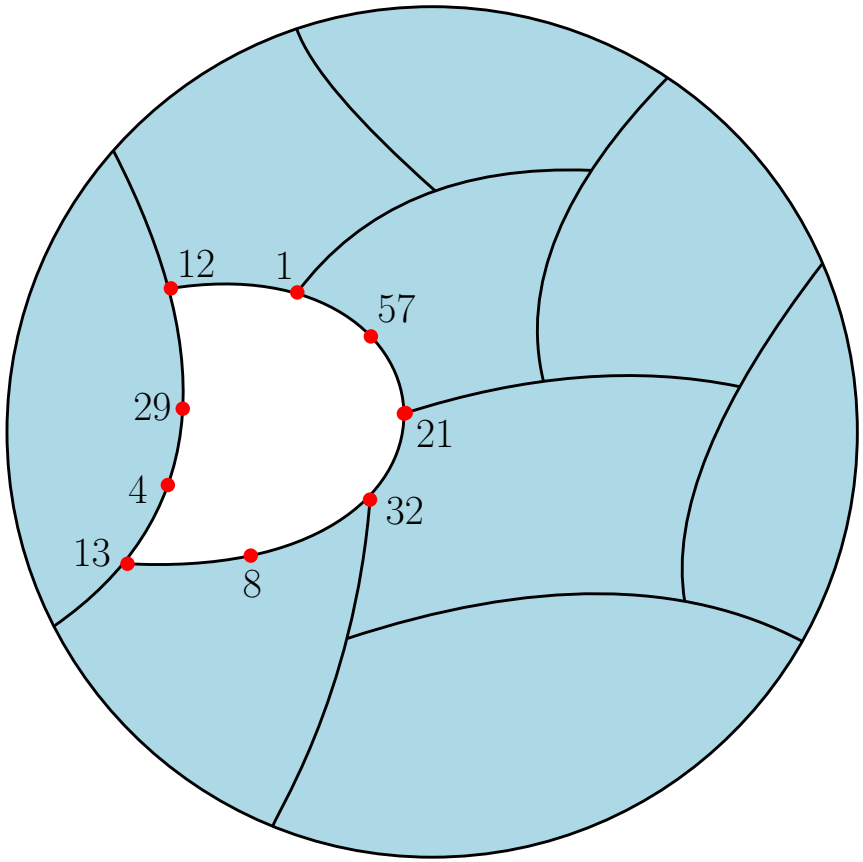

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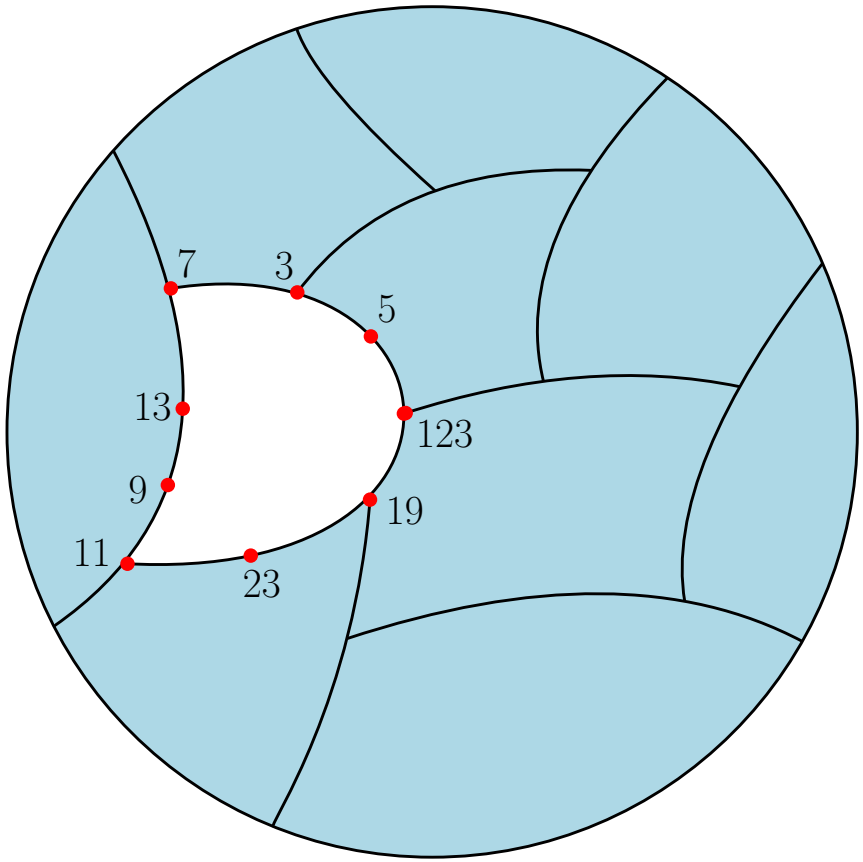

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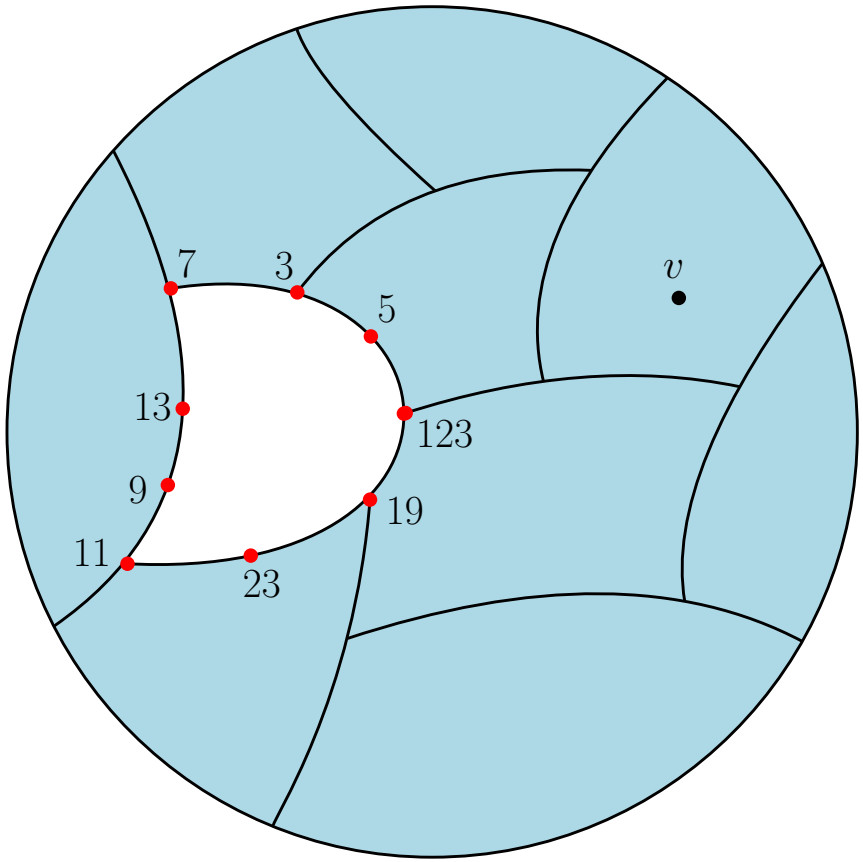

PointLocation(u, v) query: given a vertex $v \notin P$, find site $s \in \partial P$ minimizing $w_u(s) + d(s, v)$.

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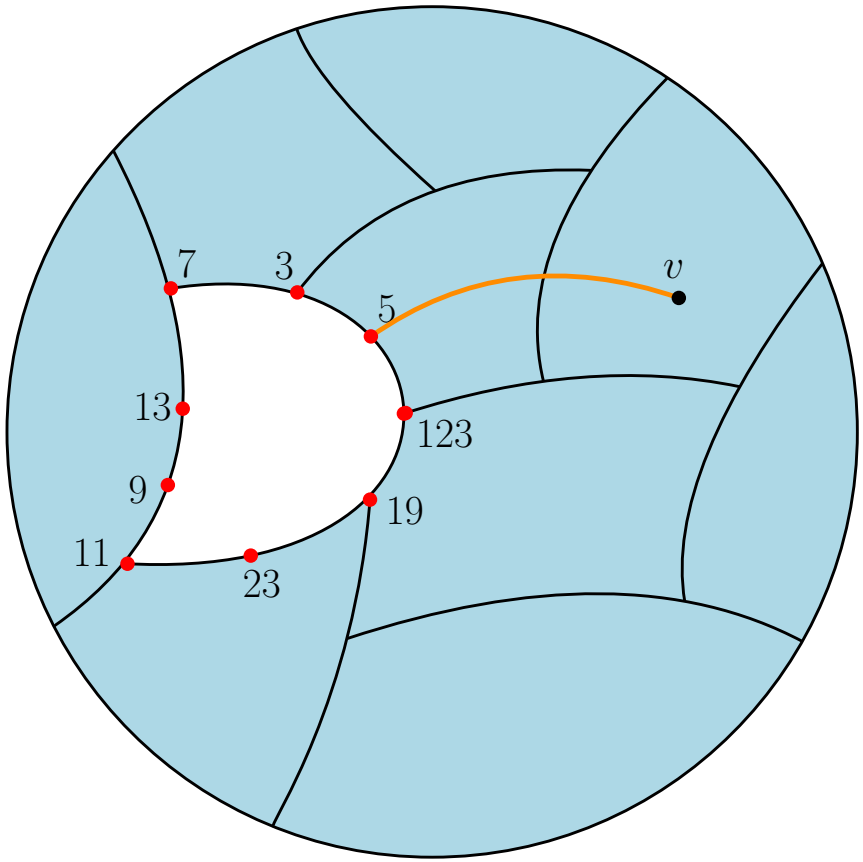

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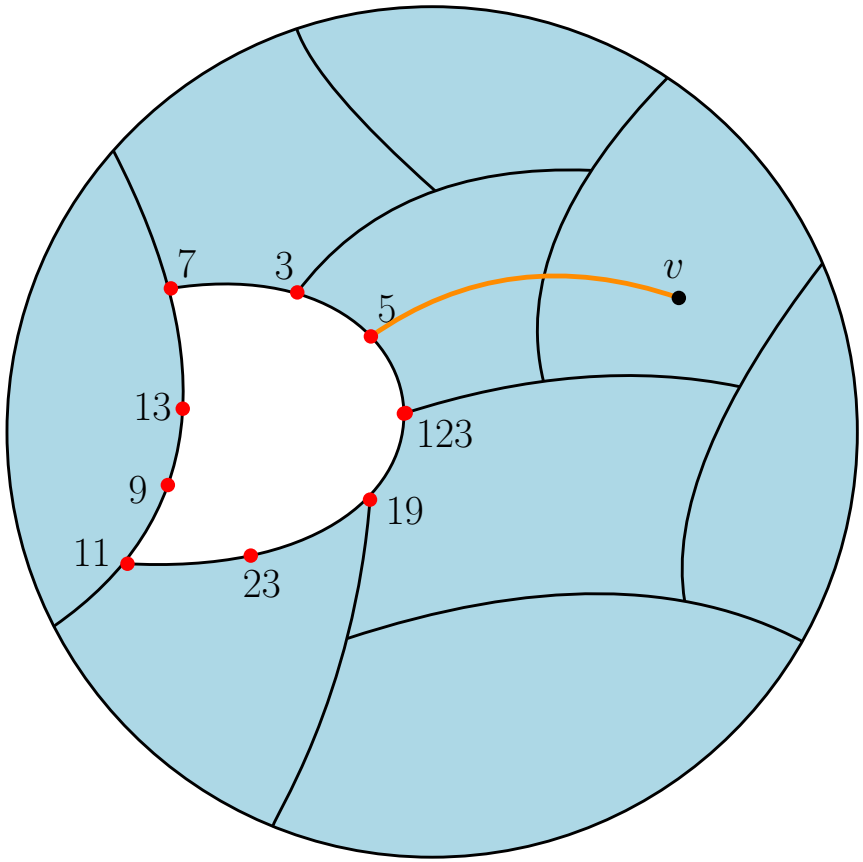

Gawrychowski et. al. [SODA'18]

Given an MSSP data structure for the outside of P, with sources ∂P, there exists an $\tilde{O}(|\partial P|)$-sized data structure for each set of additive weights for ∂P that answers point location queries in $\tilde{O}(1)$ time.

Warm-up 2: Space $\tilde{O}(n^{4/3})$, Query-time $\tilde{O}(1)$

- Compute an r-division.
- For each node $u \in P$, store $d_G(u, p)$ for $p \in \partial P$.
  Space $O(n \cdot \sqrt{r})$.
- For each piece P, store an MSSP data structure for the outside of P with sources ∂P.
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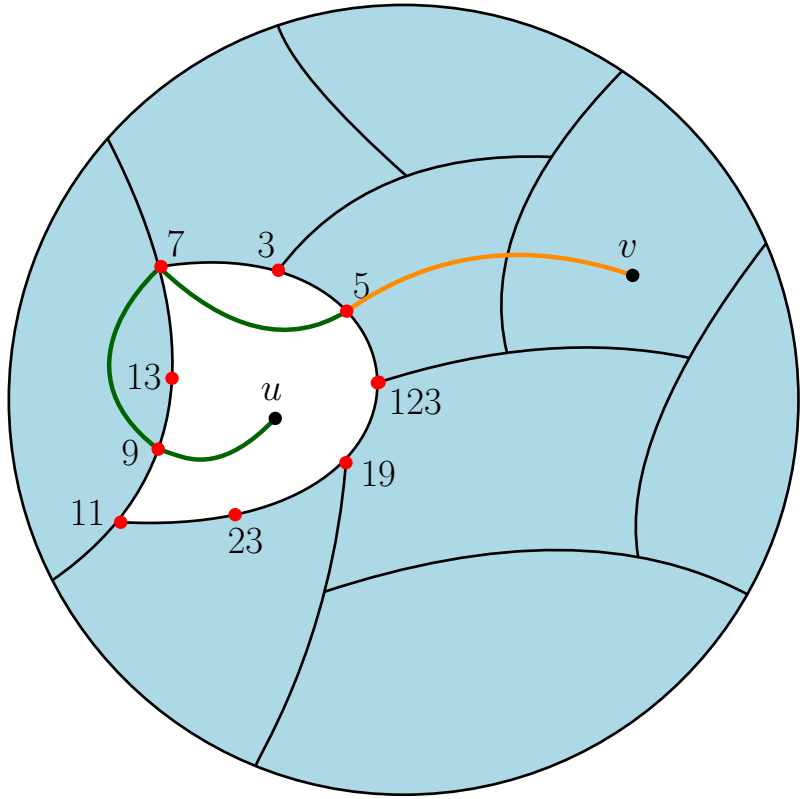

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- For each node $u \in P$, store $d_G(u, p)$ for $p \in \partial P$. Preprocess these for point location.
  Space $\tilde{O}(n \cdot \sqrt{r})$.
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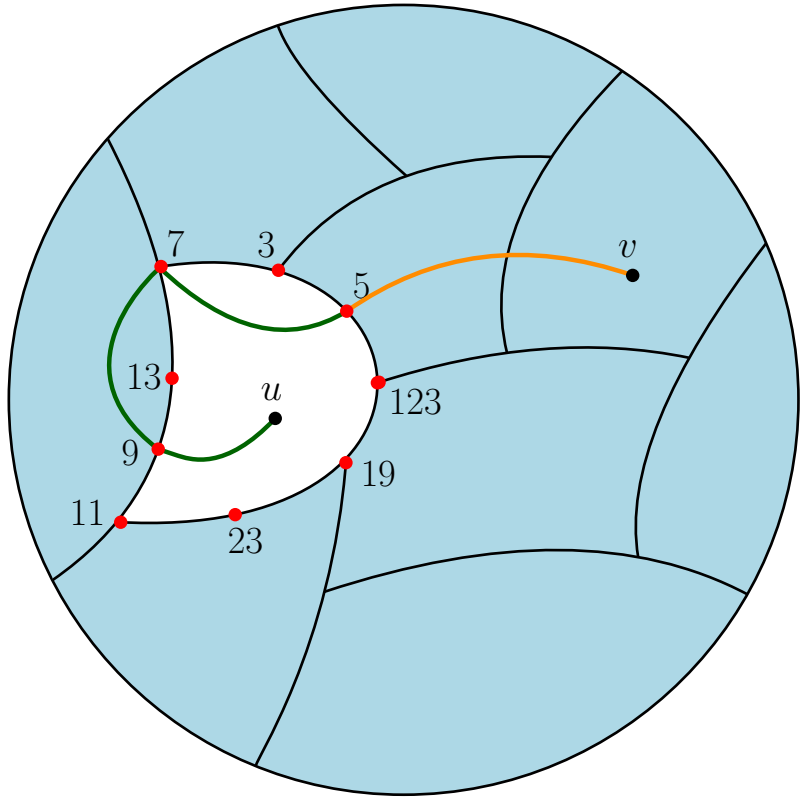
  Space $\tilde{O}(n/r \cdot n)$.



Perform point location!

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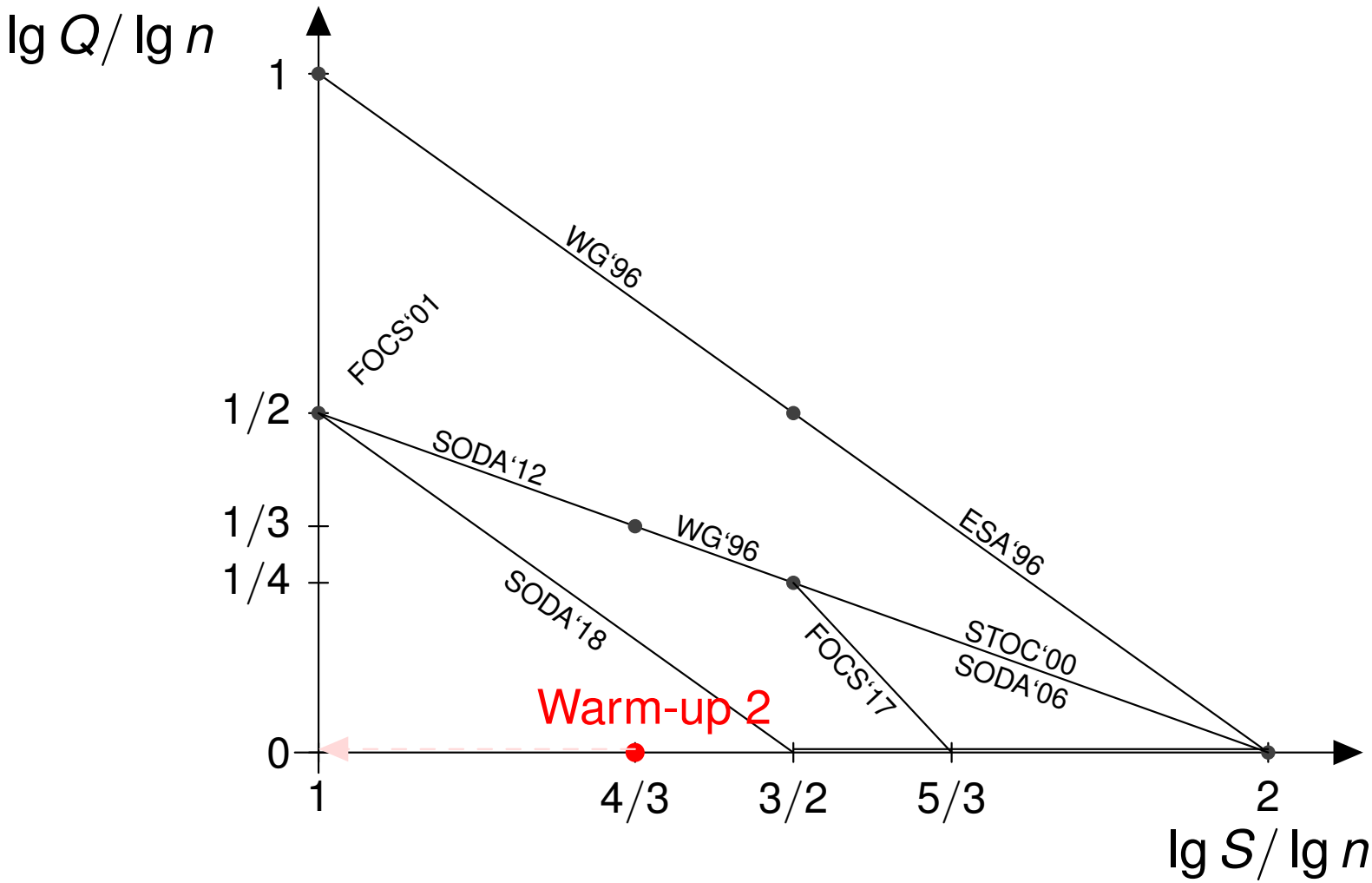
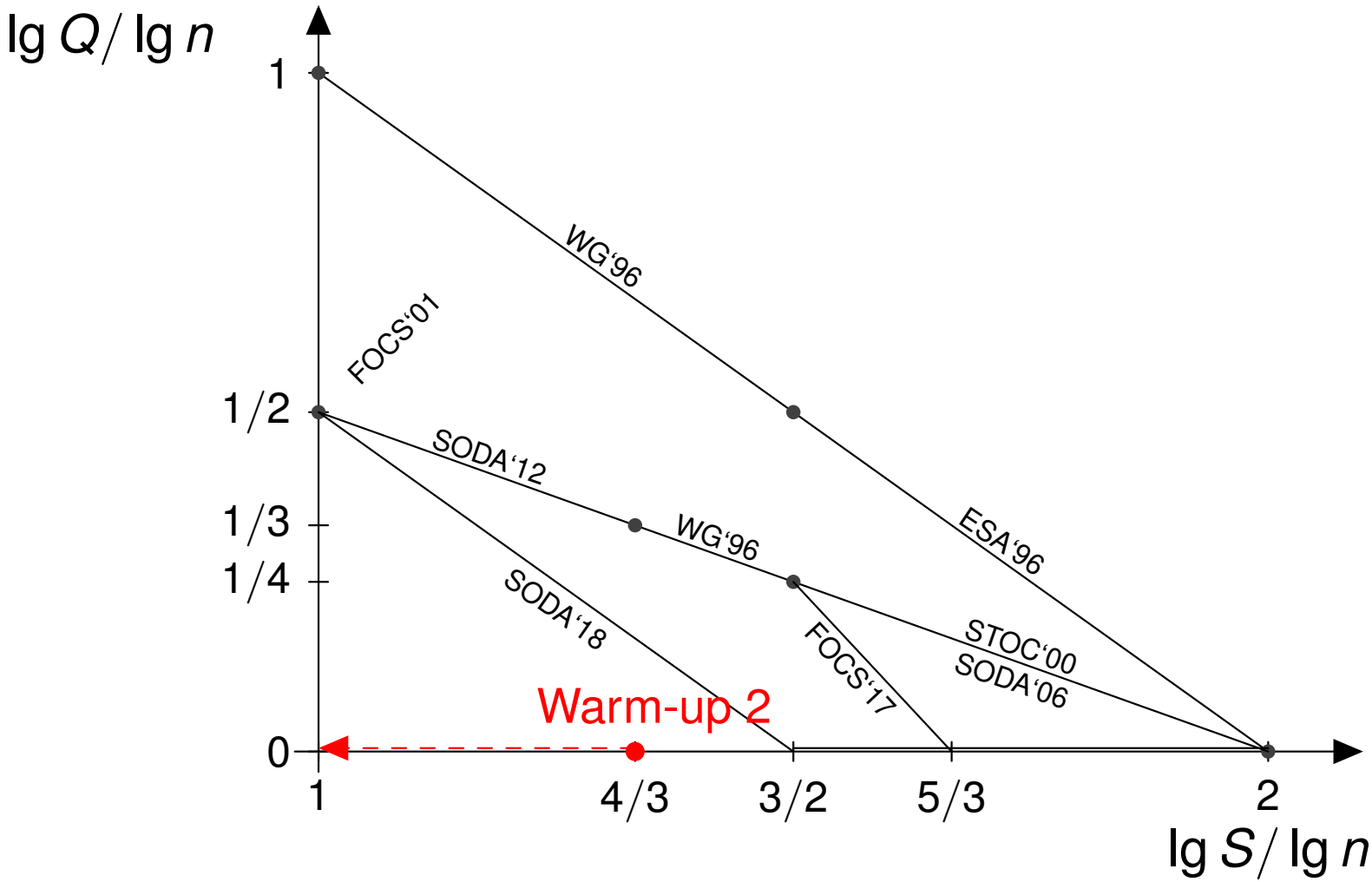


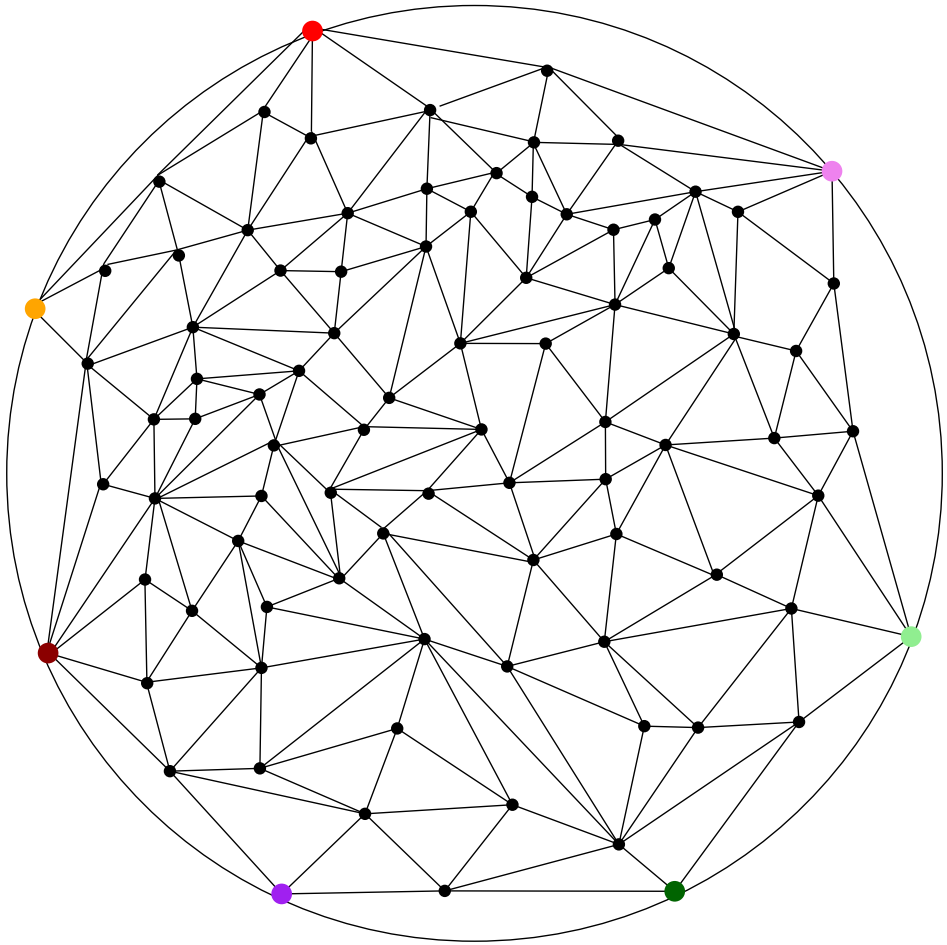
Warm-up 2: Space $\tilde{O}(n^{4/3})$, Query-time $\tilde{O}(1)$



Additively weighted Voronoi diagrams

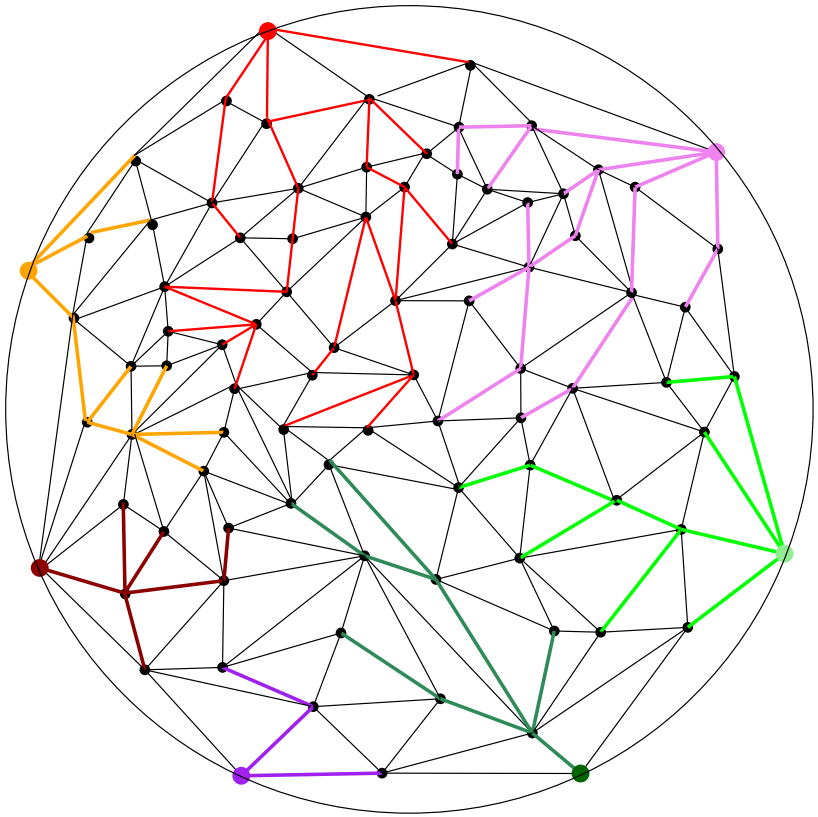
Internals of point location.

Additively weighted Voronoi diagrams



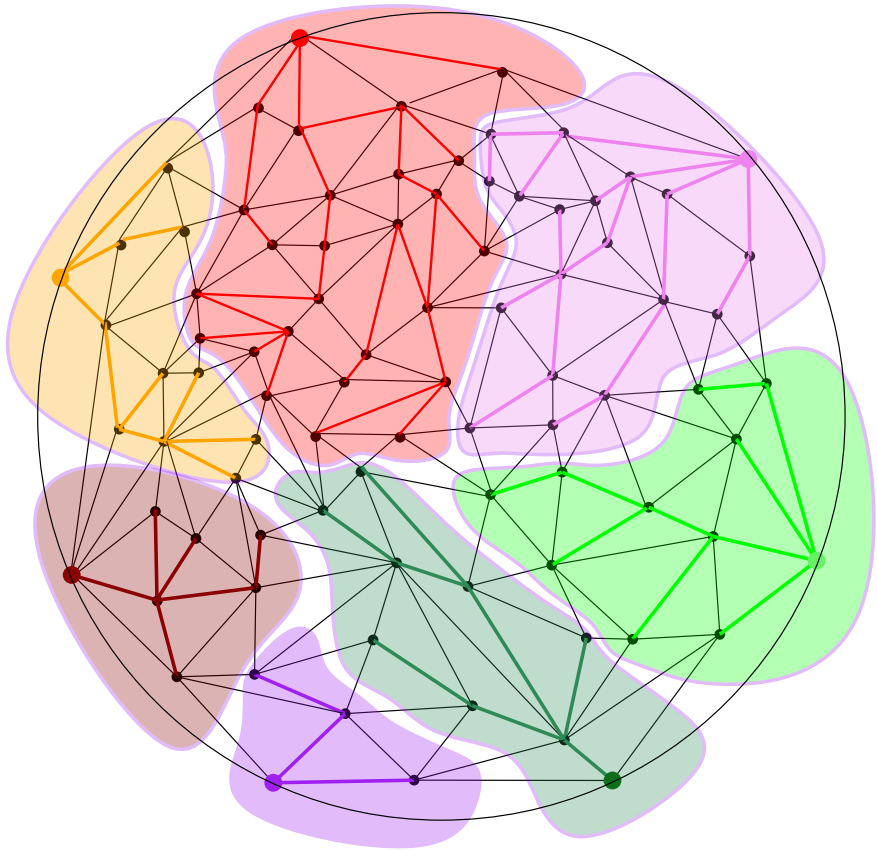
We are given weights for all nodes of ∂P, called sites.

Additively weighted Voronoi diagrams



The Voronoi cell of each site consists of all nodes closer to it with respect to the additive distances.

Additively weighted Voronoi diagrams



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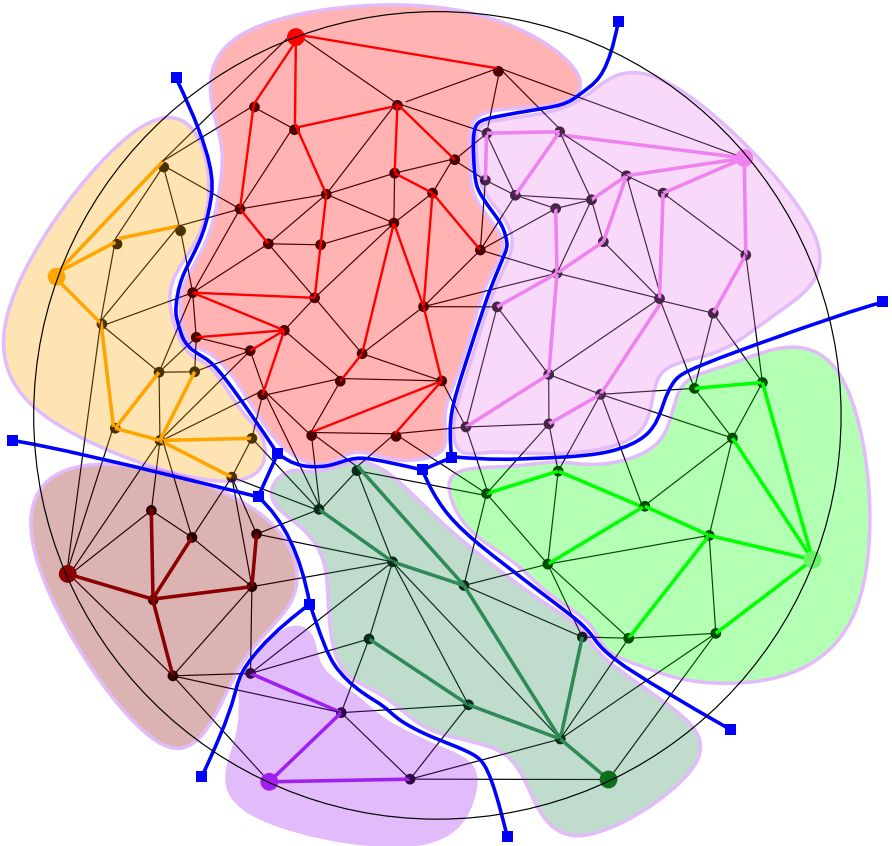
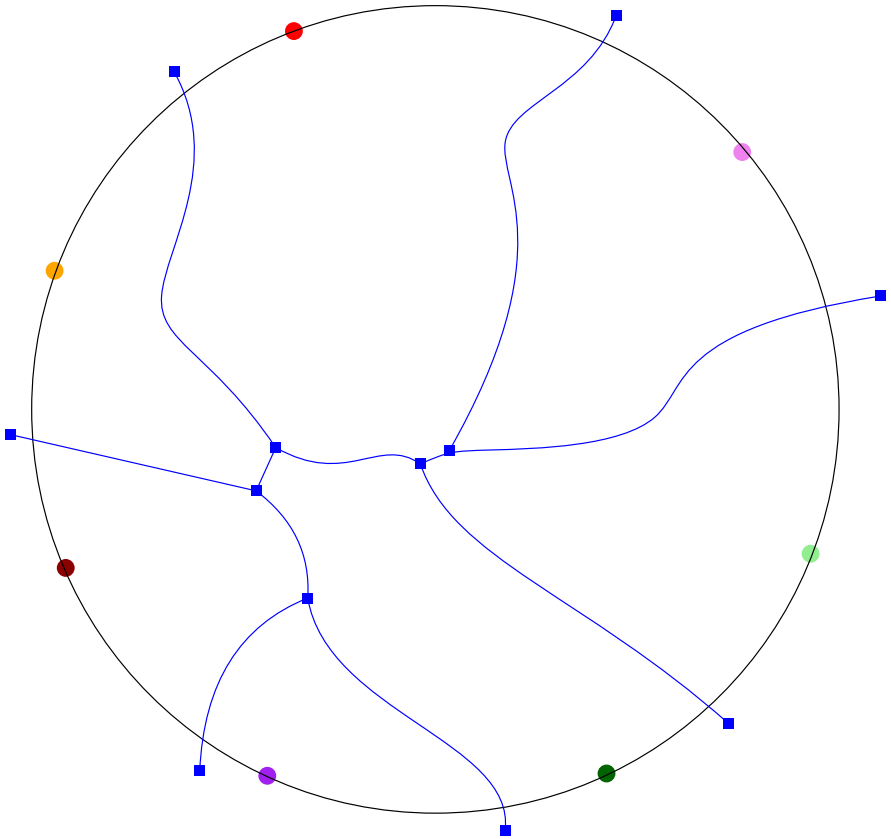
Additively weighted Voronoi diagrams



Because all sites are adjacent to one face, the diagram can be described by a tree on $O(|\partial P|) = O(\sqrt{r})$ nodes (independept of n!).

Additively weighted Voronoi diagrams



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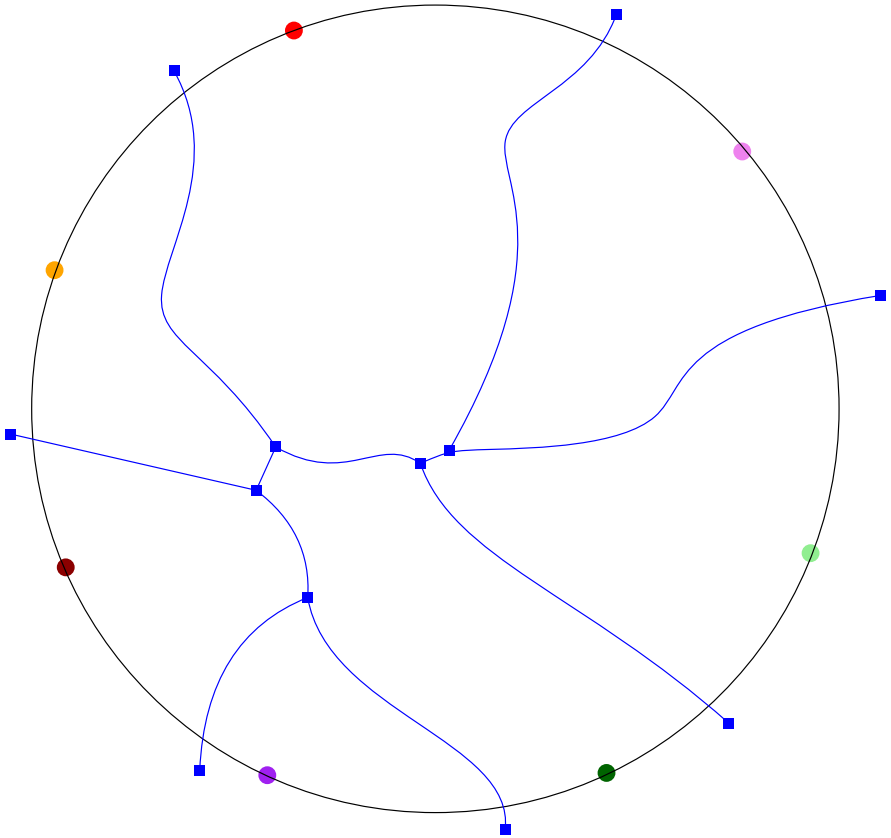
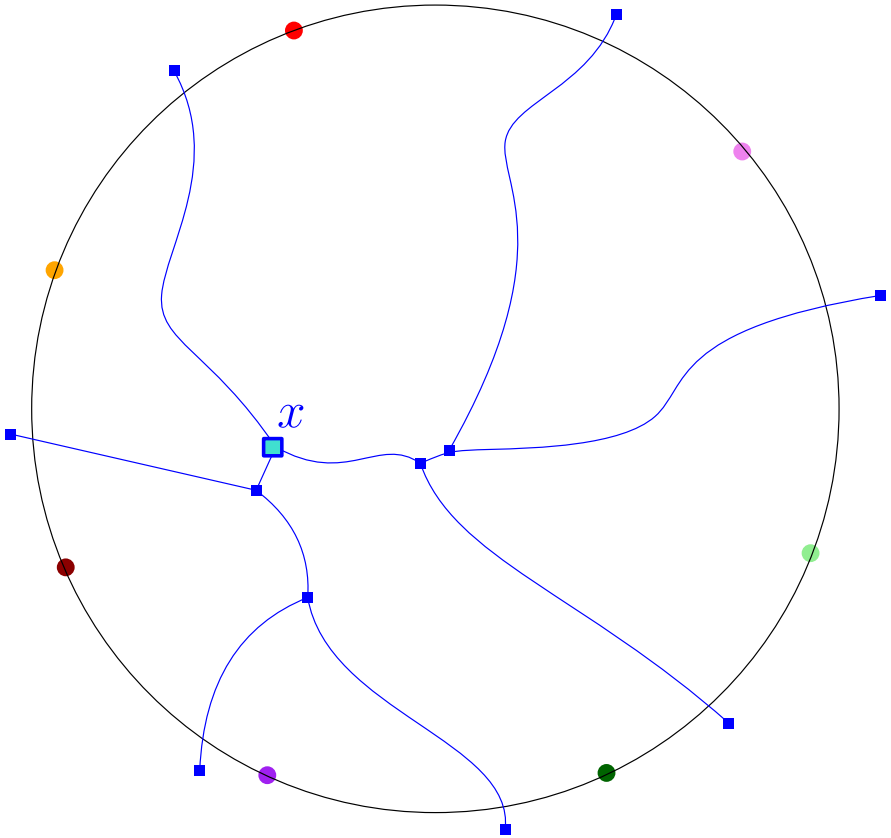
Point location



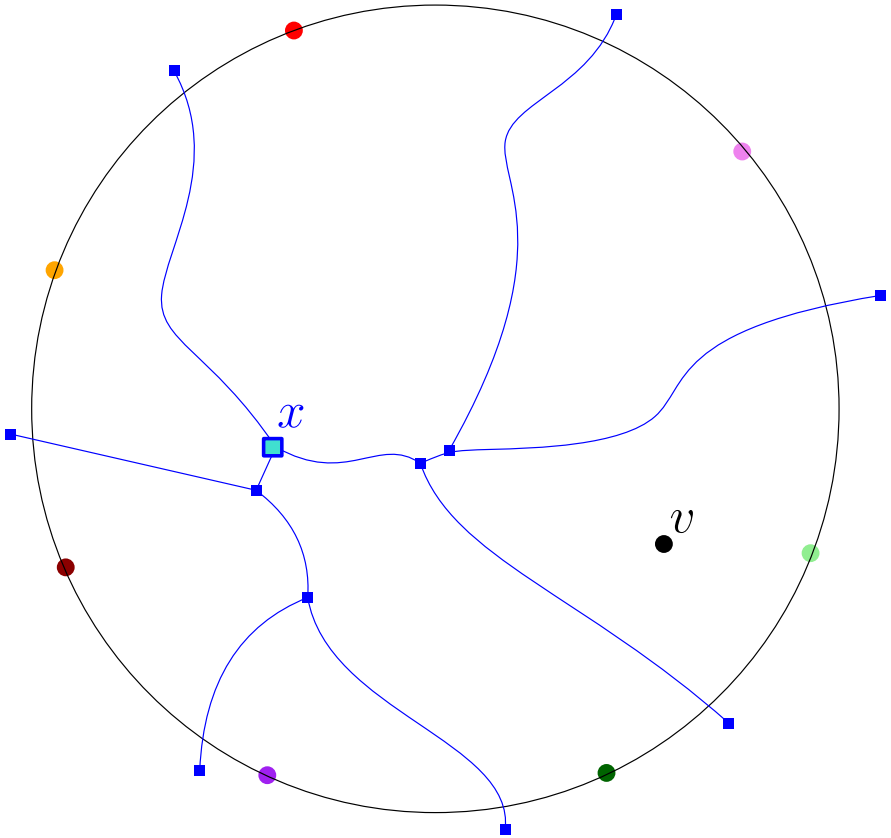
Point location

Any tree on k nodes contains a centroid node x such that every component of $T \setminus \{x\}$ is of size $\frac{2}{3}k$.



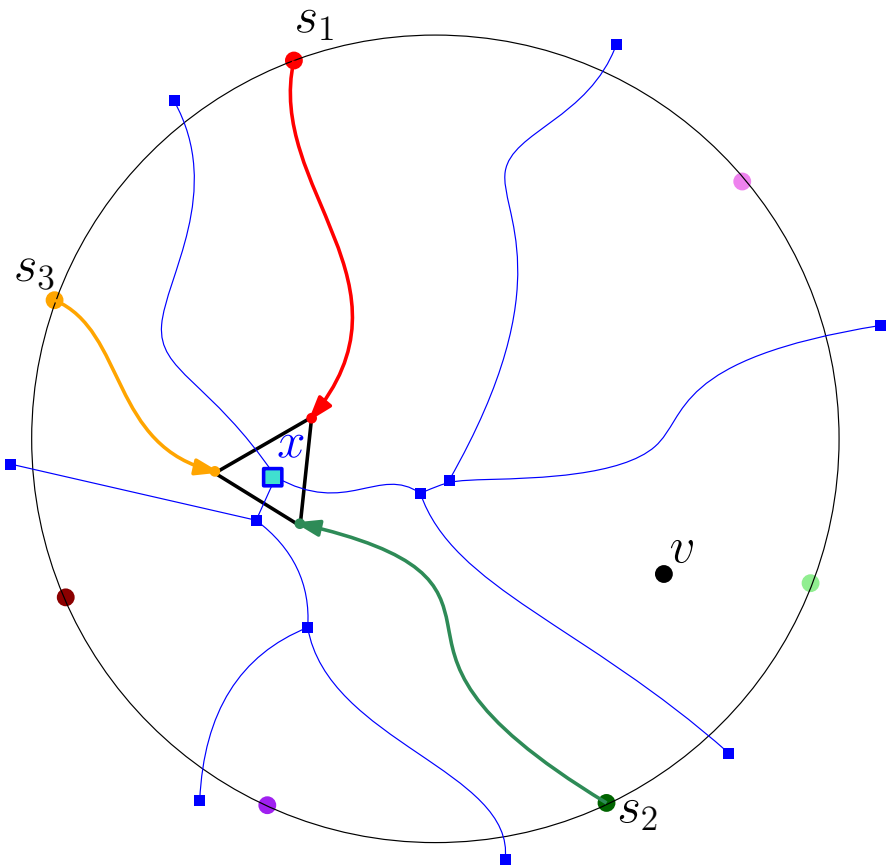
Point location

Main idea: consider the centroid. Find which subtree contains edges adjacent to the Voronoi cell containing v. Recurse.

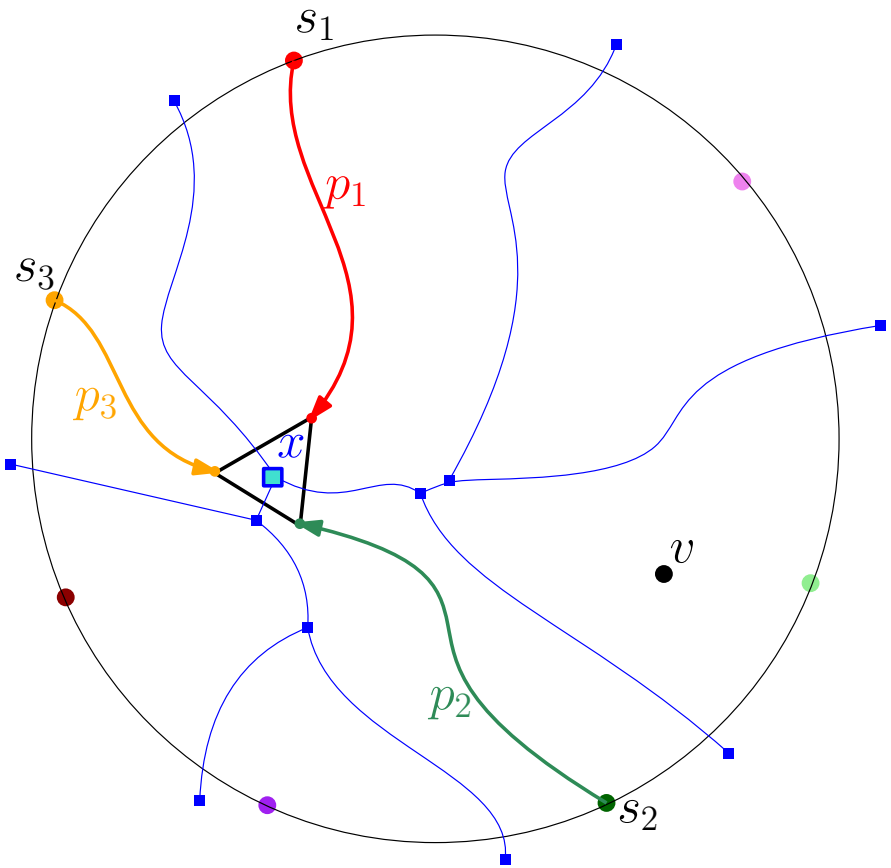


Point location

The centroid node corresponds to a trichromatic face of the original graph.



Point location



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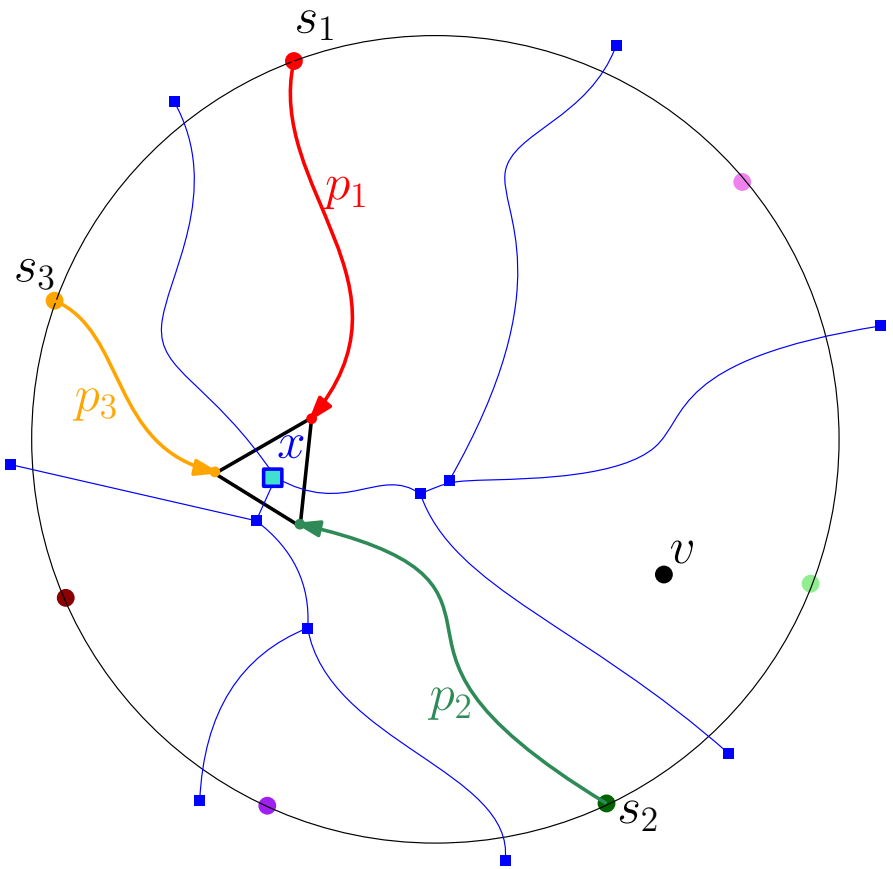
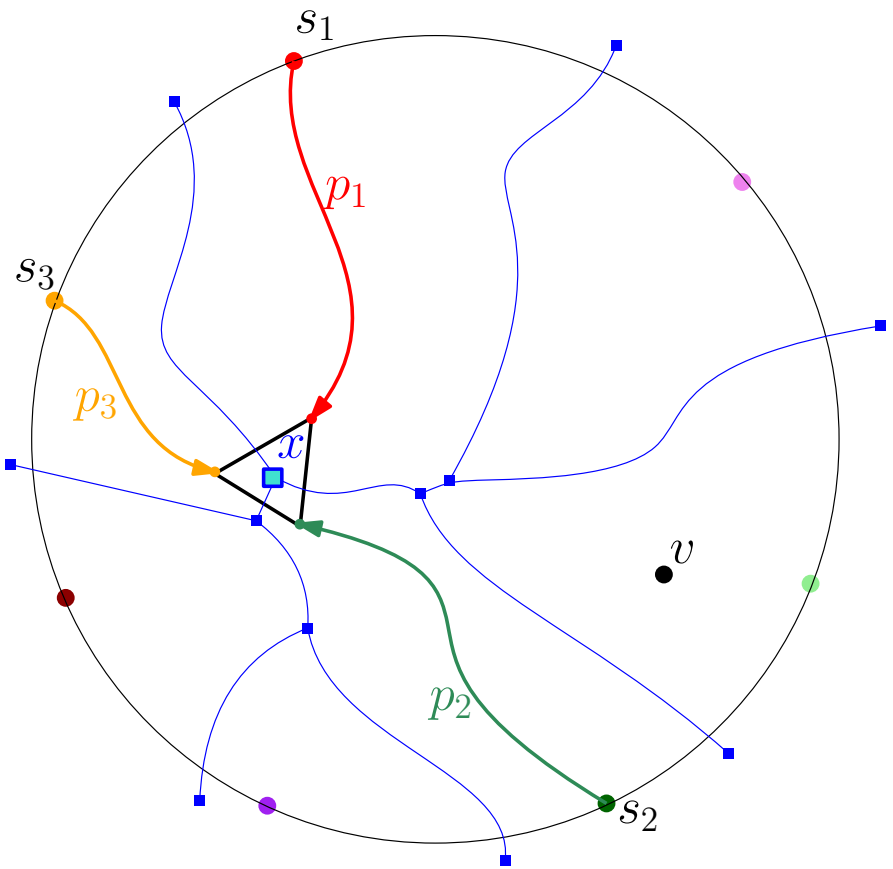


- Find which s_k among s_0, s_1, s_2 is closest to v. (distance query from boundary nodes)
- Check whether v is left or right of p_k. (left/right query)

Point location



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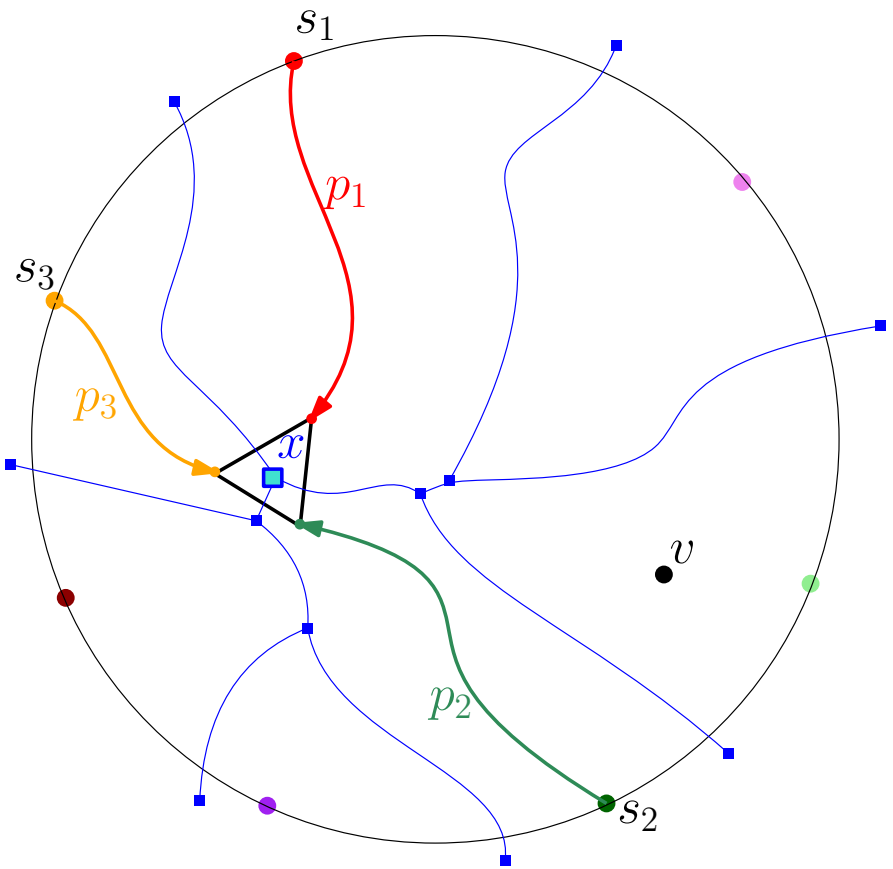

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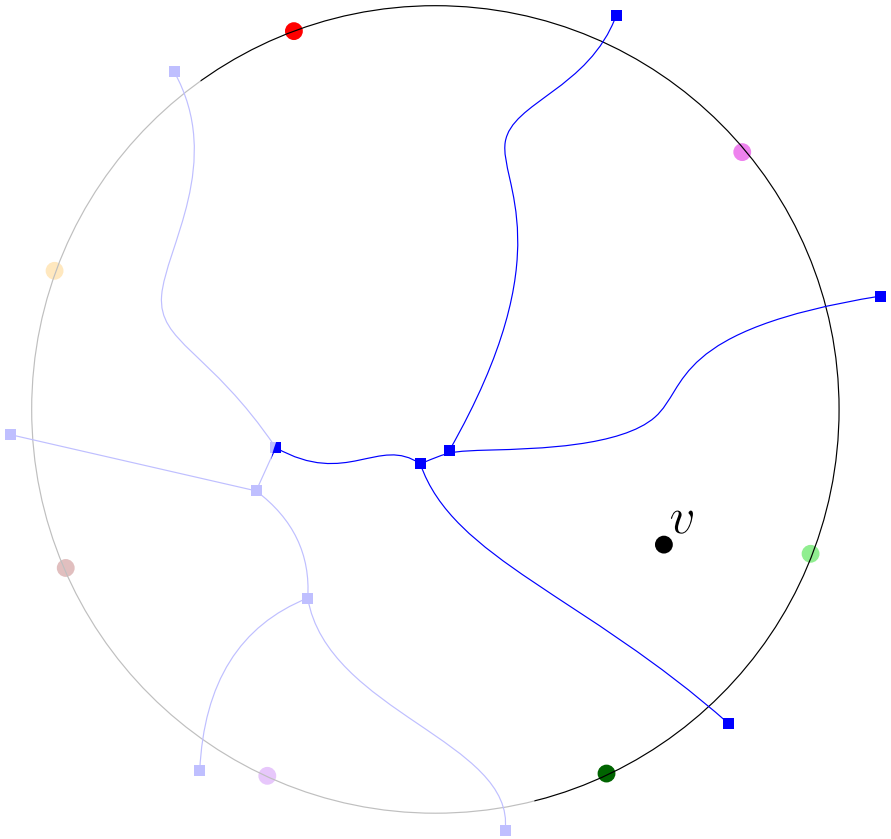

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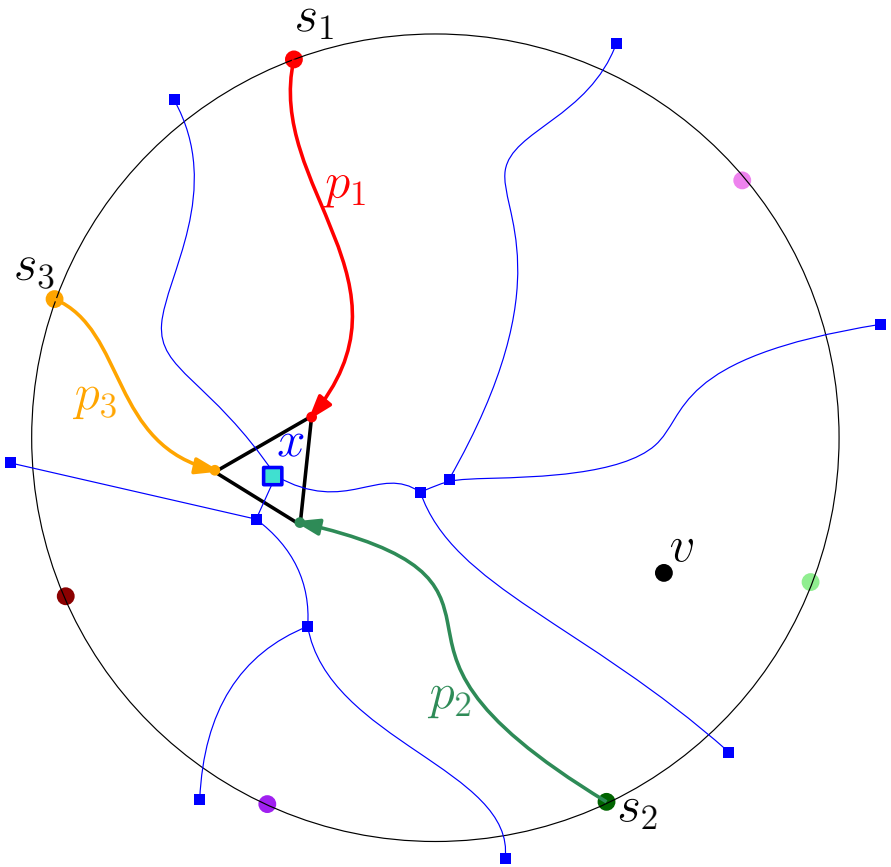
Recurse on the remaining part of the tree.

Point location



- Find which s_k among s_0, s_1, s_2 is closest to v. (distance query from boundary nodes)
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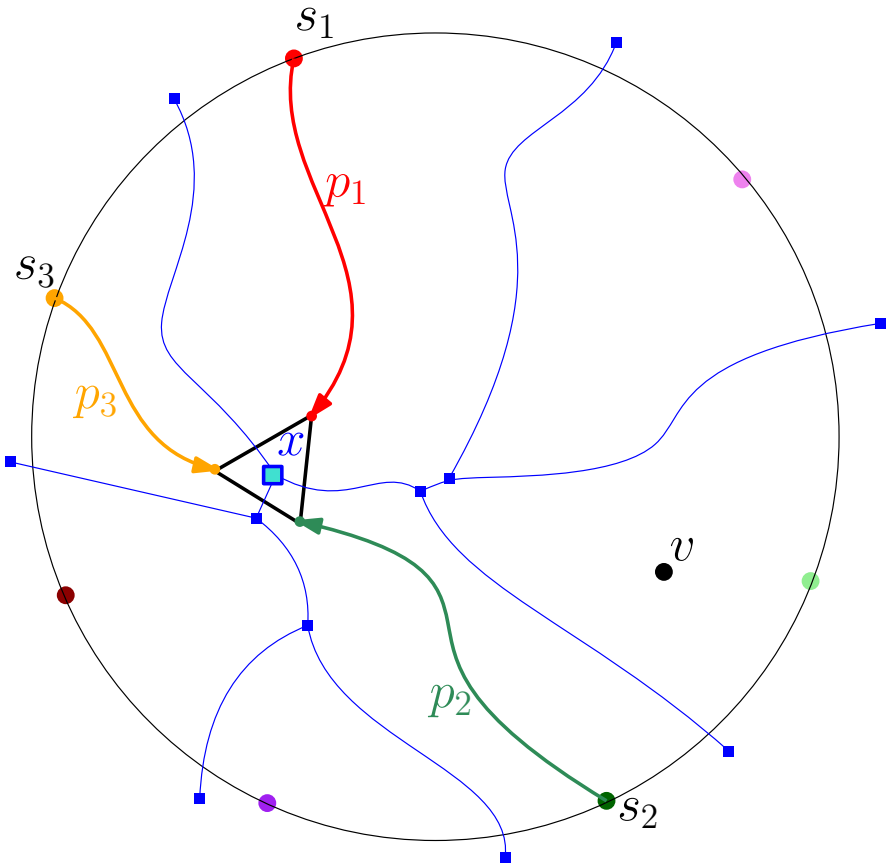
We can answer both of these queries with an MSSP data structure.

Point location



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Second goal

Find which s_k among s_0, s_1, s_2 is closest to v. (distance query from boundary nodes)

Idea: Recursion, exploiting the structure of such queries.

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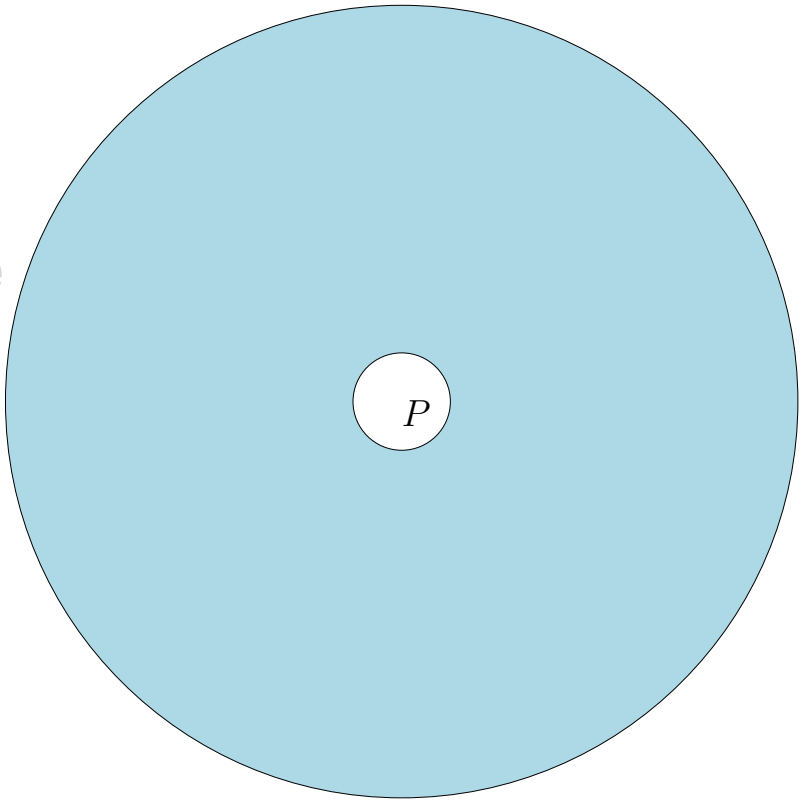
Main result: Space $\tilde{O}(n^{1+\epsilon})$, Query-time $\tilde{O}(1)$

- Compute an r-division for $r = n^{\epsilon}$.
- For each piece P, for each node $u \in P$, store a Voronoi diagram for the outside of P with sites ∂P and additive weight $d_G(u, p)$ for $p \in \partial P$. Space $\tilde{O}(n \cdot \sqrt{r})$.
- For each piece P, store the required information to answer distance queries from ∂P to nodes outside P.



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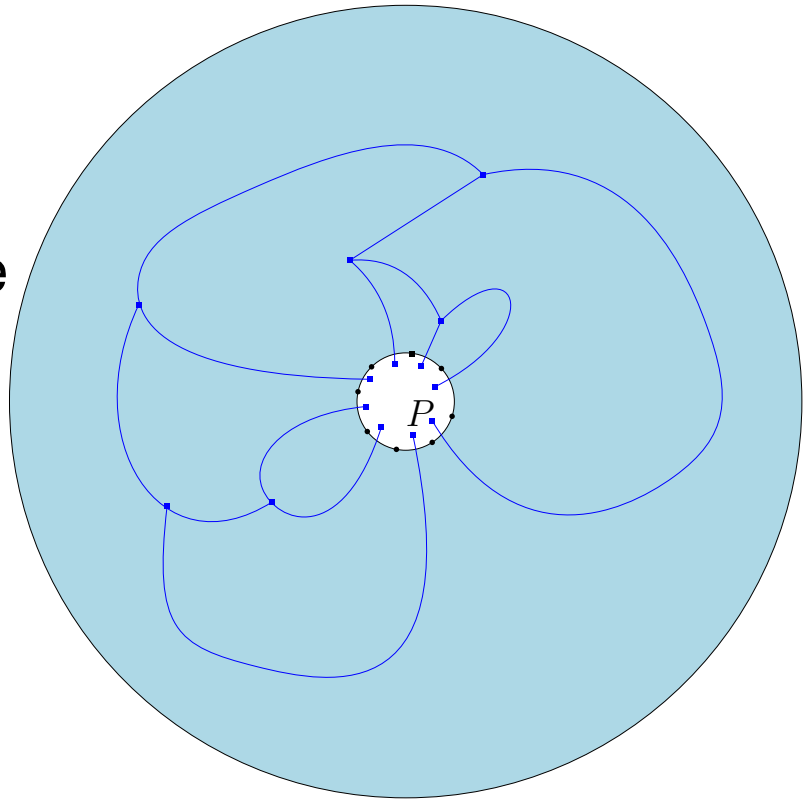

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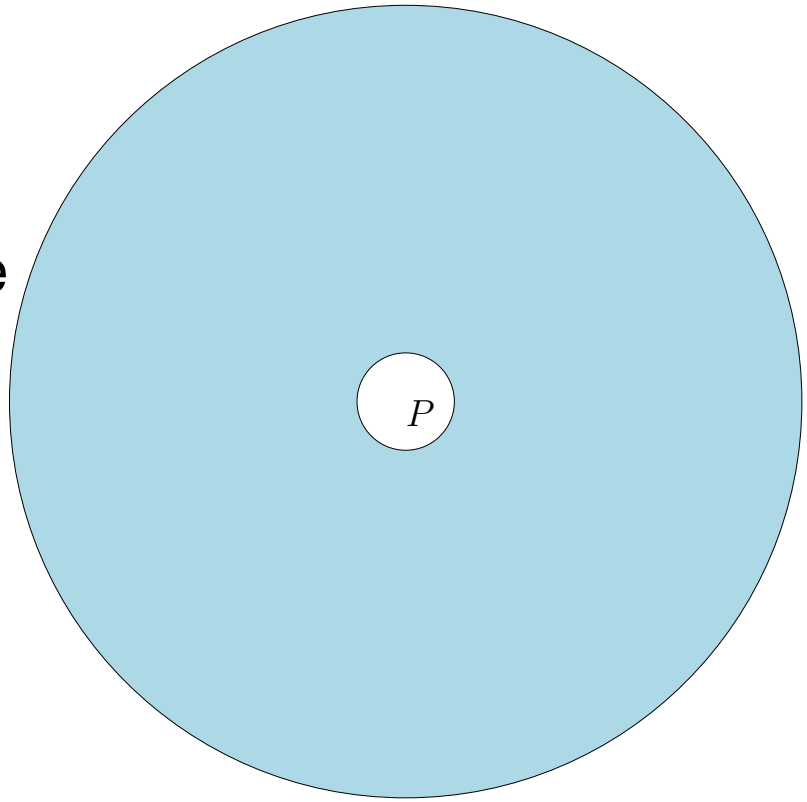

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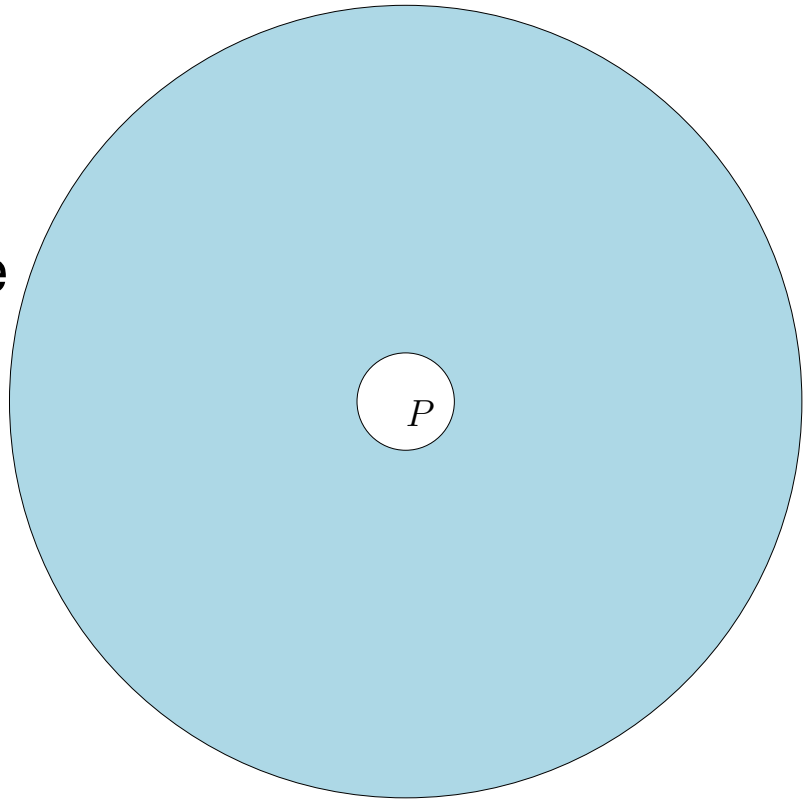
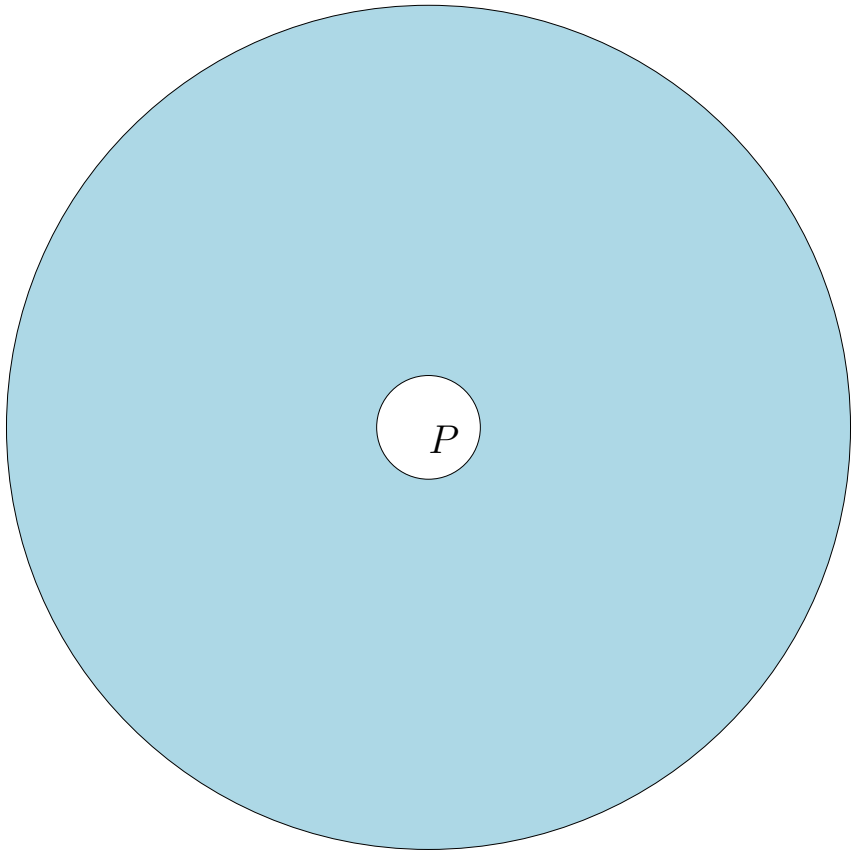


Perform point location.

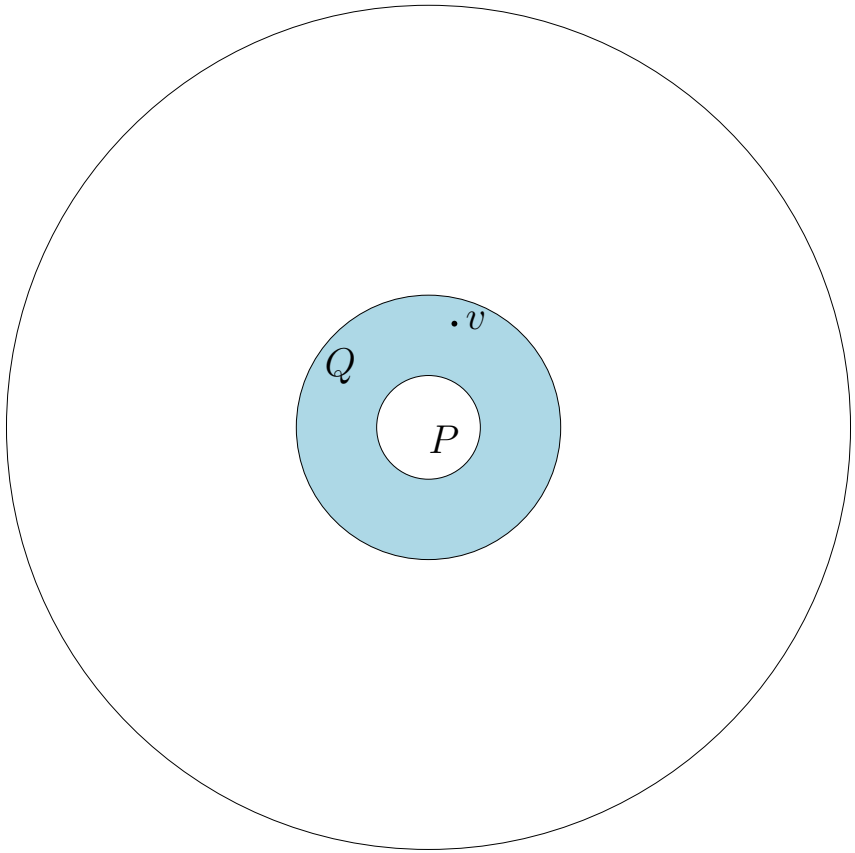
Main result: Space $\tilde{O}(n^{1+\epsilon})$, Query-time $\tilde{O}(1)$

We have $n^{1-\epsilon}$ pieces and we can thus not afford to store an $\tilde{O}(n)$-sized MSSP for each of them.



Main result: Space $\tilde{O}(n^{1+\epsilon})$, Query-time $\tilde{O}(1)$

However, we can store an MSSP for a piece Q that contains P and belongs to a coarser r'-division, for $r' = n^{2\epsilon}$. This requires $\tilde{O}(n^{1-\epsilon} \cdot n^{2\epsilon}) = \tilde{O}(n^{1+\epsilon})$ space, and handles the case $v \in Q$.



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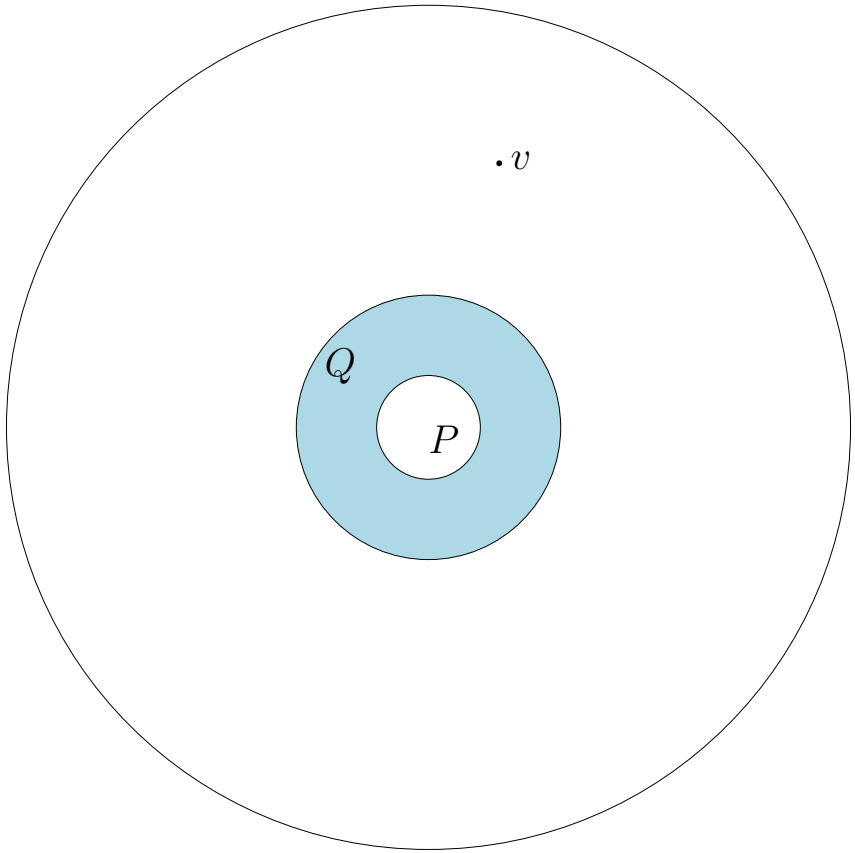
To handle the case $v \notin Q$, each $p \in \partial P$ stores a Voronoi diagram for the outside of Q with sites ∂Q. This requires $\tilde{O}(n^{1-\epsilon} \cdot n^{\epsilon/2} \cdot n^{2\epsilon/2}) = = \tilde{O}(n^{1+\epsilon/2})$ space. Recursively support point location outside Q.



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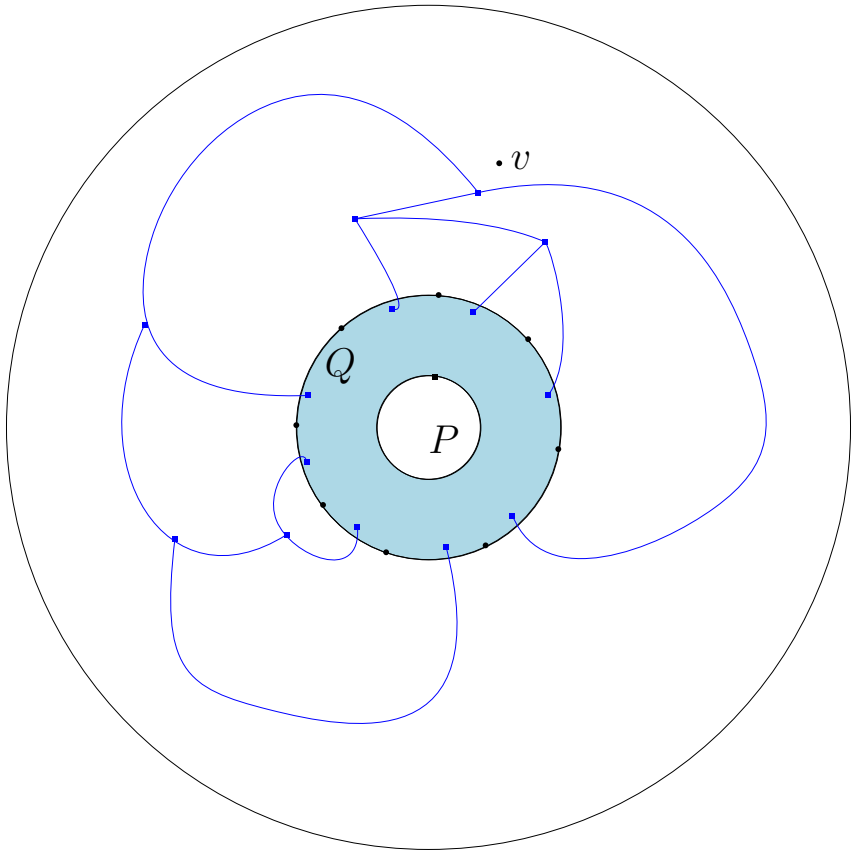
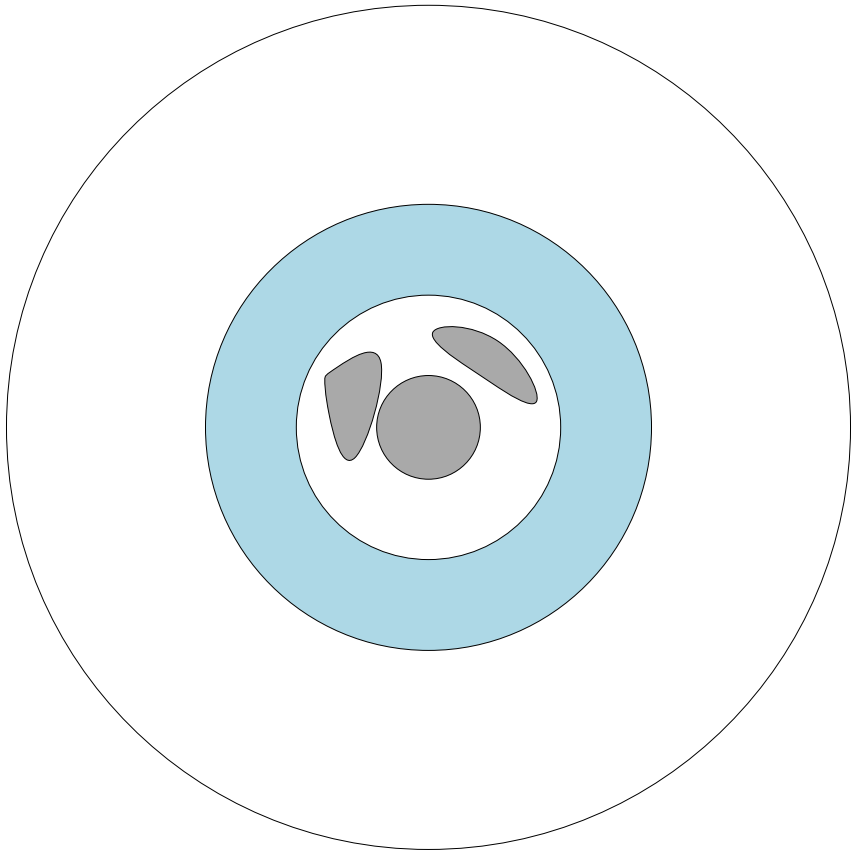


Main result: Space $\tilde{O}(n^{1+\epsilon})$, Query-time $\tilde{O}(1)$

Repeat the same reasoning for increasingly larger pieces of sizes $n^{i\cdot\epsilon}$, for $i = 1, \ldots, 1/\epsilon$. There are $n^{1-i\epsilon}$ pieces at level i, each stores MSSP and Voronoi diagrams of size $\tilde{O}(n^{(i+1)\epsilon})$. Total space is $\tilde{O}(\epsilon^{-1} n^{1+\epsilon})$.

Main result: Space $\tilde{O}(n^{1+\epsilon})$, Query-time $\tilde{O}(1)$

The idea is that smaller pieces share the cost of MSSP data structures at higher levels.



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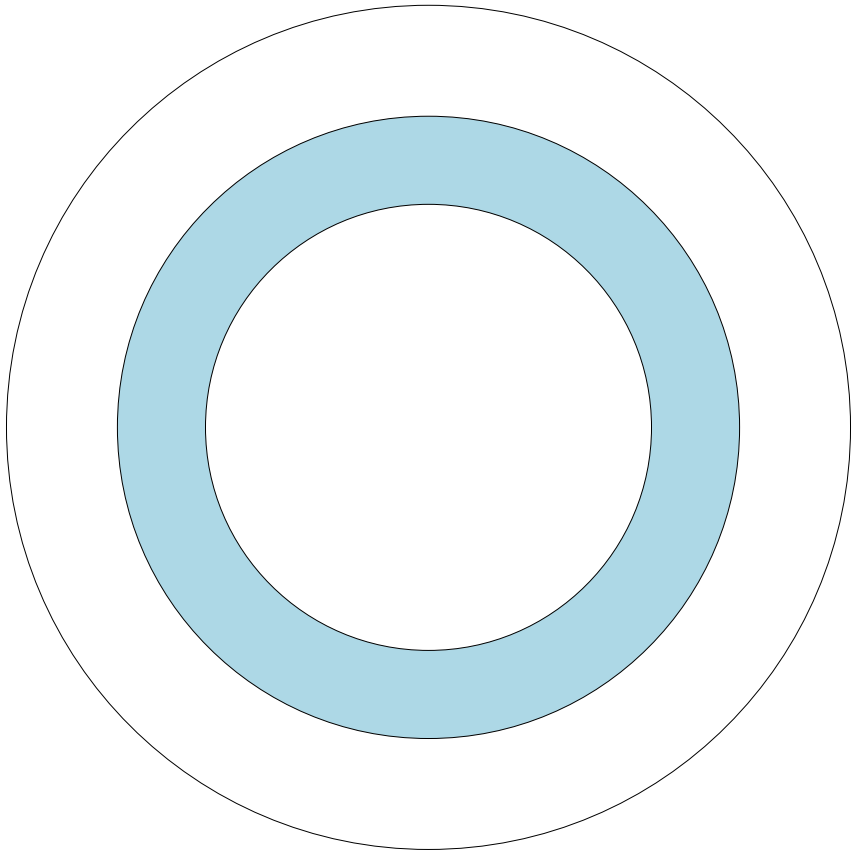

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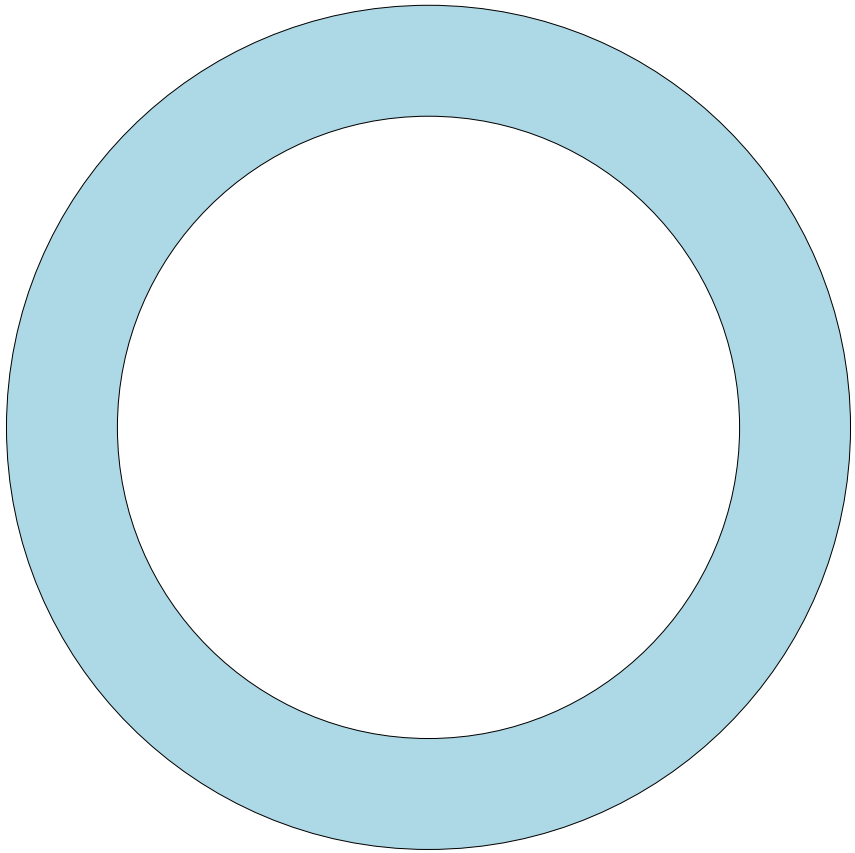
Each point location query, either gets answered at the current level, or reduces to $O(\log n)$ point location queries at a higher level.

Main result: Space $\tilde{O}(n^{1+\epsilon})$, Query-time $\tilde{O}(1)$

If not earlier, then in the top level we can answer the query in $O(\log^2 n)$ time. So the overall query time is $O(\log^{1/\epsilon} n)$.



Tradeoffs and construction time

We show the following tradeoffs for $\langle S, Q \rangle$:

1. $\langle \tilde{O}(n^{1+\epsilon}), O(\log^{1/\epsilon} n) \rangle$, for any constant $1/2 \geq \epsilon > 0$;
2. $\langle O(n \log^{2+1/\epsilon} n), \tilde{O}(n^{\epsilon}) \rangle$, for any constant $\epsilon > 0$;
3. $\langle n^{1+o(1)}, n^{o(1)} \rangle$.

Some of the issues I shoved under the rug:

- handling left/right queries.
- ∂P is not a single face of P (holes).
- constructing these oracles in $O(n^{3/2+\epsilon})$ time.

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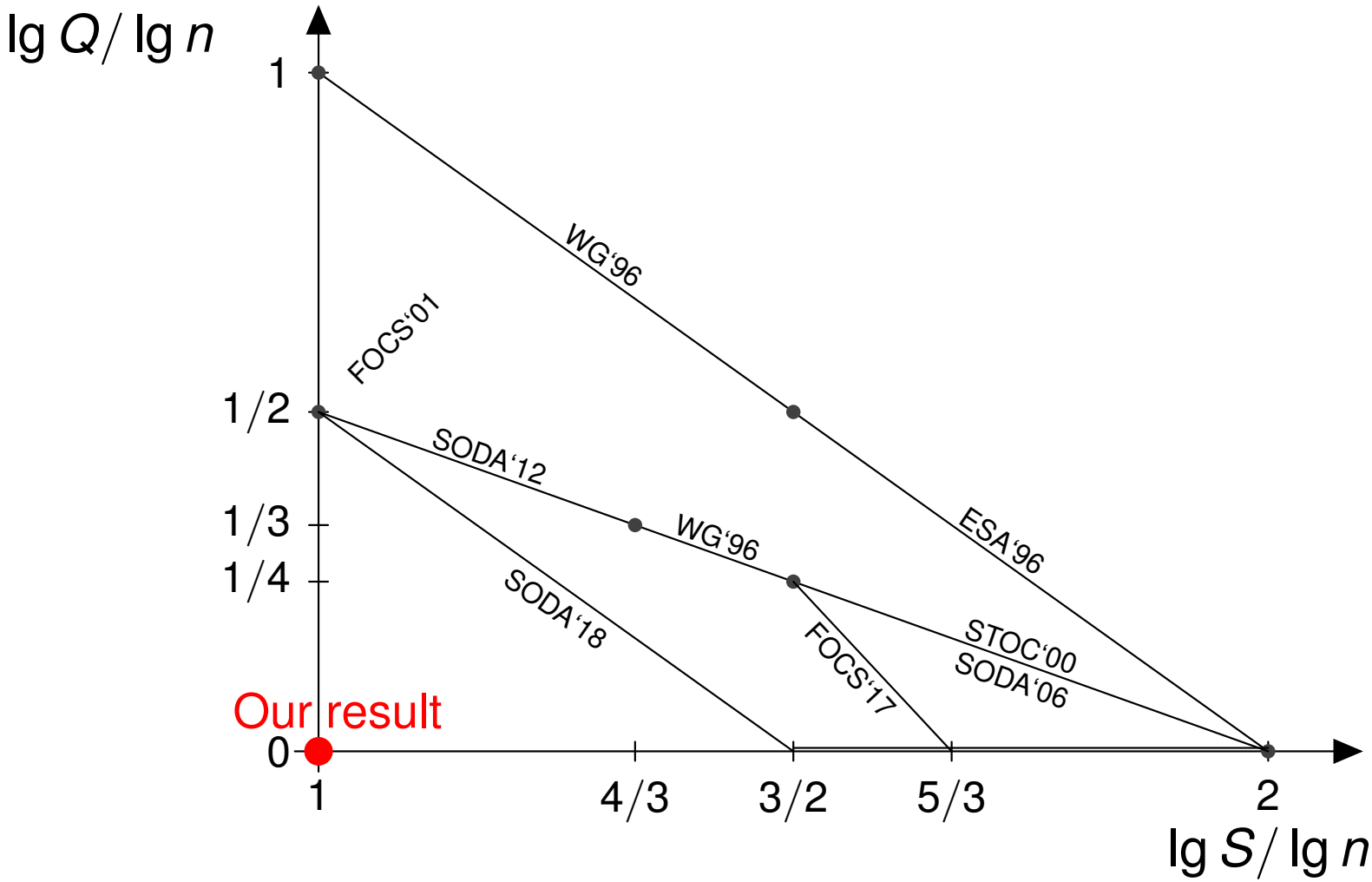
Open problems

- Can we get $\tilde{O}(n)$ space and $\tilde{O}(1)$ query time?
- Can we get the construction time to be $\tilde{O}(n)$?
- Improvements on dynamic distance oracels?
  currently:
  1. exact: UB $\tilde{O}(n^{2/3})$; LB $\tilde{O}(n^{1/2})$ (conditioned on APSP)
  2. approx.: UB $\tilde{O}(n^{1/2})$ (undirected) ; no LB.


lg Q/ lg n
1
1/2
1/3
1/4
0
1
4/3
3/2
5/3
2
lg S/ lg n
WG'96
FOCS'01
SODA'12
WG'96
ESA'96
SODA'18
FOCS'17
STOC'00
SODA'06
Our result

Questions?