High Order Asymptotic Preserving Schemes for Some Discrete-Velocity Kinetic Equations

Fengyan Li

Department of Mathematical Sciences

Rensselaer Polytechnic Institute

Troy, NY, USA

Joint work with: J. Jang (UC Riverside), J.-M. Qiu (Houston), T. Xiong (Houston)

lif@rpi.edu

Outline

- Discrete-velocity kinetic equations
- High order asymptotic preserving (AP) methods
- Theoretical results
- Numerical examples

Discrete-velocity Kinetic Equations

Consider the discrete-velocity model

$$\partial_t f + v \cdot \nabla_x f = \mathcal{C}(f) \tag{1}$$

$f = f(x, v, t)$: distribution function of particles dependent of time $t > 0$, position x, and velocity $v \in \{-1, 1\}$. $\mathcal{C}(f)$: collision operator ($\int \mathcal{C}(f) dv = 0$).

We are particularly interested in the discrete-velocity model in a diffusive scaling (under the scaling of $t' := \varepsilon^2 t$ and $x' := \varepsilon x$):

$$\varepsilon \partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \mathcal{C}(f) \tag{2}$$

The parameter $\varepsilon > 0$ can be regarded as the mean free path of the particles, and it measures the distance of the system to the equilibrium state. The smaller ε is, the closer the system is to the equilibrium.

Examples: Jin-Pareschi-Toscani (1998)

$d\mu$: discrete Lebesgue measure on $\{-1, 1\}$

$\langle\cdot\rangle$: $\langle f\rangle = \int f d\mu = \frac{1}{2}(f(x, v=-1, t) + f(x, v=1, t))$

	$\mathcal{C}(f)$	when $\varepsilon \to 0$, with $\rho = \langle f\rangle$
(a.1)	$\langle f\rangle - f$	$\partial_t \rho = \partial_{xx}\rho$
(a.2)	$\langle f\rangle - f + A\varepsilon v\langle f\rangle$	$\partial_t \rho + A\partial_x \rho = \partial_{xx}\rho$
(a.3)	$C\langle f\rangle^m(\langle f\rangle - f)$	$\partial_t \rho = \frac{C}{1-m}\partial_{xx}(\rho^{1-m}), m \neq 1$
(a.4)	$\langle f\rangle - f + C\varepsilon\left[\langle f\rangle^2 - (\langle f\rangle - f)^2\right] v$	$\partial_t \rho + C\partial_x \rho^2 = \partial_{xx}\rho$

Note:

- (a.1): telegraph equation
- $\partial_t \rho = \frac{1}{1-m}\partial_{xx}(\rho^{1-m})$ with $m < 0$: porous medium equation

In some applications, ε may differ in several orders of magnitude from the rarefied regime ($\varepsilon = O(1)$) to the hydrodynamic (diffusive) regime ($\varepsilon << 1$). It is desirable to design a class of numerical methods which work uniformly with respect to the parameter ε.

Objective: to design high order asymptotic preserving methods for the discrete-velocity kinetic equation in the diffusive scaling

Asymptotically preserving (AP) methods: *Jin, Levermore, Naldi, Pareschi, Degond, Toscani, Klar, Filbet, Carrillo, Lemou, Mieussens, Hauck, Liu,* $\cdots$

- uniformly stable with respect to ε ranging from $O(1)$ to 0;
- When $\varepsilon \to 0$, the methods are consistent for the limiting equation on fixed mesh.

Numerical challenges and considerations

$$\varepsilon \partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon}\mathcal{C}(f)$$

- Stiffness in both the convective and the collision terms
- The characteristic speed of the homogeneous hyperbolic part is $\frac{1}{\varepsilon}$. Standard schemes for hyperbolic problems with stiff relaxation (*implicit treatment for the collision term + explicit treatment for the convection term*) may impose a restrictive stability condition:

$$\Delta t \approx \varepsilon \Delta x \qquad (3)$$

- Not all stable schemes can capture the diffusive limit when $\varepsilon \to 0$ on under-resolved meshes $(\Delta t, \Delta x >> \varepsilon)$
- Easy to solve

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High Order Asymptotic Preserving Methods

Three components

- Micro-macro decomposition of the equation
- Discontinuous Galerkin (DG) spatial discretization: *numerical flux*
- Globally stiffly accurate implicit-explicit (IMEX) Runge-Kutta temporal discretization: *implicit-explicit strategy*

Major references:

- Jin-Pareschi-Toscani (1998)
- Lemou-Mieussens (2010), Liu-Mieussens (2010)
- Boscarino-Pareschi-Russo (2013)

Main results: a family of high order methods are proposed for discrete-velocity kinetic equations in the diffusive scaling.

- For the telegraph equation, when a first order temporal discretization is applied, uniform stability is established with respect to ε. Error estimates are also obtained for any ε.
- Formal asymptotic analysis shows that the proposed schemes in the limit of $\varepsilon \to 0$ provide explicit and consistent high order methods for the limiting equations.

1. Micro-macro decomposition

Consider the Hilbert space $L^2(d\mu)$ with the inner product $\langle\cdot,\cdot\rangle$:

$$\langle f_1, f_2\rangle = \int f_1 f_2 d\mu,$$

and an orthogonal projection Π: $\Pi f = \langle f\rangle$.

Let $f = \Pi f + (\mathbf{I} - \Pi)f =: \rho + \varepsilon g$, then the micro-macro decomposition of (2) is

$$\partial_t \rho + \partial_x \langle vg\rangle = 0 \tag{4a}$$

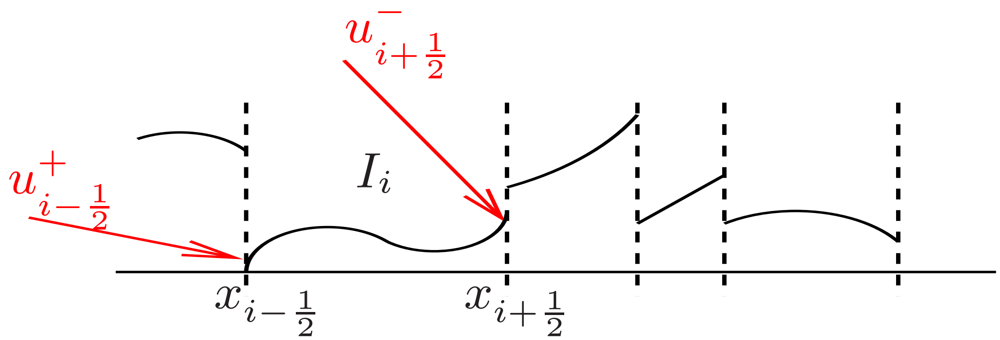
$$\partial_t g + \frac{1}{\varepsilon^2} v\partial_x \rho + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}(v\partial_x g) = \frac{1}{\varepsilon^3}\mathcal{C}(\rho + \varepsilon g) \tag{4b}$$

Note: Solving for f from (2) is equivalent to solving for ρ and g from (4).

2. Discontinuous Galerkin (DG) spatial discretization

Mesh: $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $h_i = |I_i|$

Discrete space: $U_h = U_h^k = \{u : u \in P^k(I_i),\ \forall i\}$, with $P^k(I_i)$ consisting of polynomials of degree at most k



Notations:

$$u^{\pm} = u^{\pm}(x) = \lim_{\Delta x \to 0\pm} u(x + \Delta x)$$

$$\text{average} : \quad \{u\} = \frac{1}{2}(u^- + u^+)$$

$$\text{jump} : \quad [u] = u^+ - u^-$$

Semi-discrete DG method: Look for $\rho_h(\cdot,t), g_h(\cdot,v,t) \in U_h$, such that $\forall \phi, \psi \in U_h$, and i,

$$\int_{I_i} \partial_t \rho_h \phi dx - \int_{I_i} \langle v g_h \rangle \partial_x \phi dx + \widehat{\langle v g_h \rangle}_{i+\frac{1}{2}} \phi^-_{i+\frac{1}{2}} - \widehat{\langle v g_h \rangle}_{i-\frac{1}{2}} \phi^+_{i-\frac{1}{2}} = 0, \tag{5a}$$

$$\begin{aligned} &\int_{I_i} \partial_t g_h \psi dx + \frac{1}{\varepsilon} \int_{I_i} (\mathbf{I} - \Pi) \mathcal{D}_h(g_h; v)\, \psi dx \\ &\qquad - \frac{1}{\varepsilon^2} \left(\int_{I_i} v \rho_h \partial_x \psi dx - v \widehat{\rho}_{h,i+\frac{1}{2}} \psi^-_{i+\frac{1}{2}} + v \widehat{\rho}_{h,i-\frac{1}{2}} \psi^+_{i-\frac{1}{2}} \right) \\ &\qquad = \frac{1}{\varepsilon^3} \int_{I_i} \mathcal{C}(\rho_h + \varepsilon g_h) \psi dx. \end{aligned} \tag{5b}$$

Numerical flux:

$$\text{Alternating: } \widehat{\langle vg \rangle} = \langle vg \rangle^-, \hat{\rho} = \rho^+, \text{or } \widehat{\langle vg \rangle} = \langle vg \rangle^+, \hat{\rho} = \rho^- \tag{6a}$$

$$\text{Central: } \widehat{\langle vg \rangle} = \{\langle vg \rangle\}, \hat{\rho} = \{\rho\} \tag{6b}$$

$\mathcal{D}_h(g_h; v) \in U_h$ is an upwind approximation of $v\partial_x g$, defined by

$$\int \mathcal{D}_h(g_h; v)\psi dx = \sum_i \left(-\int_{I_i} v g_h \partial_x \psi dx + \widetilde{(v g_h)}_{i+\frac{1}{2}} \psi^-_{i+\frac{1}{2}} - \widetilde{(v g_h)}_{i-\frac{1}{2}} \psi^+_{i-\frac{1}{2}} \right) \tag{7}$$

with

$$\widetilde{v g_h} := \begin{cases} v g_h^-, & \text{if } v > 0 \\ v g_h^+, & \text{if } v < 0 \end{cases} = v\{g_h\} - \frac{|v|}{2}[g_h]. \tag{8}$$

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Method in a compact form: Look for $\rho_h(\cdot,t), g_h(\cdot,v,t) \in U_h$, such that $\forall \phi, \psi \in U_h$, and i,

$$\int \partial_t \rho_h \phi dx + a_h(g_h, \phi) = 0$$

$$\int \partial_t g_h \psi dx + \frac{1}{\varepsilon} b_{h,v}(g_h, \psi) - \frac{v}{\varepsilon^2} d_h(\rho_h, \psi) = -\frac{1}{\varepsilon^2} s_{h,v}(\rho_h, g_h, \psi) - s'_{h,v}(g_h, \psi)$$

where

$$a_h(g_h, \phi) = -\sum_i \int_{I_i} \langle v g_h \rangle \partial_x \phi dx - \sum_i \widehat{\langle v g_h \rangle}_{i-\frac{1}{2}} [\phi]_{i-\frac{1}{2}},$$

$$b_{h,v}(g_h, \psi) = \int (\mathbf{I} - \Pi) \mathcal{D}_h(g_h; v) \psi dx = \int (\mathcal{D}_h(g_h; v) - \langle \mathcal{D}_h(g_h; v) \rangle) \psi dx,$$

$$d_h(\rho_h, \psi) = \sum_i \int_{I_i} \rho_h \partial_x \psi dx + \sum_i \widehat{\rho}_{h,i-\frac{1}{2}} [\psi]_{i-\frac{1}{2}},$$

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and

$$s_{h,v}(\rho_h, g_h, \psi) = \begin{cases} (g_h, \psi) & \text{for } (a.1) \\ (g_h - Av\rho_h, \psi) & \text{for } (a.2) \\ (C\rho_h^m g_h, \psi) & \text{for } (a.3) \\ (g_h - Cv\rho_h^2, , \psi) & \text{for } (a.4) \end{cases}$$

$$s'_{h,v}(g_h, \psi) = \begin{cases} 0 & \text{for } (a.1) \\ 0 & \text{for } (a.2) \\ 0 & \text{for } (a.3) \\ (Cvg_h^2, \psi) & \text{for } (a.4) \end{cases}$$

3. Globally stiffly accurate implicit-explicit (IMEX) temporal discretization

DG-IMEX1:

$$\int \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} \phi dx + a_h(g_h^n, \phi) = 0, \tag{9a}$$

$$\int \frac{g_h^{n+1} - g_h^n}{\Delta t} \psi dx + \frac{1}{\varepsilon} b_{h,v}(g_h^n, \psi) - \frac{v}{\varepsilon^2} d_h(\rho_h^{n+1}, \psi) = - \frac{1}{\varepsilon^2} s_{h,v}(\rho_h^{n+1}, g_h^{n+1}, \psi) - s'_{h,v}(g_h^n, \psi). \tag{9b}$$

Note:

(1) The terms d_h and $s_{h,v}$ are treated implicitly.

(2) One first solves ρ_h^{n+1} from (9a), and then g_h^{n+1} from (9b).

Formal asymptotic analysis: As an example, apply the DG-IMEX1 to the telegraph equation with $\mathcal{C}(f) = \langle f \rangle - f = -\varepsilon g$:

- The equation in its micro-macro formulation:

$$\partial_t \rho + \partial_x \langle vg \rangle = 0$$
$$\partial_t g + \frac{1}{\varepsilon^2} v \partial_x \rho + \frac{1}{\varepsilon} \{\mathbf{I} - \Pi\}(v \partial_x g) = -\frac{1}{\varepsilon^2} g$$

- DG-IMEX1 scheme: $\forall \phi, \psi \in U_h$,

$$\int \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} \phi dx + a_h(g_h^n, \phi) = 0$$
$$\int \frac{g_h^{n+1} - g_h^n}{\Delta t} \psi dx + \frac{1}{\varepsilon} b_{h,v}(g_h^n, \psi) - \frac{v}{\varepsilon^2} d_h(\rho_h^{n+1}, \psi) = -\frac{1}{\varepsilon^2} \int g_h^{n+1} \psi dx$$

Formal asymptotic analysis: As an example, apply the DG-IMEX1 to the telegraph equation with $\mathcal{C}(f) = \langle f \rangle - f = -\varepsilon g$:

- The equation in its micro-macro formulation:

$$\partial_t \rho + \partial_x \langle v g \rangle = 0 \tag{12a}$$

$$\partial_t g + \frac{1}{\varepsilon^2} v \partial_x \rho + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}(v \partial_x g) = -\frac{1}{\varepsilon^2} g \tag{12b}$$

- DG-IMEX1 scheme: $\forall \phi, \psi \in U_h$,

$$\int \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} \phi dx + a_h(g_h^n, \phi) = 0 \tag{13a}$$

$$\int \frac{g_h^{n+1} - g_h^n}{\Delta t} \psi dx + \frac{1}{\varepsilon} b_{h,v}(g_h^n, \psi) - \frac{v}{\varepsilon^2} d_h(\rho_h^{n+1}, \psi) = -\frac{1}{\varepsilon^2} \int g_h^{n+1} \psi dx \tag{13b}$$

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When $\varepsilon \to 0$,

- The limiting equation:

$$\partial_t \rho + \partial_x \langle vg \rangle = 0 \tag{14a}$$

$$-v\partial_x \rho = g \tag{14b}$$

- DG-IMEX1 scheme in the limit: $\forall \phi, \psi \in U_h$, $\forall i$

$$\int_{I_i} \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} \phi dx - \int_{I_i} \langle v g_h^n \rangle \partial_x \phi dx + \widehat{\langle v g_h^n \rangle}_{i+\frac{1}{2}} \phi^-_{i+\frac{1}{2}} - \widehat{\langle v g_h^n \rangle}_{i-\frac{1}{2}} \phi^+_{i-\frac{1}{2}} = 0$$

$$v \left(\int_{I_i} \rho_h^n \partial_x \psi dx - \widehat{\rho_h^n}_{i+\frac{1}{2}} \psi^-_{i+\frac{1}{2}} + \widehat{\rho_h^n}_{i-\frac{1}{2}} \psi^+_{i-\frac{1}{2}} \right) = \int_{I_i} g_h^n \psi dx$$

Remark: *In the limit of $\varepsilon \to 0$, the proposed scheme becomes a consistent and explicit discretization for the limiting equation on fixed mesh.*

For the two-velocity model, we can further write the limiting equation and scheme in terms of macro quantities ρ and $j := \langle vg \rangle$

- The limiting equation:

$$\partial_t \rho + \partial_x j = 0$$
$$-\partial_x \rho = j$$

This is $\partial_t \rho = \partial_{xx} \rho$.

- DG-IMEX1 scheme in the limit: $\forall \phi, \psi \in U_h$, $\forall i$

$$\int_{I_i} \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} \phi dx - \int_{I_i} j_h^n \partial_x \phi dx + \widehat{j_h^n}_{i+\frac{1}{2}} \phi^-_{i+\frac{1}{2}} - \widehat{j_h^n}_{i-\frac{1}{2}} \phi^+_{i-\frac{1}{2}} = 0$$
$$\int_{I_i} \rho_h^n \partial_x \psi dx - \widehat{\rho_h^n}_{i+\frac{1}{2}} \psi^-_{i+\frac{1}{2}} + \widehat{\rho_h^n}_{i-\frac{1}{2}} \psi^+_{i-\frac{1}{2}} = \int_{I_i} j_h^n \psi dx$$

This is a local DG method for $\partial_t \rho = \partial_{xx} \rho$. *Cockburn-Shu (1998)*

IMEX Runge-Kutta methods: *in the double Butcher Tableau representation below, the left is for the explicit term, and the right is for the implicit term*

$$
\begin{array}{c|c}
\tilde{c} & \tilde{A} \\
\hline
 & \tilde{b}^T
\end{array}
\qquad
\begin{array}{c|c}
c & A \\
\hline
 & b^T
\end{array}
$$

$\tilde{A}$: lower triangular with zero diagonal entries

A: lower triangular with nonzero diagonal entries *(diagonally implicit RK (DIRK))*

1st order:

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 1 & 0 \\
\hline
 & 1 & 0
\end{array}
\qquad
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 0 & 1 \\
\hline
 & 0 & 1
\end{array}
$$

Globally stiffly accurate: the last row of A is the same as b; the last row of $\tilde{A}$ is the same as $\tilde{b}$; in addition, the last entries of c and $\tilde{c}$ are 1.

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{b}^T \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Remark: *The implicit-explicit strategy and the globally stiffly accurate property of the time discretizations ensure that numerical solutions from both inner stages and discrete times will be relaxed to approximate the limiting problem when $\varepsilon \to 0$.*

Boscarino-Pareschi-Russo (2013)

2nd order: ARS(2, 2, 2) (with $\gamma = 1 - \frac{1}{\sqrt{2}}$ and $\delta = 1 - \frac{1}{2\gamma}$)

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \gamma & \gamma & 0 & 0 \\ 1 & \delta & 1-\delta & 0 \\ \hline & \delta & 1-\delta & 0 \end{array} \qquad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \gamma & 0 & \gamma & 0 \\ 1 & 0 & 1-\gamma & \gamma \\ \hline & 0 & 1-\gamma & \gamma \end{array}$$

3rd order: ARS(4, 4, 3)

$$\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 2/3 & 11/18 & 1/18 & 0 & 0 & 0 \\ 1/2 & 5/6 & -5/6 & 1/2 & 0 & 0 \\ 1 & 1/4 & 7/4 & 3/4 & -7/4 & 0 \\ \hline & 1/4 & 7/4 & 3/4 & -7/4 & 0 \end{array} \qquad \begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 2/3 & 0 & 1/6 & 1/2 & 0 & 0 \\ 1/2 & 0 & -1/2 & 1/2 & 1/2 & 0 \\ 1 & 0 & 3/2 & -3/2 & 1/2 & 1/2 \\ \hline & 0 & 3/2 & -3/2 & 1/2 & 1/2 \end{array}$$

Formal asymptotic analysis: the fully-discrete DG-IMEX scheme in the limit of $\varepsilon \to 0$, becomes a consistent scheme for the limiting equation. This scheme involves high order DG discretization in space where the numerical fluxes are naturally from the DG-IMEX schemes, and an explicit time discretization which is the explicit part of the IMEX scheme.

Theoretical Results

Let $||\phi|| = ||\phi||_{L^2(\Omega_x)}$ and $|||\psi||| = (\langle||\psi||^2\rangle)^{1/2}$. Without loss of generality, the mesh is assumed to be uniform with $h = h_i, \forall i$.

Inverse inequality: there exist constants $C_{\rm inv} = C_{\rm inv}(k), \hat{C}_{\rm inv} = \hat{C}_{\rm inv}(k)$, s.t. $\forall w \in P^k([a,b])$,

$$|w(y)|^2(b-a) \leq C_{\rm inv} \int_a^b w(x)^2 dx, \qquad \text{with } y = a \text{ or } b$$

$$(b-a)^2 \int_a^b |w_x(x)|^2 dx \leq \hat{C}_{\rm inv} \int_a^b w(x)^2 dx.$$

Theorem (Stability) *When the DG-IMEX1 method is applied to the telegraph equation with* $\mathcal{C}(f) = \langle f \rangle - f = -\varepsilon g$*, given* $\langle g_h^0 \rangle = 0$*, the following stability result holds for the numerical solution,*

$$||\rho_h^{n+1}||^2 + \varepsilon^2 |||g_h^n|||^2 \leq ||\rho_h^n||^2 + \varepsilon^2 |||g_h^{n-1}|||^2, \ \forall n \tag{15}$$

under the CFL condition

$$\Delta t \leq \Delta t_{stab} := \begin{cases} \frac{1}{2\hat{C}_{inv} + 8C_{inv}^2}(h + 2C_{inv} \min(\varepsilon, \frac{2C_{inv}}{\hat{C}_{inv}} h))h, & \textit{for } k \geq 1, \\ \frac{1}{4C_{inv}^2}(h + 2C_{inv}\varepsilon)h = \frac{1}{2}\varepsilon h + \frac{1}{4}h^2, & \textit{for } k = 0. \end{cases} \tag{16}$$

Note: (1) The stability result is uniform with respect to ε.

(2) For $k = 0$, numerical stability requires $\Delta t = O(h^2)$ in diffusive regime ($\varepsilon << 1$), and $\Delta t = O(\varepsilon h)$ in rarefied regime ($\varepsilon = O(1)$).

(3) For $k \geq 1$, numerical stability requires $\Delta t = O(h^2)$. It is conjectured that the restrictive timestep in the rarefied regime ($\varepsilon = O(1)$) can be improved and/or removed with higher order time discretizations. (supported by numerics)

Theorem (Error estimates) *When the DG-IMEX1 method is applied to the telegraph equation with* $\mathcal{C}(f) = \langle f \rangle - f = -\varepsilon g$*, given* $\langle g_h^0 \rangle = 0$*, the following error estimates hold for the numerical solution,*

$$||\rho(t^n) - \rho_h^n||^2 + \varepsilon^2 |||g(t^{n-1}) - g_h^{n-1}|||^2$$

$$\leq C_\star \begin{cases} \left((1+\varepsilon^4)\Delta t^2 + (1+\varepsilon^2)h^{2k+2} + \varepsilon h^{2k+\sigma(k)}\right) & \textit{(alternating flux)} \\ \left((1+\varepsilon^4)\Delta t^2 + \varepsilon^2 h^{2k+2} + h^{2k+\sigma(k)-1} + \varepsilon h^{2k+\sigma(k)}\right) & \textit{(central flux)} \end{cases}$$

for $n : n\Delta t \leq T$ *under the CFL condition* $\Delta t < \Delta t_{stab}$*. Here*

$$\sigma(k) = \begin{cases} 1 & \textit{for } k \geq 1, \\ 2 & \textit{for } k = 0. \end{cases} \tag{17}$$

The positive constant $C_\star$ *is independent of* $h, \Delta t$*, and* n*. It depends on* T*,* k*, and some Sobolev norms of the exact solution (hence* ε*).*

lif@rpi.edu

Spatial accuracy orders established by the error estimates

	$k \geq 1, \varepsilon \neq 0$	$k \geq 1, \varepsilon = 0$	$k = 0$
alternative	$k + \frac{1}{2}$	$k + 1$	1
central	k	k	$\frac{1}{2}$

Numerical Examples

DG(k+1)-IMEX(k+1): DG method with k-th order polynomials; $(k+1)$-th order IMEX time discretization

How to choose time step?

$$\Delta t = C_{conv}\varepsilon h + C_{diff}h^2 \tag{18}$$

with

- DG1-IMEX1: $C_{conv} = 0.5$ and $C_{diff} = 0.25$ (based on our analysis)
- DG2-IMEX2: $C_{conv} = 0.5$ and $C_{diff} = 0.01$
- DG3-IMEX3: $C_{conv} = 0.25$ and $C_{diff} = 0.006$

Define $j := \langle vg \rangle$

Example 1 (telegraph equation): $\mathcal{C}(f) = \langle f \rangle - f$. The exact solution is

$$\begin{cases} \rho(x,t) = \frac{1}{r}\exp(rt)\sin(x), \quad r = \frac{-2}{1+\sqrt{1-4\varepsilon^2}}, \\ j(x,t) = \exp(rt)\cos(x) \end{cases} \tag{19}$$

Table 1: L^1 error and order of ρ and j, $T = 1.0$, DG1-IMEX1 with the alternating flux.

	N	L^1 error of ρ	order	L^1 error of j	order
$\varepsilon = 0.5$	10	6.04E-02	–	7.46E-02	–
	20	2.19E-02	1.46	3.38E-02	1.14
	40	9.20E-03	1.25	1.60E-02	1.08
	80	4.19E-03	1.14	7.81E-03	1.03
$\varepsilon = 10^{-2}$	10	3.79E-02	–	8.05E-02	–
	20	1.78E-02	1.09	3.77E-02	1.09
	40	8.79E-03	1.02	1.85E-02	1.03
	80	4.36E-03	1.01	9.22E-03	1.01
$\varepsilon = 10^{-6}$	10	3.79E-02	–	8.03E-02	–
	20	1.79E-02	1.08	3.76E-02	1.09
	40	8.82E-03	1.02	1.85E-02	1.02
	80	4.38E-03	1.01	9.21E-03	1.01

Table 2: L^1 error and order of ρ and j, $T = 1.0$, DG2-IMEX2 with the alternating flux.

	N	L^1 error of ρ	order	L^1 error of j	order
$\varepsilon = 0.5$	10	1.35E-03	–	2.36E-03	–
	20	3.00E-04	2.17	4.90E-04	2.27
	40	7.23E-05	2.05	1.14E-04	2.10
	80	1.79E-05	2.01	2.76E-05	2.04
$\varepsilon = 10^{-2}$	10	4.83E-03	–	4.94E-03	–
	20	1.19E-03	2.02	1.19E-03	2.06
	40	2.96E-04	2.01	2.97E-04	2.00
	80	7.40E-05	2.00	7.40E-05	2.00
$\varepsilon = 10^{-6}$	10	4.82E-03	–	4.93E-03	–
	20	1.19E-03	2.02	1.18E-03	2.06
	40	2.96E-04	2.00	2.96E-04	2.00
	80	7.40E-05	2.00	7.40E-05	2.00

Table 3: L^1 error and order of ρ and j, $T = 1.0$, DG3-IMEX3 with the alternating flux.

	N	L^1 error of ρ	order	L^1 error of j	order
$\varepsilon = 0.5$	10	6.33E-05	–	9.48E-05	–
	20	7.54E-06	3.07	1.15E-05	3.04
	40	9.31E-07	3.02	1.44E-06	3.00
	80	1.16E-07	3.01	1.80E-07	3.00
$\varepsilon = 10^{-2}$	10	2.53E-04	–	2.46E-04	–
	20	3.11E-05	3.03	3.11E-05	2.98
	40	3.89E-06	3.00	3.89E-06	3.00
	80	4.87E-07	3.00	4.87E-07	3.00
$\varepsilon = 10^{-6}$	10	2.53E-04	–	2.46E-04	–
	20	3.11E-05	3.03	3.11E-05	2.98
	40	3.89E-06	3.00	3.89E-06	3.00
	80	4.87E-07	3.00	4.87E-07	3.00

Let k be the polynomial degree of the discrete space. It is observed that

- with alternating flux: $(k+1)$-th order
- with central flux: k-th order for odd k, and $(k+1)$-th order for even k

Example 2 (telegraph equation): $\mathcal{C}(f) = \langle f \rangle - f$. Consider a Riemann problem with the initial condition

$$\begin{cases} \rho_L = 2.0, \quad j_L = 0.0, \qquad -1 < x < 0; \\ \rho_R = 1.0, \quad j_R = 0.0, \qquad 0 < x < 1. \end{cases} \tag{20}$$

lif@rpi.edu

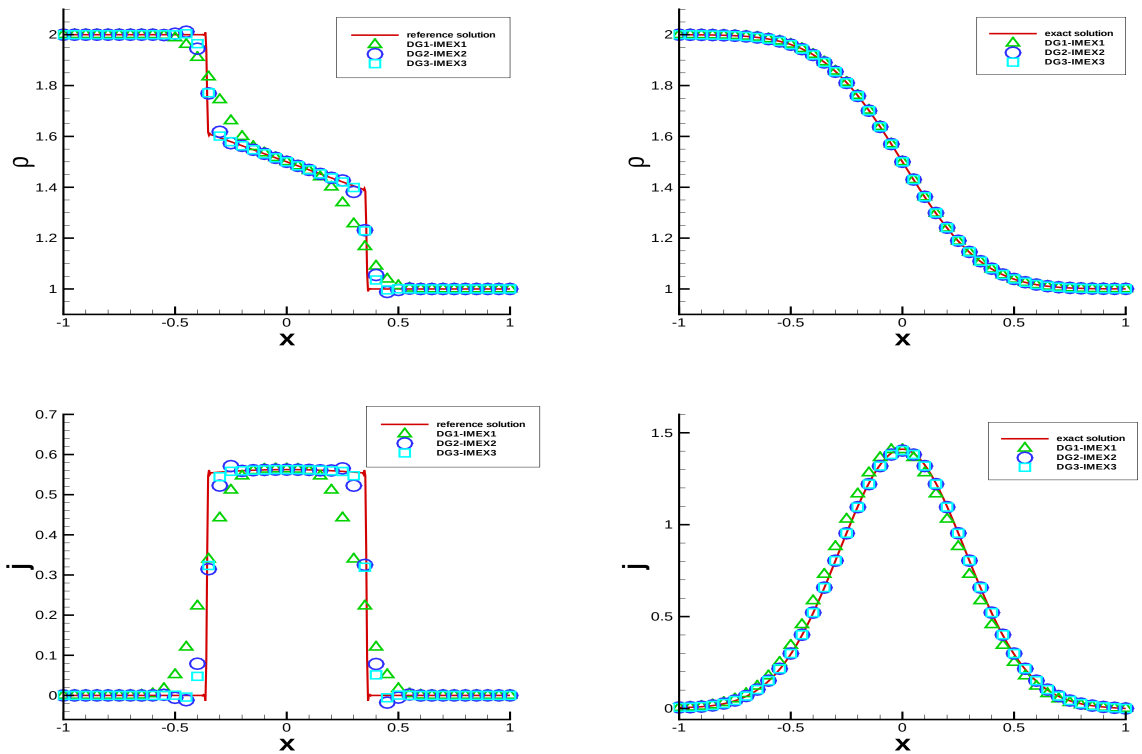

Figure 1: Numerical solution of DG(k+1)-IMEX(k+1) ($k = 0, 1, 2$) with the alternating flux and $h = 0.05$. Left: $\varepsilon = 0.7$ at $T = 0.25$; Right: $\varepsilon = 10^{-6}$ at $T = 0.04$. Top: ρ; Bottom: j. The reference solution is from DG3-IMEX3 with $h = 0.004$.

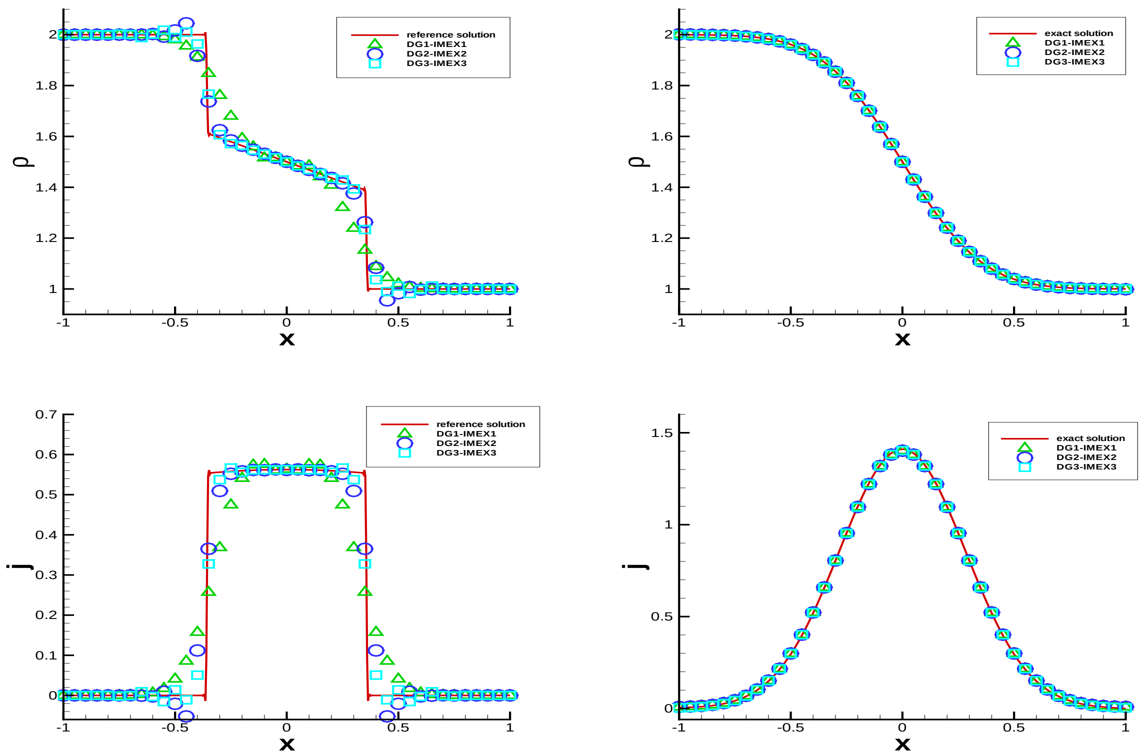

Figure 2: Numerical solution of DG(k+1)-IMEX(k+1) ($k = 0, 1, 2$) with the central flux and $h = 0.05$. Left: $\varepsilon = 0.7$ at $T = 0.25$; Right: $\varepsilon = 10^{-6}$ at $T = 0.04$. Top: ρ; Bottom: j. The reference solution is from DG3-IMEX3 with $h = 0.004$.

Example 3 (viscous Burgers' equation): The initial condition is given as two local Maxwellian

$$\begin{cases} \rho_L = 2.0, & -10 < x < 0 \\ \rho_R = 1.0, & 0 < x < 10 \end{cases} \tag{21}$$

with $j = \rho^2/(1+\sqrt{1+\rho^2\varepsilon^2})$.

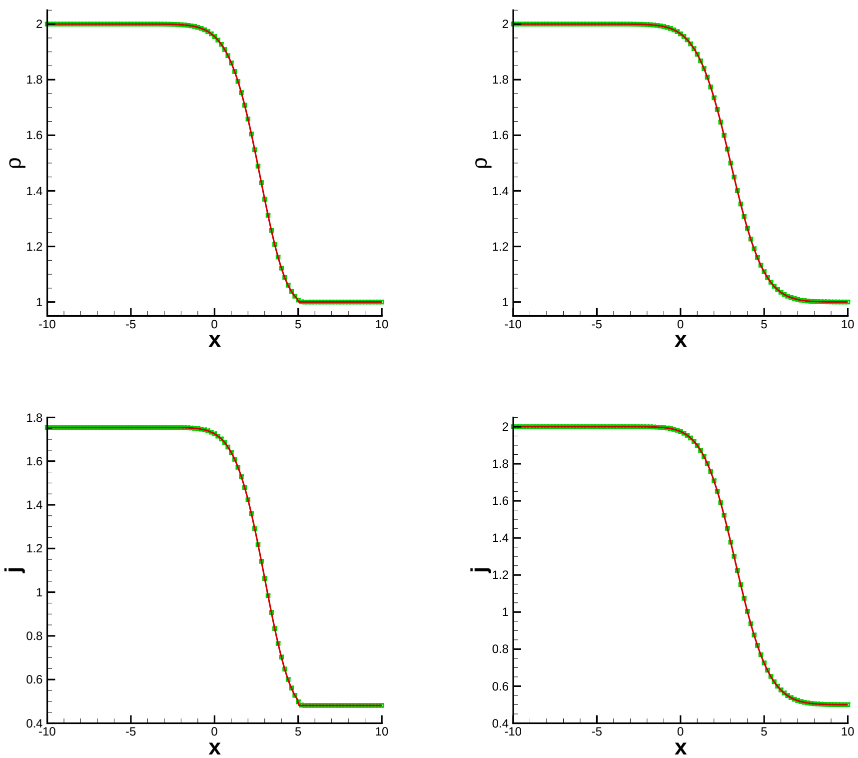

Figure 3: Numerical solution of DG3-IMEX3 with the alternating flux with $h = 0.5$ at $T = 2.0$. Left: $\varepsilon = 0.4$; Right: $\varepsilon = 10^{-6}$. Top: ρ; Bottom: j. Symbol: numerical solution; Solid line: reference solution.

Example 4 (porous medium equation):
$\mathcal{C}(f) = \langle f \rangle^m (\langle f \rangle - f) = -\varepsilon \rho^m g$ with $m = -1$. The initial condition is chosen to be the same as the Barenblatt solution for the limiting equation

$$\begin{cases} \rho(x,t) = \frac{1}{R(t)}\left[1 - \left(\frac{x}{R(t)}\right)^2\right], \quad j(x,t) = \rho \frac{4x}{R(t)^3}, \quad |x| < R(t), \\ \rho(x,t) = 0, \quad j(x,t) = 0, \quad |x| > R(t), \end{cases}$$

where $R(t) = [12(t+1)]^{1/3}$, $t \geq 0$.

Note:

- Nodal DG method is used.
- Alternating flux needs to be suitably chosen around the interface of $\rho = 0$.

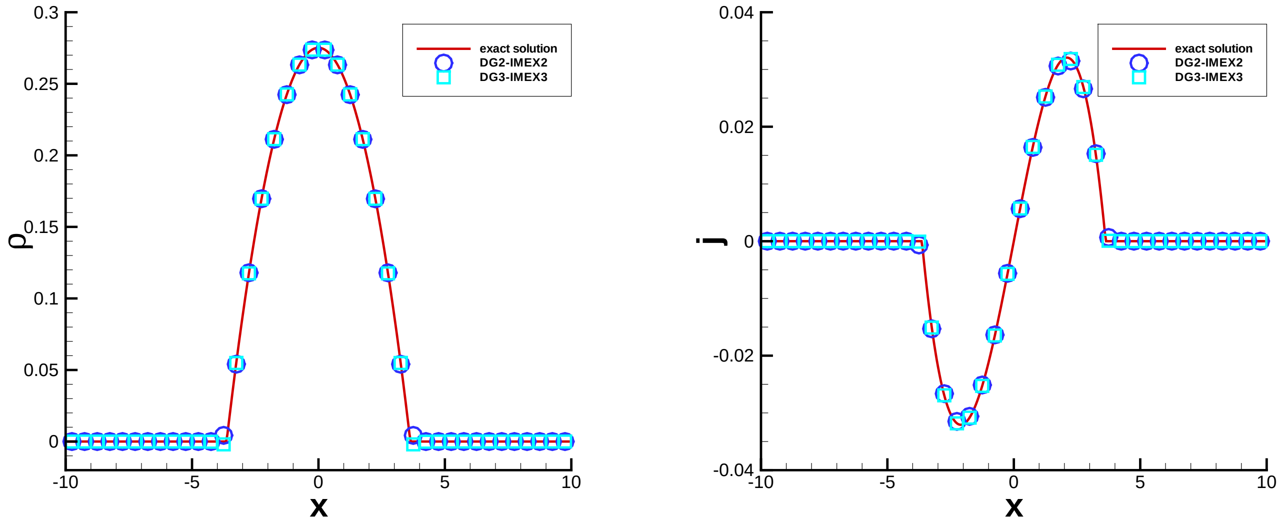

Figure 4: Numerical solution of DG(k+1)-IMEX(k+1) ($k = 1, 2$) with the alternating flux (*Right half domain:* $\widehat{\langle vg\rangle} = \langle vg\rangle^-$, $\hat{\rho} = \rho^+$, *left half domain* $\widehat{\langle vg\rangle} = \langle vg\rangle^+$, $\hat{\rho} = \rho^-$); $\varepsilon = 10^{-6}$ with $h = 0.5$ at $T = 3.0$ compared with the exact solution. Left: ρ; Right: j.

Note: with $k = 0$, the scheme in the center element is inconsistent.

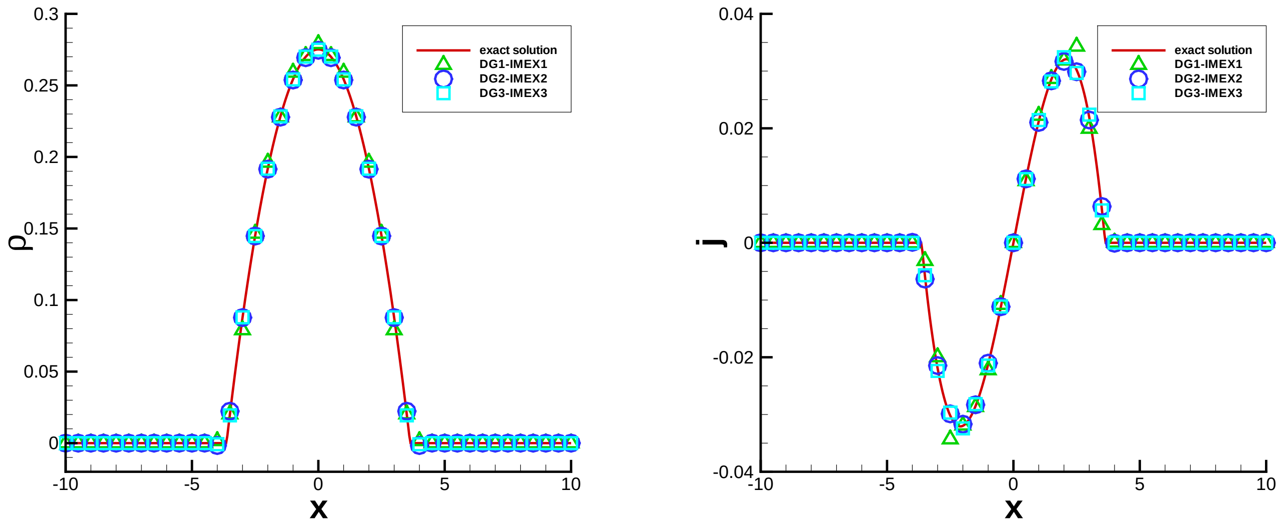

Figure 5: Numerical solution of DG(k+1)-IMEX(k+1) ($k = 0, 1, 2$) with the central flux. $\varepsilon = 10^{-6}$ with $h = 0.5$ at $T = 3.0$ compared with the exact solution. Left: ρ; Right: j.

Summary: a family of high order methods are proposed for discrete-velocity kinetic equations in the diffusive scaling.

- For the telegraph equation, stability is established for DG-IMEX1 uniformly with respect to ε. Error estimates are also obtained for any ε.
- Formal asymptotic analysis shows that the proposed schemes in the limit of $\varepsilon \to 0$ provide explicit and consistent high order methods for the limiting equations.
- Numerical experiments demonstrate the performance of the methods with higher order temporal accuracy and other collision operators.
- The proposed methods and most analysis can be naturally extended to kinetic models with $v \in \{v_1, \cdots, v_N\}$, or $v \in [-v_*, v_*]$.

Current and future effort: kinetic equations in other scalings and (or) with general collision operators; boundary conditions; analysis