A priori and *a posteriori* analyses of the DPG method

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Principal Collaborator in DPG research: Leszek Demkowicz.

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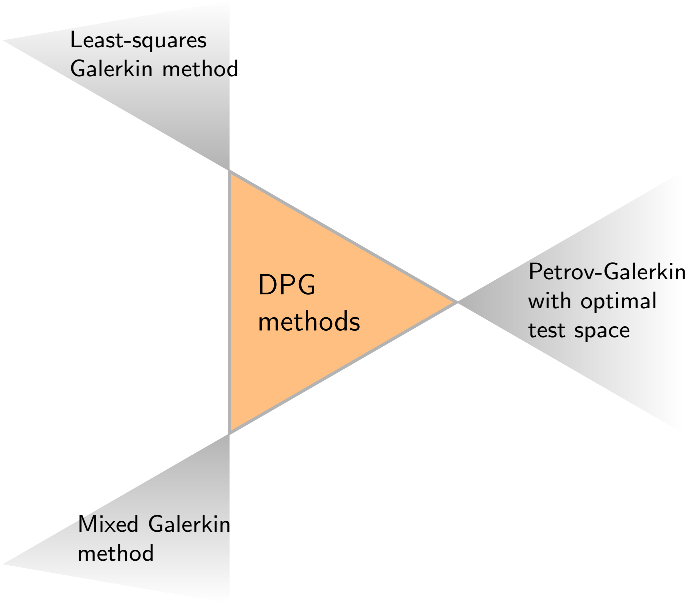
Three avenues to DPG methods



"Petrov-Galerkin" schemes (PG)

PG schemes are distinguished by different trial and test (Hilbert) spaces.

The problem: P.D.E.+ boundary conditions.

$\downarrow$

Variational form: Find x in a trial space X satisfying

$$b(x, y) = \ell(y)$$

for all y in a test space Y.

$\downarrow$

Discretization: Find x_h in a discrete trial space $X_h \subset X$ satisfying

$$b(x_h, y_h) = \ell(y_h)$$

for all y_h in a discrete test space $Y_h \subset Y$.

For PG schemes, $X_h \neq Y_h$ in general.

Elements of theory

- *Variational formulation:*

$$\left[\begin{array}{c}\text{Exact inf-sup condition}\\ C\|x\|_X \le \sup\limits_{y\in Y} \dfrac{|b(x,y)|}{\|y\|_Y}\end{array}\right] + \left[\begin{array}{c}\text{a uniqueness}\\ \text{condition}\end{array}\right] \implies \text{wellposedness}$$

- *Babuška-Brezzi theory:*

$$\left[\begin{array}{c}\text{Discrete inf-sup condition}\\ C\|x_h\|_X \le \sup\limits_{y_h\in Y_h} \dfrac{|b(x_h,y_h)|}{\|y_h\|_Y}\end{array}\right] \implies \|x - x_h\|_X \le C \inf_{w_h\in X_h} \|x - w_h\|_X.$$

- *Difficulty:* Exact inf-sup condition $\not\Longrightarrow$ Discrete inf-sup condition

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- *Difficulty:* Exact inf-sup condition $\not\Longrightarrow$ Discrete inf-sup condition

- Is there a way to find a stable test space for *any* given trial space (thus giving a stable method automatically)?

The ideal method

Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ such that

$$b(x_h, y) = \ell(y), \qquad \forall y \in Y_h^{\mathrm{opt}} \stackrel{\mathrm{def}}{=} T(X_h),$$

where $T : X \mapsto Y$ is defined by

$$(Tw, y)_Y = b(w, y), \qquad \forall w \in X,\ y \in Y.$$

[Demkowicz+G 2011]

Rationale:

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[Demkowicz+G 2011]

Rationale:

- *Q:* Which function y maximizes $\dfrac{|b(x,y)|}{\|y\|_Y}$ for any given x ?
- *A:* $y = Tx$ is the maximizer. $\leftarrow$ Optimal test function.

DPG Idea: If the discrete test space contains the optimal test functions,

exact inf-sup condition $\implies$ discrete inf-sup condition.

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where $T : X \mapsto Y$ is defined by

$$(Tw, y)_Y = b(w, y), \qquad \forall w \in X, \; y \in Y.$$

[A.1] $\{w \in X : \; b(w, y) = 0 \quad \forall y \in Y\} = \{0\}.$

[A.2] $\exists \; C_1, C_2 > 0$ such that $\quad C_1 \|y\|_Y \le \sup_{w \in X} \dfrac{|b(w, y)|}{\|w\|_X} \le C_2 \|y\|_Y.$

Theorem (DPG Quasioptimality)

[A.1–A.2] $\implies \|x - x_h\|_X \le \dfrac{C_2}{C_1} \inf_{w_h \in X_h} \|x - w_h\|_X.$

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But ... can we really compute Tx?

- For a few problems, Tx can be calculated in closed form.
- When Tx cannot be hand calculated, we overcome two difficulties:
  - Redesign formulation so that T is local (by hybridization).
  - Approximate T by a computable (finite-rank) T^r.

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- The ideal DPG method = iDPG method

Trivial Example 1

Standard FEM is an iDPG method

Problem
Given $F \in H^{-1}(\Omega)$,
find $u \in H_0^1(\Omega)$ solving:

$$\int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v = F(v), \quad \forall v \in H_0^1(\Omega).$$

Recall

Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ such that

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- Set $X = Y = H_0^1(\Omega)$ and

$$(v, y)_Y = \int_\Omega \vec{\nabla} v \cdot \vec{\nabla} y.$$

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Trivial Example 1

Standard FEM is an iDPG method

Problem:
Given $F \in H^{-1}(\Omega)$,
find $u \in H^1_0(\Omega)$ solving:

$$\int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v = F(v), \quad \forall v \in H^1_0(\Omega).$$

- Set $X = Y = H^1_0(\Omega)$ and
$$(v, y)_Y = \int_\Omega \vec{\nabla} v \cdot \vec{\nabla} y.$$
- Then $(\cdot,\cdot)_Y = b(\cdot,\cdot) \implies T =$ identity, so
$$Y_h^{\text{opt}} = X_h.$$

Recall:

Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ such that

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Trivial Example 2

L^2-based least squares method is an ideal DPG method

Problem: Given an $f \in L^2(\Omega)$ and a linear continuous bijective $A : X \to L^2(\Omega)$, find $u \in X$ satisfying $\boxed{Au = f}$.

Recall: Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ such that

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- Set $Y = L^2(\Omega)$, $\quad b(x, y) = (Ax, y)_Y, \quad \ell(y) = (f, y)_Y$.
- Then $(Tw, y)_Y = (Aw, y) \implies T = A \implies Y_h^{\text{opt}} = AX_h \implies$ iDPG equations become Normal equations:

$$(Ax_h, Aw_h)_Y = (f, Aw_h)_Y \qquad \forall w_h \in X_h.$$

Recall: Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ such that

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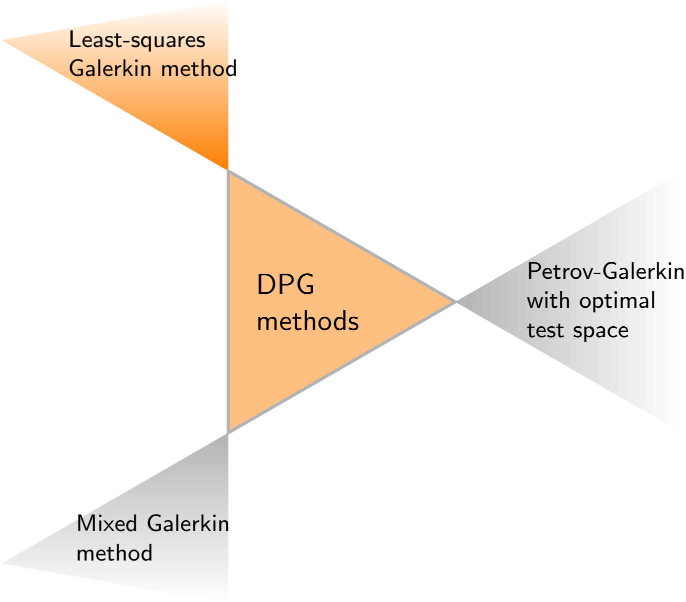
The least-squares avenue



Definitions

- *Riesz map:*

$$R_Y : Y \to Y^* : \qquad (R_Y y)(v) = (y, v)_Y, \qquad \forall y, v \in Y.$$

- *Operator generated by the form:*

$$B : X \to Y^* : \qquad Bx(y) = b(x, y), \quad \forall x \in X, y \in Y.$$

- *Trial-to-Test operator* $T : X \mapsto Y$ was defined by

$$(Tw, y)_Y = b(w, y), \qquad \forall w \in X,\ y \in Y.$$
$$\implies \qquad T = R_Y^{-1} \circ B.$$

- *Energy norm* on X:

$$|||z|||_X \stackrel{\text{def}}{=} \|Tz\|_Y.$$

Residual minimization

Theorem (DPG methods are least-squares methods)

The following are equivalent statements:

i) $x_h \in X_h$ *is the unique solution of the ideal DPG method.*

ii) x_h *is the best approximation to* x *from* X_h *in the energy norm:*

$$\|x - x_h\|_X = \inf_{z_h \in X_h} \|x - z_h\|_X$$

iii) x_h *minimizes residual in the following sense:*

$$\boxed{x_h = \arg\min_{z_h \in X_h} \|\ell - B z_h\|_{Y^*}.}$$

Proof of $(i) \iff (ii)$.

$$\begin{aligned} b(x - x_h, y_h) = 0 \quad \forall y_h \in Y_h^{\text{opt}} &\iff b(x - x_h, T z_h) = 0 \qquad \forall z_h \in X_h \\ &\iff (T(x - x_h), T z_h)_Y = 0 \quad \forall z_h \in X_h. \end{aligned}$$

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Proof of $(ii) \iff (iii)$.

$$\begin{aligned}\|x - z_h\|_X &= \|T(x - z_h)\|_Y = \|R_Y^{-1} B(x - z_h)\|_Y \\ &= \|B(x - z_h)\|_{Y^*} = \|\ell - B z_h)\|_{Y^*}.\end{aligned}$$

Example 3: An ODE

Pavlovian integration by parts, or not?

1D transport eq. $\left[\begin{array}{ll} u' = f & \text{in } (0,1), \\ u(0) = u_0 & \text{(inflow b.c.)} \end{array}\right.$

<u>Variational form:</u> $\left[\begin{array}{l} \text{Find } u \text{ in } H^1, \text{ satisfying } u(0) = u_0, \& \\ \underbrace{\int_0^1 u' v}_{b(u,v)} = \underbrace{\int_0^1 f v}_{l(v)}, \qquad \forall v \text{ in } L^2. \end{array}\right.$

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<u>Ultra-weak form:</u>

$$\left[\begin{array}{l} \text{Find } u \in L^2, \text{ and a number } \hat{u}_1 \in \mathbb{R}, \text{ satisfying} \\ \underbrace{-\int_0^1 u v' + \hat{u}_1 v(1)}_{b(\,(u,\hat{u}_1),\, v)} = \underbrace{\int_0^1 f v + u_0 v(0)}_{l(v)}, \quad \forall v \in H^1. \end{array}\right.$$

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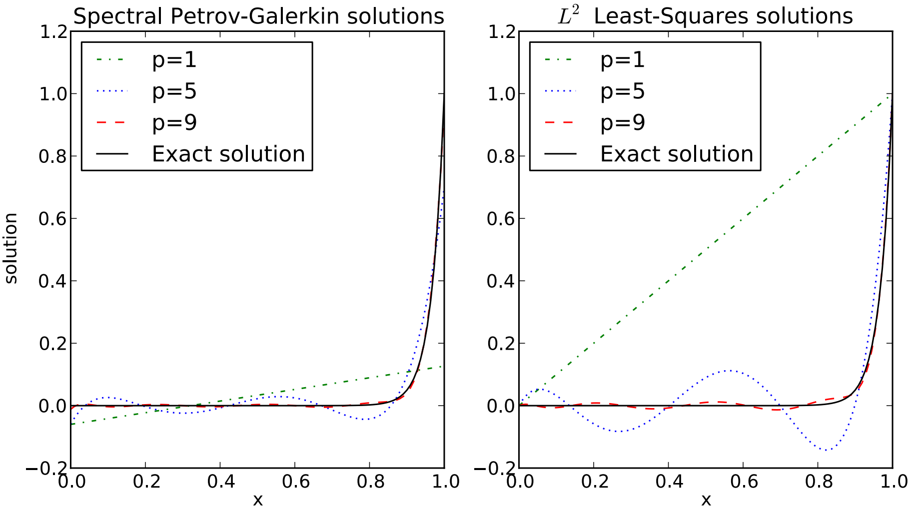
Variational form:
(DPG gives LS with $Au = u'$.)

$$\left[\begin{array}{l} \text{Find } u \text{ in } H^1, \text{ satisfying } u(0) = u_0, \& \\ \underbrace{\int_0^1 u' v}_{b(u,v)} = \underbrace{\int_0^1 f v}_{l(v)}, \qquad \forall v \text{ in } L^2. \end{array}\right.$$

Ultra-weak form:
(Here DPG gives something new.)

$$\left[\begin{array}{l} \text{Find } u \in L^2, \text{ and a number } \hat{u}_1 \in \mathbb{R}, \text{ satisfying} \\ \underbrace{-\int_0^1 u v' + \hat{u}_1 v(1)}_{b(\,(u,\hat{u}_1),\, v)} = \underbrace{\int_0^1 f v + u_0 v(0)}_{l(v)}, \quad \forall v \in H^1. \end{array}\right.$$

One-dimensional results using spectral trial space



[Click here to download FEniCS code.]

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- *A priori* error analysis
  - Ideal DPG method . . . . ✓
  - Practical DPG method . . . . ←
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  - Example 1 (Standard FEM) . . . . ✓
  - Example 2 (L^2-based least-squares) . . . . ✓
  - Example 3 (An ODE) . . . . ✓
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The practical method

Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ such that

$$b(x_h, y) = \ell(y), \qquad \forall y \in Y_h^{\text{opt}} \stackrel{\text{def}}{=} T(X_h),$$

where $T : X \mapsto Y$ is defined by

$$(Tw, y)_Y = b(w, y), \qquad \forall w \in X, \; y \in Y.$$

Pick any $X_h \subseteq X$. The practical DPG method finds $x_h^r \in X_h$, using a (finite-dimensional) $Y^r \subseteq Y$, such that

$$b(x_h^r, y) = \ell(y), \qquad \forall y \in Y_h^r \stackrel{\text{def}}{=} T^r(X_h),$$

where $T^r : X \mapsto Y^r$ is defined by

$$(T^r w, y)_Y = b(w, y), \qquad \forall w \in X, \; y \in Y^r.$$

The practical method

Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ such that

$$x_h = \arg\min_{z_h \in X_h} \|\ell - B z_h\|_{Y^*}.$$

Pick any $X_h \subseteq X$. The practical DPG method finds $x_h^r \in X_h$, using a (finite-dimensional) $Y^r \subseteq Y$, such that

$$x_h^r = \arg\min_{z_h \in X_h} \|\ell - B z_h\|_{(Y^r)^*}.$$

Error analysis of the practical DPG method

[A.1] $\{w \in X : \; b(w, y) = 0 \quad \forall y \in Y\} = \{0\}.$

[A.2] $\exists\, C_1, C_2 > 0$ such that $\quad C_1 \|y\|_Y \le \sup_{w \in X} \dfrac{|b(w,y)|}{\|w\|_X} \le C_2 \|y\|_Y.$

[A.3] $\exists\, \Pi : Y \mapsto Y^r$ and $C_\Pi > 0$ such that for all $w_h \in X_h$ and $y \in Y$,

$$b(w_h, y - \Pi y) = 0, \qquad \|\Pi y\|_Y \le C_\Pi \|y\|_Y.$$

Theorem (*A priori* estimates for practical DPG method [G+Qiu 2013])

[A.1–A.3] $\implies \|x - x_h^r\|_X \le \dfrac{C_2 C_\Pi}{C_1} \inf_{w_h \in X_h} \|x - w_h\|_X.$

The 'D' in 'DPG'

For the residual minimization in

$$x_h = \arg\min_{z_h \in X_h} \|\ell - Bz_h\|_{Y^*}$$

to be feasible, the *dual norm $\|\cdot\|_{Y^*}$ must be easily computable!*

- "Negative-norm least-squares" uses multigrid or operators spectrally equivalent to the dual norm. [Bramble+Pasciak+Lazarov 1997]

- DPG methods reformulate problems to *localize* the dual norm computation (to parallel element-by-element computations). DPG methods have *discontinuous* test function space

$$Y = \prod_{K \in mesh} Y(K),$$

which have locally invertible Riesz maps.

Example 4: The Dirichlet problem

A new weak form for the old Laplacian

Find u:
$$\begin{cases} -\Delta u = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Let Ω_h be a mesh of Ω and $K \in \Omega_h$ be a mesh element. Then:

$$\int_K \vec{\nabla}\, u \cdot \vec{\nabla}\, v - \int_{\partial K} (n \cdot \vec{\nabla}\, u) v = \int_K f\, v.$$

This allows test function $v \in Y$ to be in a "broken" Sobolev space

$$Y = H^1(\Omega_h) := \prod_{K \in \Omega_h} H^1(K).$$

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$$\int_K \vec{\nabla} u \cdot \vec{\nabla} v - \int_{\partial K} (n \cdot \vec{\nabla} u) v = \int_K f v.$$

$$\sum_{K \in \Omega_h} \left[\int_K \vec{\nabla} u \cdot \vec{\nabla} v - \int_{\partial K} \hat{q}_n v \right] = \int_\Omega f v.$$

This allows test function $v \in Y$ to be in a “broken” Sobolev space

$$Y = H^1(\Omega_h) := \prod_{K \in \Omega_h} H^1(K).$$

Functional setting for the Laplacian

Want X and Y to make $B : X \to Y^*$ a continuous bijection, i.e., the form

$$b(x, y) = (Bx)(y) \quad \text{on} \quad X \times Y$$

must satisfy a uniqueness and inf-sup condition.

- Set $b(\,(u, \hat{q}_n), v) = \displaystyle\sum_{K \in \Omega_h} \left[\int_K \vec{\nabla}\, u \cdot \vec{\nabla}\, v \; - \int_{\partial K} \hat{q}_n \; v \right].$
- We seek u in $H_0^1(\Omega)$ and $\hat{q}_n$ in $H^{-1/2}(\partial\Omega_h)$.

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Definition (of $H^{-1/2}(\partial\Omega_h)$, the space of numerical fluxes)

Define the element-by-element trace operator tr_n by

$$\mathrm{tr}_n : H(\mathrm{div}, \Omega) \to \prod_{K \in \Omega_h} H^{-1/2}(\partial K), \qquad \mathrm{tr}_n \, r|_{\partial K} = r \cdot n|_{\partial K}.$$

and set $H^{-1/2}(\partial\Omega_h) = \mathrm{ran}(\mathrm{tr}_n)$.

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- We seek u in $H_0^1(\Omega)$ and $\hat{q}_n$ in $H^{-1/2}(\partial\Omega_h)$.

Theorem

With $\boxed{X = H_0^1(\Omega) \times H^{-1/2}(\partial\Omega_h)}$ *and* $\boxed{Y = H^1(\Omega_h)}$, *the operator B is a continuous bijection and has a continuous inverse.*

[Demkowicz+G 2013]

Discrete spaces for the Laplacian

Trial subspace $X_h \subseteq X \equiv H_0^1(\Omega) \times H^{-1/2}(\partial\Omega_h)$: Approximate

u by Lagrange FE of degree $\le p+1$, $\forall\, K \in \Omega_h$,

$\hat{q}_n$ by polynomials of degree $\le p$, $\forall$ mesh edges.

Test subspace $Y^r \subseteq H^1(\Omega_h)$: Set, for some $r \ge 0$,

$$Y^r = \{v : v|_K \in P_r(K), \quad \forall K \in \Omega_h\}.$$

Discrete spaces for the Laplacian

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Recall

Pick any $X_h \subseteq X$. The practical DPG method finds $x_h^r \in X_h$, using a (finite-dimensional) $Y^r \subseteq Y$, such that

$$b(x_h^r, y) = \ell(y), \qquad \forall y \in Y_h^r \stackrel{\text{def}}{=} T^r(X_h),$$

where $T^r : X \mapsto Y^r$ is defined by

$$(T^r w, y)_Y = b(w, y), \qquad \forall w \in X,\ y \in Y^r.$$

Discrete spaces for the Laplacian

Trial subspace $X_h \subseteq X \equiv H_0^1(\Omega) \times H^{-1/2}(\partial\Omega_h)$: Approximate

u	by Lagrange FE of degree $\le p+1$,	$\forall\, K \in \Omega_h$,
$\hat{q}_n$	by polynomials of degree $\le p$,	$\forall$ mesh edges.

Test subspace $Y^r \subseteq H^1(\Omega_h)$: Set, for some $r \ge 0$,

$$Y^r = \{v : v|_K \in P_r(K), \quad \forall K \in \Omega_h\}.$$

Computation of T^r is local:

$$\text{Apply:} \qquad (T^r w, y)_Y = b(w, y)$$

$$\Longrightarrow \quad (T^r(u, \hat{q}_n), y)_{H^1(\Omega_h)} = b(\,(u, \hat{q}_n), y), \qquad \forall\, y \in Y^r.$$

$$\Longrightarrow \quad (T^r(u, \hat{q}_n), y)_{H^1(K)} = \int_K \vec{\nabla}\, u \cdot \vec{\nabla}\, y \;-\; \int_{\partial K} \hat{q}_n\, y, \qquad \forall K \in \Omega_h.$$

Discrete spaces for the Laplacian

Trial subspace $X_h \subseteq X \equiv H_0^1(\Omega) \times H^{-1/2}(\partial\Omega_h)$: Approximate

u by Lagrange FE of degree $\leq p+1$, $\forall\, K \in \Omega_h$,

$\hat{q}_n$ by polynomials of degree $\leq p$, $\forall$ mesh edges.

Test subspace $Y^r \subseteq H^1(\Omega_h)$: Set, for some $r \geq 0$,

$$Y^r = \{v : v|_K \in P_r(K), \quad \forall K \in \Omega_h\}.$$

To prove optimal convergence, we must choose r so that **[A.3]** holds.

Recall

[A.3] $\exists\, \Pi : Y \mapsto Y^r$ and $C_\Pi > 0$ such that for all $w_h \in X_h$ and $y \in Y$,

$$b(w_h, y - \Pi y) = 0, \qquad \|\Pi y\|_Y \leq C_\Pi \|y\|_Y.$$

Discrete spaces for the Laplacian

Trial subspace $X_h \subseteq X \equiv H_0^1(\Omega) \times H^{-1/2}(\partial\Omega_h)$: Approximate

u by Lagrange FE of degree $\le p+1$, $\forall\, K \in \Omega_h$,

$\hat{q}_n$ by polynomials of degree $\le p$, $\forall$ mesh edges.

Test subspace $Y^r \subseteq H^1(\Omega_h)$: Set, for some $r \ge 0$,

$$Y^r = \{v : v|_K \in P_r(K), \quad \forall K \in \Omega_h\}.$$

Theorem (Verification of [A.3])

Let Ω_h be a simplicial shape-regular finite element mesh in N-space dimensions. For any $p \ge 0$, whenever $\boxed{r \ge p + N}$, there exists a continuous $\Pi : Y \to Y^r$ such that for all $(w_h, \hat{s}_{n,h}) \in X_h$,

$$\int_K \vec{\nabla} w_h \cdot \vec{\nabla}(v - \Pi v) - \int_{\partial K} \hat{s}_{n,h}\,(v - \Pi v) = 0, \qquad \forall K \in \Omega_h.$$

Next

- Three avenues to DPG methods
  - Petrov-Galerkin with optimal test functions .... ✓
  - Least-squares Galerkin method .... ✓
  -
- *A priori* error analysis
  - Ideal DPG method .... ✓
  - Practical DPG method .... ✓
- *A posteriori* error analysis
- Fast solvers .... ←
- Examples
  - Example 1 (Standard FEM) .... ✓
  - Example 2 (L^2-based least-squares) .... ✓
  - Example 3 (An ODE) .... ✓
  - Example 4 (Diffusion) .... ✓
  -

Preconditioning

$$\begin{aligned} \text{Abstractly,} \quad b(x_h^r, y) &= \ell(y) && \forall y \in Y_h^r = T^r(X_h), \\ \implies \quad b(x_h^r, T^r z_h) &= \ell(T^r z_h) && \forall z_h \in X_h \\ \implies \quad (T^r x_h^r, T^r z_h)_Y &= \ell(T^r z_h) && \forall z_h \in X_h. \end{aligned}$$

Lemma

$$\textbf{[A.1–A.3]} \quad \implies \quad \frac{C_1}{C_\Pi} \|x\|_X \le \|T^r x\|_Y \le C_2 \|x\|_X$$

for all $x \in X_h$.

This implies that any preconditioner spectrally equivalent to the $(\cdot,\cdot)_X$*-inner product is also a preconditioner for the practical DPG method.*

Example: A BDDC preconditioner

$$b(\,(u, \hat{q}_n), v) = \sum_{K \in \Omega_h} \left[\int_K \vec{\nabla}\, u \cdot \vec{\nabla}\, v \; - \int_{\partial K} \hat{q}_n \, v \right]$$

$$X = H_0^1(\Omega) \times H^{-1/2}(\partial \Omega_h),$$

Implementation in NGSolve with Lukas Kogler & Joachim Schöberl

1. Statically condense the stiffness matrix to $u|_{\partial \Omega_h}$ and $\hat{q}_n$.
2. Apply a BDDC preconditioner as follows:
   1. Do a wire basket coarse solve.
   2. Add inverses of small blocks of $u|_{\partial \Omega_h}$-unknowns on each interface.
   3. Add inverses of small blocks of $\hat{q}_n$-unknowns on each interface.

Example: A BDDC preconditioner

$$b(\,(u,\hat{q}_n),v) = \sum_{K\in\Omega_h} \left[\int_K \vec{\nabla}\,u\cdot\vec{\nabla}\,v\; - \int_{\partial K}\;\hat{q}_n\; v\;\right]$$

$$X = H_0^1(\Omega)\times H^{-1/2}(\partial\Omega_h),$$

Implementation in NGSolve with Lukas Kogler & Joachim Schöberl

$p+1$	diagonal	BDDC
4	142	60
5	159	65
6	180	77
7	202	78
8	209	88
9	243	90

- Used a small fixed 8 x 8 mesh
- Number of preconditioned conjugate gradient iterations are reported.

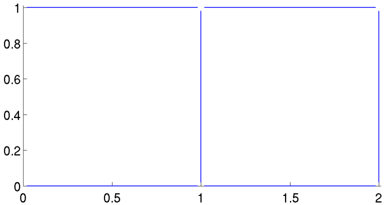
Next

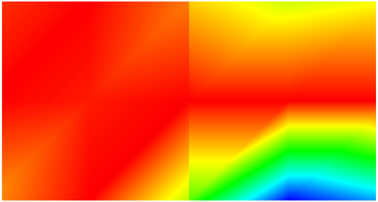
- Three avenues to DPG methods
  - Petrov-Galerkin with optimal test functions . . . . ✓
  - Least-squares Galerkin method . . . . ✓
  - Mixed Galerkin method . . . . ←
- *A priori* error analysis
  - Ideal DPG method . . . . ✓
  - Practical DPG method . . . . ✓
- *A posteriori* error analysis . . . . ←
- Fast solvers . . . . ✓
- Examples
  - Example 1 (Standard FEM) . . . . ✓
  - Example 2 (L^2-based least-squares) . . . . ✓
  - Example 3 (An ODE) . . . . ✓
  - Example 4 (Diffusion) . . . . ✓
  -

Built-in error estimator in DPG methods

Results for Carter's flat plate problem: *(courtesy of Jesse Chan)*

Adaptivity shows no preasymptotics.





Iteration 0

Supersonic flow impinging over a flat plate (Ma $= 3$, Re $= 1000$). Used Petrov-Galerkin implementation in Camillia package with h-adaptivity, $p = 2$, starting with a mesh of just two elements.

Built-in error estimator in DPG methods

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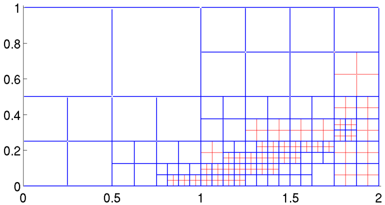

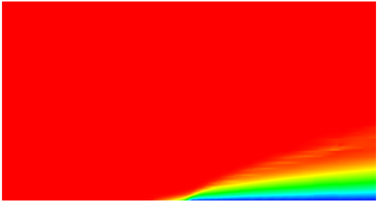

Iteration 5

Supersonic flow impinging over a flat plate (Ma = 3, Re = 1000). Used Petrov-Galerkin implementation in Camillia package with h-adaptivity, $p = 2$, starting with a mesh of just two elements.

Built-in error estimator in DPG methods

Results for Carter's flat plate problem: *(courtesy of Jesse Chan)*

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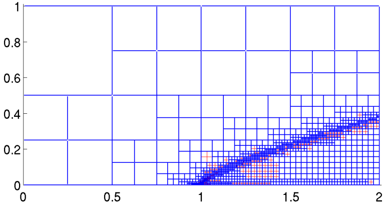
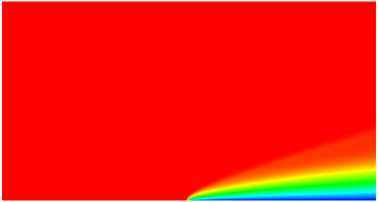
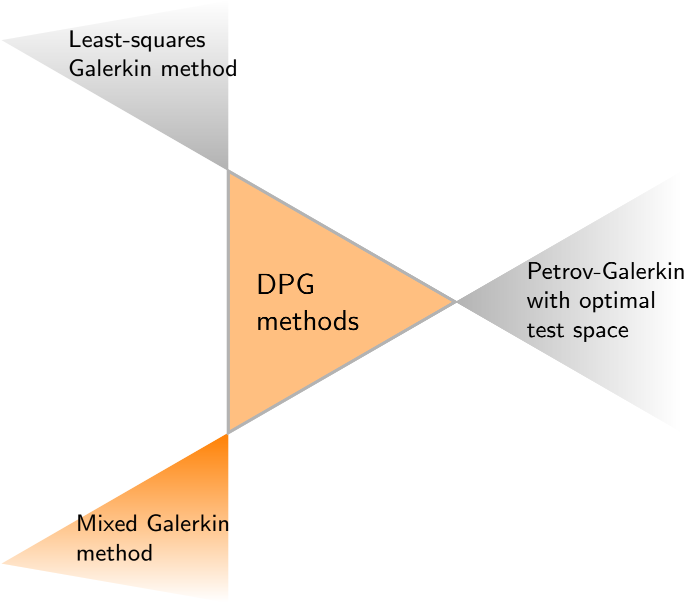




Iteration 10

Supersonic flow impinging over a flat plate (Ma $= 3$, Re $= 1000$). Used Petrov-Galerkin implementation in Camillia package with h-adaptivity, $p = 2$, starting with a mesh of just two elements.

The mixed method approach



Error representation function

Residual: $\rho = \ell - Bx_h.$

Error representation function: $\varepsilon^r = R_{Y^r}^{-1}(\ell - Bx_h).$

It can be practically computed by

$$(\varepsilon^r, y)_Y = \ell(y) - b(x_h, y), \qquad \forall y \in Y^r.$$

Error estimator: $\eta = \|\varepsilon^r\|_Y.$ [Demkowicz+G+Niemi 2012]

Petrov-Galerkin solve → ε^r by local postprocessing

Least-squares → ε^r is Riesz inverse of residual

Mixed method → ε^r is one of the variables

DPG as a Mixed method

Theorem (Reinterpretation of DPG as a mixed method)

The following are equivalent statements:

i) $x_h \in X_h$ *solves the practical DPG method.*

ii) $x_h \in X_h$ *and* $\varepsilon^r \in Y^r$ *solve the mixed formulation*

$$(\varepsilon^r, y)_Y + b(x_h, y) = \ell(y) \qquad \forall y \in Y^r, \tag{1a}$$

$$b(z_h, \varepsilon^r) = 0 \qquad \forall z_h \in X_h. \tag{1b}$$

Proof.

$(i) \implies (ii)$: Eq. (1a) is just the defintion of ε^r. For (1b),

$$\begin{aligned} b(z_h, \varepsilon^r) &= (T^r z_h, \varepsilon^r)_Y = (T^r z_h, R_{Y^r}^{-1}(\ell - Bx_h))_Y = (T^r z_h, T^r(x - x_h))_Y \\ &= b(x - x_h, T^r z_h) = 0. \end{aligned}$$

$(ii) \implies (i)$: Similar. □

DPG as a Mixed method

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$$b(z_h, \varepsilon^r) = 0 \qquad \forall z_h \in X_h. \tag{1b}$$

- [Dahmen+Huang+Schwab+Welper 2012] studied similar mixed formulations and found techniques other than localization by discontinuous spaces to make the method practical.

Recall the previous assumptions

[A.1] $\{w \in X : \; b(w, y) = 0 \quad \forall y \in Y\} = \{0\}.$

[A.2] $\exists \; C_1, C_2 > 0$ such that $\quad C_1 \|y\|_Y \le \sup_{w \in X} \dfrac{|b(w, y)|}{\|w\|_X} \le C_2 \|y\|_Y.$

[A.3] $\exists \; \Pi : Y \mapsto Y^r$ and $C_\Pi > 0$ such that for all $w_h \in X_h$ and $y \in Y$,

$$b(w_h, y - \Pi y) = 0, \qquad \|\Pi y\|_Y \le C_\Pi \|y\|_Y.$$

Optimal *a priori* estimates followed from these assumptions.

We now show that a posteriori error estimators also follow from the same assumptions **[A.1–A.3].**

A posteriori error estimates

Theorem (Reliability & Efficiency of DPG error estimator)

Suppose **[A.1–A.3]** *hold. Let* $F \in Y^*$,

- $x = B^{-1}F$, [Carstensen+Demkowicz+G 2014]
- $x_h \in X_h$ *be the DPG solution,*
- $\eta = \|F - Bx_h\|_{(Y^r)^*} = \|\varepsilon^r\|_Y$ *be the error estimator,*
- $\operatorname{osc}(F) \stackrel{\text{def}}{=} \|F \circ (1 - \Pi)\|_{Y^*}$.

Then

$$C_1^2 \|x - x_h\|_X^2 \le \eta^2 + (C_\Pi \, \eta + \operatorname{osc}(F))^2, \qquad \leftarrow \textit{Reliability}$$

$$\eta^2 \le C_2^2 \|x - x_h\|_X^2. \qquad \leftarrow \textit{Efficiency}$$

- "Efficiency" is trivial in least-square methods.
- Proof of "Reliability" uses Π critically.

A posteriori error estimates

Theorem (Reliability & Efficiency of DPG error estimator)

Suppose **[A.1–A.3]** *hold. Let* $F \in Y^*$,

- $x = B^{-1}F$, [Carstensen+Demkowicz+G 2014]
- $\tilde{x}_h \in X_h$ ~~*be the DPG solution,*~~
- $\tilde{\eta} = \|F - B\tilde{x}_h\|_{(Y^r)^*} = \|\varepsilon^r\|_Y$ *be the error estimator,*
- $\text{osc}(F) \stackrel{\text{def}}{=} \|F \circ (1 - \Pi)\|_{Y^*}$.

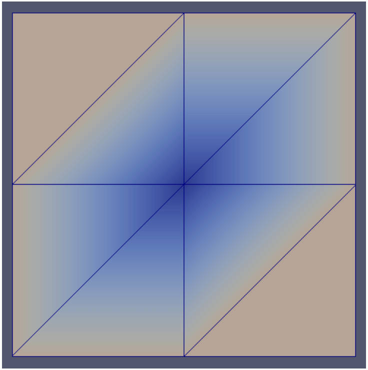
Then

$$C_1^2 \, \|x - \tilde{x}_h\|_X^2 \le \tilde{\eta}^2 + (C_\Pi \, \tilde{\eta} + \text{osc}(F))^2 \,, \qquad \leftarrow \textit{Reliability}$$

$$\tilde{\eta}^2 \le C_2^2 \, \|x - \tilde{x}_h\|_X^2. \qquad \leftarrow \textit{Efficiency}$$

- "Efficiency" is trivial in least-square methods.
- Proof of "Reliability" uses Π critically.

Error estimator in the Laplace example

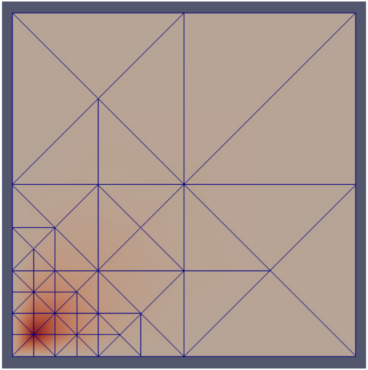


Iteration 0

- Results for Dirichlet problem with $f(x, y) = e^{-100(x^2+y^2)}$ aside.
- No need to code an error estimator for driving adaptivity in DPG methods.
- The mixed formulation is standard Galerkin, so it is easily implementable in codes without support for Petrov-Galerkin forms.

[Click here to download `FEniCS` code for this experiment.]

Error estimator in the Laplace example



Iteration 6

- Results for Dirichlet problem with $f(x, y) = e^{-100(x^2+y^2)}$ aside.
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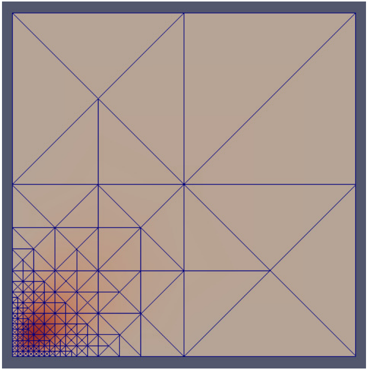
Error estimator in the Laplace example



Iteration 11

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Example 5: Stresses in Stokes flow

Second order system:

$$\frac{1}{2}\Delta\vec{u} - \vec{\nabla}\, p = \vec{f} \quad \text{in } \Omega,$$
$$\nabla\cdot\vec{u} = 0 \quad \text{in } \Omega.$$

No slip B.C.: $\vec{u} = \vec{0}$ on $\partial\Omega$.
For uniqueness: $(p, 1)_\Omega = 0$.

Convert to first order system:

$$\sigma + p\,\delta - \varepsilon(\vec{u}) = 0, \quad \text{(definition of true fluid stress } \sigma\text{)}$$
$$\nabla\cdot\sigma = \vec{f}. \quad \text{(since } \nabla\cdot\sigma = \tfrac{1}{2}\Delta\vec{u} - \vec{\nabla}\, p\text{)}$$

Example 5: Stresses in Stokes flow

Second order system:

$$\begin{aligned} \tfrac{1}{2}\Delta\vec{u} - \vec{\nabla}\, p &= \vec{f} \qquad \text{in } \Omega, \\ \nabla\cdot\vec{u} &= 0 \qquad \text{in } \Omega. \end{aligned}$$

No slip B.C.: $\vec{u} = \vec{0}$ on $\partial\Omega$.
For uniqueness: $(p, 1)_\Omega = 0$.

Convert to first order system:

Apply deviatoric:

$$D\tau = \tau - \frac{\operatorname{tr}\tau}{N}\delta$$

$$\begin{aligned} \sigma + \; p\,\delta - \varepsilon(\vec{u}) &= 0, \\ \nabla\cdot\sigma &= \vec{f}. \end{aligned}$$

Example 5: Stresses in Stokes flow

Second order system:

$$\frac{1}{2}\Delta\vec{u} - \vec{\nabla}\, p = \vec{f} \quad \text{in } \Omega,$$
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For uniqueness: $(p, 1)_\Omega = 0$.

Convert to first order system:

Apply deviatoric:

$$D\tau = \tau - \frac{\operatorname{tr}\tau}{N}\delta$$

$$D\sigma \quad - \varepsilon(\vec{u}) = 0,$$
$$\nabla \cdot \sigma = \vec{f}.$$

And $(\operatorname{tr}\sigma, 1)_\Omega = 0$.

Example 5: Stresses in Stokes flow

Second order system:

$$\tfrac{1}{2}\Delta\vec{u} - \vec{\nabla}\, p = \vec{f} \quad \text{in } \Omega,$$
$$\nabla\cdot\vec{u} = 0 \quad \text{in } \Omega.$$

No slip B.C.: $\vec{u} = \vec{0}$ on $\partial\Omega$.
For uniqueness: $(p, 1)_\Omega = 0$.

Convert to first order system:

$$D\sigma \qquad - \varepsilon(\vec{u}) = 0,$$
$$\nabla\cdot\sigma = \vec{f}. \qquad \text{And} \qquad (\mathrm{tr}\,\sigma, 1)_\Omega = 0.$$

DPG form with $x = (\sigma, \vec{u}, \hat{u}, \hat{\sigma}_n, \alpha)$ and $y = (\tau, \vec{v}, \omega)$:

$$b(x, y) = (D\sigma, \tau)_\Omega + (\vec{u}, \nabla\cdot\tau)_{\Omega_h} - \langle \hat{u}, \tau n\rangle_{\partial\Omega_h} + (\alpha, \mathrm{tr}\,\tau)_\Omega$$
$$+ (\sigma, \varepsilon(\vec{v}))_{\Omega_h} - \langle \hat{\sigma}_n, \vec{v}\rangle_{\partial\Omega_h} + (\mathrm{tr}\,\sigma, \omega)_\Omega.$$

Spaces for Stokes example

DPG form with $x = (\sigma, \vec{u}, \hat{u}, \hat{\sigma}_n, \alpha)$ and $y = (\tau, \vec{v}, \omega)$:

$$b(x, y) = (D\sigma, \tau)_\Omega + (\vec{u}, \nabla \cdot \tau)_{\Omega_h} - \langle \hat{u}, \tau n \rangle_{\partial\Omega_h} + (\alpha, \operatorname{tr} \tau)_\Omega + (\sigma, \varepsilon(\vec{v}))_{\Omega_h} - \langle \hat{\sigma}_n, \vec{v} \rangle_{\partial\Omega_h} + (\operatorname{tr} \sigma, \omega)_\Omega.$$

Trial and test spaces:

$$X = L^2(\Omega; \mathbb{S}) \times L^2(\Omega)^N \times H_0^{1/2}(\partial\Omega_h)^N \times H^{-1/2}(\partial\Omega_h)^N \times \mathbb{R},$$
$$Y = H(\mathrm{div}, \Omega_h; \mathbb{S}) \times H^1(\Omega_h)^N \times \mathbb{R}.$$

Discrete spaces:

$$X_h = \{(\sigma, \vec{u}, \hat{u}, \hat{\sigma}_n, \alpha) \in X : \sigma|_K \in P_p(K; \mathbb{S}),\ \vec{u}|_K \in P_p(K)^N, \forall \text{elements } K, \hat{u}|_F \in P_{p+1}(F)^N,\ \hat{\sigma}_n|_F \in P_p(F)^N,\ \forall \text{interfaces } F, \alpha \in \mathbb{R}\},$$
$$Y^r = \{(\tau, \vec{v}, \omega) \in Y :\ \omega \in \mathbb{R}, \tau|_K \in P_{p+2}(K; \mathbb{S}),\ \vec{v}|_K \in P_{p+N}(K)^N,\ \forall \text{elements } K\}.$$

A priori and *a posteriori* estimates for Stokes example

Theorem

Suppose Ω_h is a shape-regular simplicial mesh of Ω and $p \geq 0$.
Then **[A.1–A.3]** *holds for the Stokes example.*
Consequently, $\exists$ mesh-independent constants $c_1, \ldots, c_4 > 0$ such that

$$\|x - x_h\|_X \leq c_1 \min_{\xi_h \in X_h} \|x - \xi_h\|_X,$$

$$c_4 \|x - x_h\|_X^2 - c_2 \operatorname{osc}(F)^2 \leq \eta^2 \leq c_3 \|x - x_h\|_X^2.$$

- Verification of **[A.3]** uses degrees of freedom of symmetric matrix polynomials in [G+Guzmán 2011].
- Proof proceeds by taking the incompressible limit of a similar elasticity discretization.

Stokes solution on L-shaped domain

Osborn's singular solution:

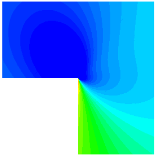
$$u = \operatorname{curl}\left(a_+ s_+ + a_- s_- + c_+ - c_-\right),$$

where

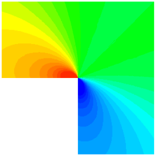
$$s_\pm = r^{1+z}\sin((z\pm 1)\theta), \quad c_\pm = r^{1+z}\cos((z\pm 1)\theta),$$
$$a_\pm = -z\cot(3z\pi/2)/(z\pm 1),$$
$$z^2 = \sin^2(3z\pi/2) \quad [z = \text{root with smallest real part}].$$

Results from an h-adaptive algorithm with η as estimator and $p = 2$:

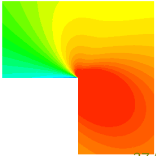
σ_{xy}



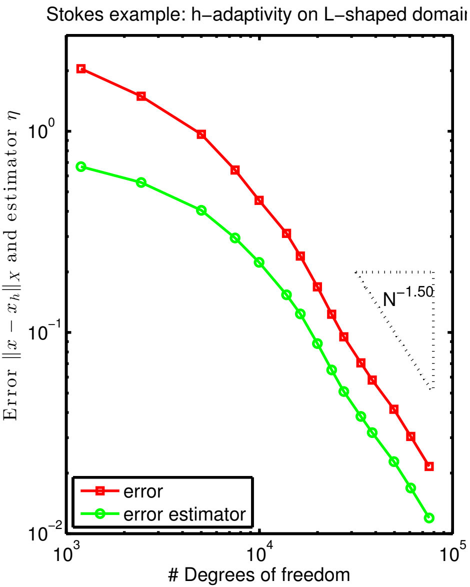
σ_{xx}



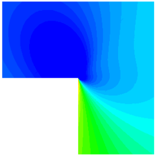
σ_{yy}



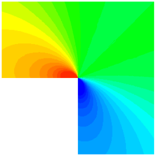
Stokes solution on L-shaped domain



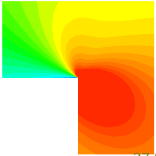
σ_{xy}



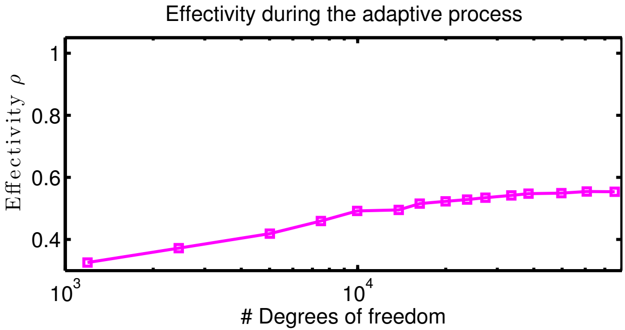
σ_{xx}



σ_{yy}

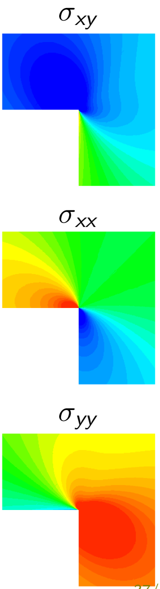


Stokes solution on L-shaped domain

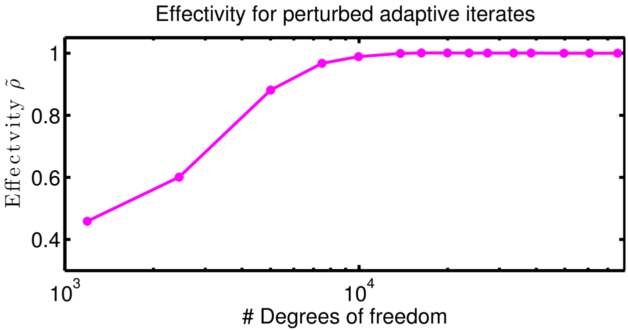


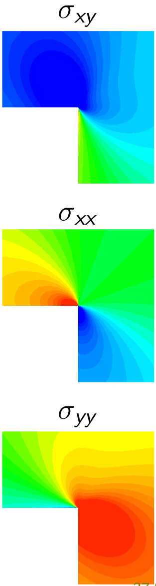
Effectivity index ρ.

$$\rho = \frac{\eta}{\|x - x_h\|_X}$$



Stokes solution on L-shaped domain





After x_h randomly perturbed by 5%.

$$\tilde{\rho} = \frac{\eta}{\|x - \tilde{x}_h\|_X}$$

Conclusion

- Three avenues to DPG methods
  - Petrov-Galerkin with optimal test functions .... ✓
  - Least-squares Galerkin method .... ✓
  - Mixed Galerkin method .... ✓
- *A priori* error analysis
  - Ideal DPG method .... ✓
  - Practical DPG method .... ✓
- *A posteriori* error analysis .... ✓
- Fast solvers .... ✓
- Examples
  - Example 1 (Standard FEM) .... ✓
  - Example 2 (L^2-based least-squares) .... ✓
  - Example 3 (An ODE) .... ✓
  - Example 4 (Diffusion) .... ✓
  - Example 5 (Stokes) .... ✓