Approximating the Fisher Information for a Partially-Observable Growing Population

Ali Eshragh

(Joint work with Nigel Bean and Joshua Ross)

School of Mathematical and Physical Sciences & CARMA
The University of Newcastle, Australia

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Motivation

- **Epidemiology**

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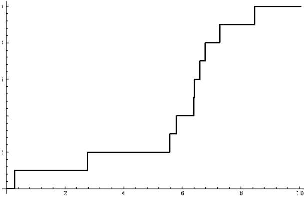
- A **Growing** Population



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- It is **Markovian**, that is

  $$\Pr(X_{t_{n+1}} = x_{n+1} | X_{t_n} = x_n, \ldots, , X_{t_1} = x_1) \; = \; \Pr(X_{t_{n+1}} = x_{n+1} | X_{t_n} = x_n) \, ,$$

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  for all possible values of n and $t_1, \ldots, t_{n+1}$.
- The **transition probability** is equal to

  $$\Pr(X_{s+t} = j | X_s = i) = \binom{j-1}{i-1} e^{-\lambda t i} (1 - e^{-\lambda t})^{j-i} .$$

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$$= \prod_{i=2}^{n} \Pr(X_{t_i} = x_i | X_{t_{i-1}} = x_{i-1}, \ldots, X_{t_1} = x_1) \Pr(X_{t_1} = x_1)$$

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- It can be shown that

$$\mathcal{FI}_{(X_{t_1},\ldots,X_{t_n})}(\lambda) = E_{\mathcal{L}}\left[\left(\frac{d}{d\lambda}\ln(\mathcal{L}(X_{t_1},\ldots,X_{t_n};\lambda))\right)^2\right].$$

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- Hence, $(t_1^*, \ldots, t_n^*) \in \text{argmax}\{\mathcal{FI}_{(X_{t_1},\ldots,X_{t_n})}(\lambda)\}$.

Fisher Information and Optimal Observation Times

Proposition (Becker and Kersting, 1983)

The **Fisher information** *for a SBP with the parameter* λ*, the initial value of* x_0 *and the observation times of* $(t_1, \ldots, t_n)$ *is as follows:*

$$\mathcal{FI}_{(X_{t_1},\cdots,X_{t_n})}(\lambda) \quad = \quad x_0 \sum_{i=1}^{n} \frac{(t_i - t_{i-1})^2}{e^{-\lambda t_{i-1}} - e^{-\lambda t_i}} .$$

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Optimal Observation Times (Becker and Kersting, 1983)

$$\mathbf{t_i^* \approx \frac{3}{\lambda}\log\left(1 + \frac{i}{n}(e^{\frac{\lambda\tau}{3}} - 1)\right) \quad for\ i = 1, \ldots, n}$$

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- $\text{POSBP}(\lambda, 1) \equiv \text{SBP}(\lambda)$.

Markovian or non-Markovian?

Theorem (Bean, Elliott, Eshragh and Ross; 2014)

The POSBP $\{Y_t : t \in \mathrm{R}_0^+\}$ *with parameters* (λ, p) *is* **not Markovian**.

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- However,

$$\Pr(Y_{t_1} = y_{t_1}, \ldots, Y_{t_n} = y_{t_n} | X_{t_1} = x_{t_1}, \ldots, X_{t_n} = x_{t_n})$$

$$= \prod_{i=1}^{n} \Pr(Y_{t_i} = y_{t_i} | X_{t_i} = x_{t_i}) \, .$$

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where $q := 1 - p$ and $\upsilon_{i-1,i} := e^{-\lambda(t_i - t_{i-1})}$.

Theoretical Result

Proposition (Bean, Eshragh and Ross; 2014)

For a POSBP with n observations and time horizon τ, the FI is an **increasing** *function of t_n. Hence, the* **optimal observation time** *for the last observation, that is t_n^*, is equal to $\boldsymbol{\tau}$.*

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Proposition (Bean, Eshragh and Ross; 2014)

If $t_1^*, \ldots, t_n^*$ are optimal observation times for a POSBP with parameters (λ, p) and time-horizon τ, then $\frac{\mathbf{t_1^*}}{\boldsymbol{\tau}}, \ldots, \frac{\mathbf{t_n^*}}{\boldsymbol{\tau}}$ are **optimal observation times** *for a POSBP with parameters $(\lambda\tau, p)$ and time-horizon $\mathbf{1}$.*

Truncated Summation

- The Fisher Information:

$$\mathcal{FI}_{(Y_{t_1},\ldots,Y_{t_n})}(\lambda) \quad = \quad \sum_{y_{t_1},\ldots,y_{t_n}} \frac{\left(\frac{d\mathcal{L}(y_{t_1},\ldots,y_{t_n};\,\lambda)}{d\lambda}\right)^2}{\mathcal{L}(y_{t_1},\ldots,y_{t_n};\,\lambda)}.$$

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- Here, the likelihood function $\mathcal{L}(y_{t_1},\ldots,y_{t_n};\lambda)$ is equal to

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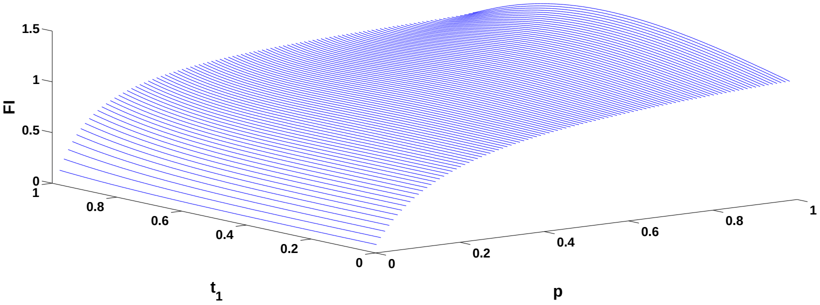
where $\upsilon_{i-1,i} := e^{-\lambda(t_i - t_{i-1})}$.

- By exploiting **Chebyshev's inequality**, we have

$$\begin{aligned}\Pr\left(E[Z]-12\sqrt{Var(Z)} \le Z \le E[Z]+12\sqrt{Var(Z)}\right) &\ge 1-\frac{1}{12^2}\\ &= 99.3\%.\end{aligned}$$

Results for $\lambda = 2$, $n = 2$ and $t_2^* = \tau = 1$

- Fisher Information vs. t_1 and p



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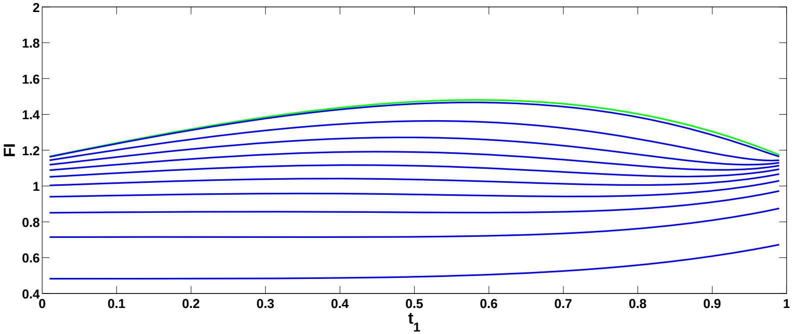
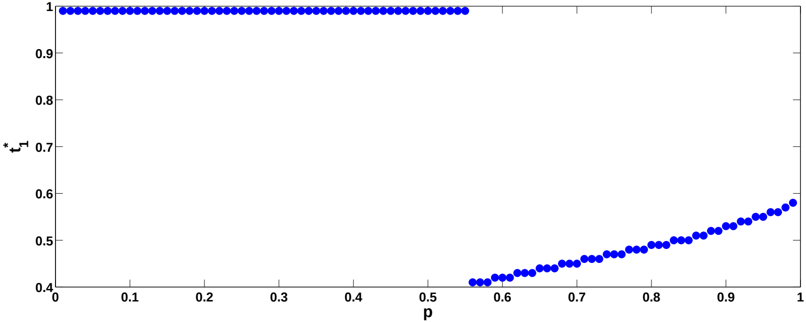
- The Fisher Information vs. t_1



Results for $\lambda = 2$, $n = 2$ and $t_2^* = \tau = 1$

- Optimal observation time t_1^* vs. p



The Chain Rule

- The likelihood function

$$\mathcal{L}(y_{t_1}, y_{t_2}|\lambda) = \Pr(Y_{t_2} = y_{t_2}|Y_{t_1} = y_{t_1}, \lambda)\Pr(Y_{t_1} = y_{t_1}|\lambda).$$

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- Accordingly,

$$\log\left(\mathcal{L}(y_{t_1}, y_{t_2}|\lambda)\right) = \log\left(\Pr(Y_{t_2} = y_{t_2}|Y_{t_1} = y_{t_1}, \lambda)\right) + \log\left(\Pr(Y_{t_1} = y_{t_1}|\lambda)\right).$$

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- Accordingly,

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- The Fisher Information:

$$\mathcal{FI}_{(Y_{t_1}, Y_{t_2})}(\lambda) = \mathcal{FI}_{(Y_{t_2}|Y_{t_1})}(\lambda) + \mathcal{FI}_{(Y_{t_1})}(\lambda).$$

Two-Parameter Geometric Distribution

Definition

A discrete random variable V has the "**Two-Parameter Geometric**" distribution with parameters $\alpha \in [0,1)$ and $\beta \in (0,1)$, denoted by **TPG**(α, β), if its **probability mass function (p.m.f.)** is

$$P_V(v) = \begin{cases} \alpha & \text{for } v = 0 \\ (1-\alpha)\beta(1-\beta)^{v-1} & \text{for } v = 1, 2, \ldots . \end{cases}$$

Three-Parameter Negative Binomial Distribution

Definition

Suppose $V_1, \ldots, V_r$ are **i.i.d.** random variables with common TPG(α, β) distribution. If $\mathbf{W} := \sum_{i=1}^{r} \mathbf{V_i}$, then W has "**Three-Parameter Negative Binomial**" distribution with parameters $\mathbf{r}$, α and β, denoted by $\mathbf{TPNB}(\mathbf{r}, \alpha, \beta)$.

Three-Parameter Negative Binomial Distribution

Definition

Suppose $V_1, \ldots, V_r$ are **i.i.d.** random variables with common TPG(α, β) distribution. If $\mathbf{W} := \sum_{i=1}^{r} \mathbf{V}_i$, then W has "**Three-Parameter Negative Binomial**" distribution with parameters $\mathbf{r}$, α and β, denoted by $\mathbf{TPNB(r, \alpha, \beta)}$.

Proposition (Bean, Eshragh and Ross; 2014)

If W follows the $TPNB(r, \alpha, \beta)$ distribution, then its **p.m.f.** *is*

$$P_W(w) = \begin{cases} \alpha^r & \text{for } w = 0 \\ \displaystyle\sum_{\xi=1}^{\min\{r,w\}} \binom{w-1}{\xi-1} \beta^{\xi} (1-\beta)^{w-\xi} \binom{r}{\xi} (1-\alpha)^{\xi} \alpha^{r-\xi} & \text{for } w \geq 1 \end{cases}$$

The Distribution of Y_t

Theorem (Bean, Eshragh and Ross; 2014)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p) *and the* **initial population size** $x_0 \geq 1$. *For any real value* $t > 0$, *the random variable* Y_t *follows the* $\mathbf{TPNB}(\mathbf{x_0}, (\mathbf{1}-\mathbf{p})\boldsymbol{\beta}_\mathbf{t}, \boldsymbol{\beta}_\mathbf{t})$ *distribution where*

$$\beta_{\mathbf{t}} := \frac{\mathbf{e}^{-\lambda \mathbf{t}}}{\mathbf{p} + (\mathbf{1}-\mathbf{p})\mathbf{e}^{-\lambda \mathbf{t}}}.$$

The Fisher Information for a Single Observation

Proposition (Bean, Eshragh and Ross; 2014)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p)*. The Fisher Information of a single observation* Y_{t_1} *for parameter* λ *is equal to*

$$\mathcal{FI}_{\mathbf{Y}_1}(\lambda) = \frac{p t_1^2 \left(p + (1-p)(1-e^{-\lambda t_1})e^{-\lambda t_1}\right)}{(1-e^{-\lambda t_1})(p + (1-p)e^{-\lambda t_1})^2}.$$

The Distribution of $(Y2|Y1 = y_{t_1})$

Theorem (Bean, Eshragh and Ross; 2014)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p). *Then*

$$\mathbf{W} \overset{\mathbf{d}}{=} (\mathbf{Y_{t_2}} | \mathbf{Y_{t_1}} = \mathbf{y_{t_1}}) + \mathbf{V}$$

where $(Y_{t_2}|Y_{t_1} = y_{t_1})$ *and* V *are mutually independent and*

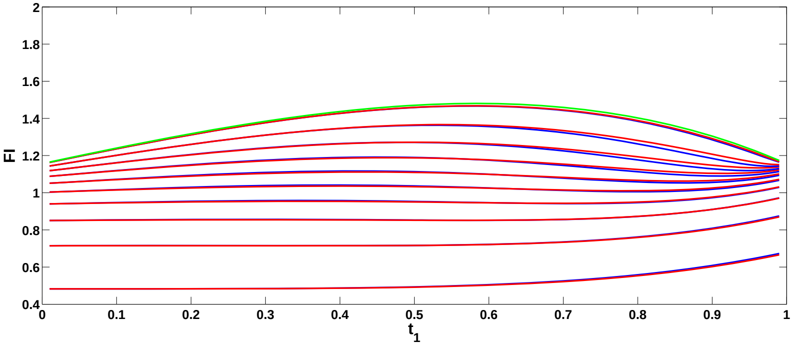
$$W \sim TPNB(y_{t_1} + 1, (1-p)\beta^{\circ}, \beta^{\circ})$$

and

$$V \sim TPG((1-p)\beta_{t2-t1}, \beta_{t2-t1}).$$

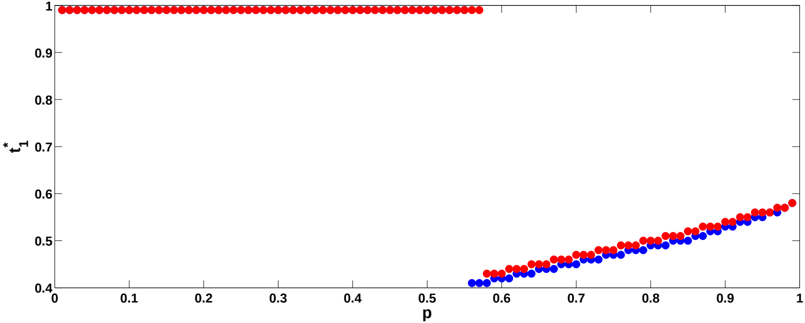
Results for $\lambda = 2$, $n = 2$ and $t_2^* = \tau = 1$

- The Fisher Information (blue) and its Approximation (red) vs. t_1



Results for $\lambda = 2$, $n = 2$ and $t_2^* = \tau = 1$

- Optimal observation time t_1^* vs. p



Bounds for the Fisher Information

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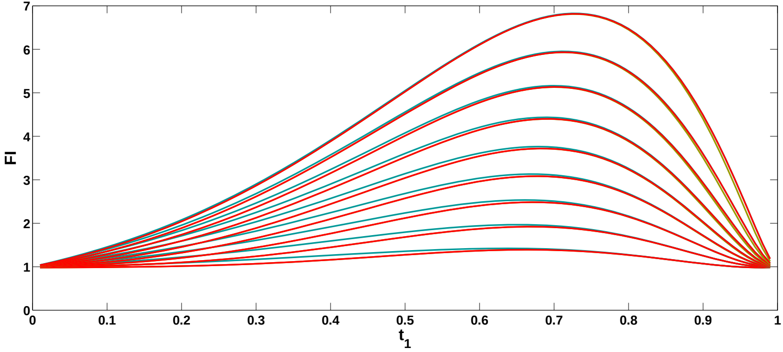
Theorem (Bean, Eshragh and Ross; 2014)

The lower and upper bounds for the Fisher Information **approach together** *as λ tends to infinity.*

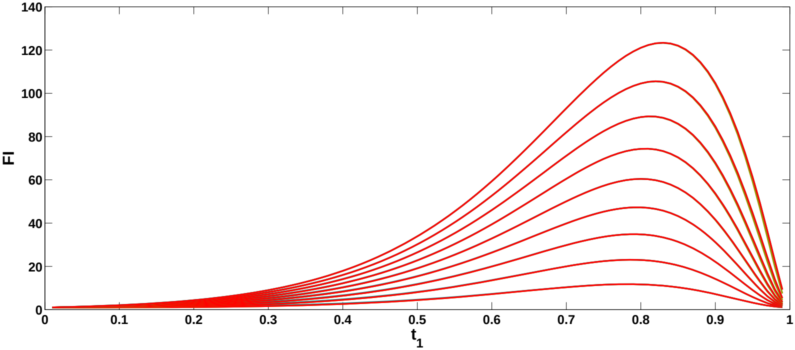
Results for $\lambda = 6$, $n = 2$ and $t_2^* = 1$

- Lower (brown) and Upper (green) Bounds for The Fisher Information and its Approximation (red) vs. t_1



Results for $\lambda = 10$, $n = 2$ and $t_2^* = 1$

- Lower (brown) and Upper (green) Bounds for The Fisher Information and its Approximation (red) vs. t_1



Further Developments

- Developing analogous approximation for **higher values** of n .

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- Finding the Fisher Information to estimate parameter **p** along with λ, both together.

End

Thank you $\cdots$ Questions?