New Congruences and Relations for the Fishburn Numbers

F.G. Garvan
url: qseries.org/fgarvan

University of Florida

Wednesday, July 23, 2014

ABSTRACT

The Fishburn numbers occur as coefficients of Konstevich's strange quantum modular form. We show how we used the computer to discover and prove new congruences and relations for the Fishburn numbers and their relatives.

INTRODUCTION

What are the Fishburn numbers?

The Fishburn numbers $\xi(n)$ are defined by the formal power series

$$\sum_{n=0}^{\infty} \xi(n) q^n = F(1-q),$$

where

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n = \sum_{n=0}^{\infty} (1-q)(1-q^2)\cdots(1-q^n),$$

and

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

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```
> with(qseries):
> F:=(q,T)->add(aqprod(q,q,n),n=0..T):
> series(F(1-q,10),q,10);
```

$$1+q+2\,q^2+5\,q^3+15\,q^4+53\,q^5+217\,q^6 \\ +1014\,q^7+5335\,q^8+31240\,q^9+O\left(q^{10}\right)$$

```
> seq(coeff(%,q,j),j=0..9);
```

$$1,1,2,5,15,53,217,1014,5335,31240$$

login

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the on-line encyclopedia of integer sequences®
founded in 1964 by N. J. A. Sloane

Invitation: celebrating 50 years of OEIS, 250000 sequences, and Sloane's 75th, there will be a conference at DIMACS, Rutgers, Oct 9-10 2014.

1, 1, 2, 5, 15, 53, 217, 1014, 5335, 31240 Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,1,2,5,15,53,217,1014,5335,31240**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A022493 Fishburn numbers: number of linearized chord diagrams of degree n; also number of nonisomorphic interval orders on n unlabeled points. +20 22

1, 1, 2, 5, 15, 53, 217, 1014, 5335, 31240, 201608, 1422074, 10886503, 89903100, 796713190, 7541889195, 75955177642, 810925547354, 9148832109645, 108759758865725, 1358836180945243, 17801039909762186, 243992799075850037, 3492329741309417600, 52105418376516869150, 809029109658971635142 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

P. Levande, Fishburn diagrams, Fishburn numbers and their refined generating functions, Journal of Combinatorial Theory, Series A 120 (2013) 194-217. - From N. J. A. Sloane, Dec 23 2012
J. A. Reeds and P. C. Fishburn, Counting split interval orders, Order, Vol. 18, No. 2, 2001, pp. 129-135.
D. Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function, Topology 40(5) (2001), 945-960.

LINKS

Alois P. Heinz, Table of n, a(n) for n = 0..185
George E. Andrews, James A. Sellers, Congruences for the Fishburn Numbers, arXiv:1401.5345 [math.NT], 2014
D. Bar-Natan, Sergei Duzhin, Bibliography of Vassiliev Invariants
Mireille Bousquet-Melou, Anders Claesson, Mark Dukes, Sergey Kitaev, (2+2)-free posets, ascent sequences and pattern avoiding permutations. arxiv:0806.0666 [From Mark Dukes (dukes(AT)hi.is), May 12 2009]
Graham Brightwell and Mitchel T. Keller, Asymptotic enumeration of labelled interval orders, arXiv 1111.6766
Anders Claesson, Mark Dukes, Sergey Kitaev, A direct encoding of Stoimenow's matchings as ascent sequences, arXiv:0910.1619 [math.CO]
Anders Claesson, Mark Dukes and Martina Kubitzke, Partition and composition matrices, arXiv:1006.1312.
Julie Christophe, Jean-Paul Doignon and Samuel Fiorini, Counting Biorders, J. Integer Seqs., Vol. 6, 2003.
Paul Levande, Two New Interpretations of the Fishburn Numbers and their Refined Generating Functions, arXiv:1006.3013v1.
A. Stoimenow, Enumeration of chord diagrams and an upper bound for

ZAGIER's STRANGE IDENTITY

$$\sum_{n=0}^{\infty}(1-q)(1-q^2)\cdots(1-q^n) = -\frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}}$$

Neither side of this identity makes sense simultaneously.

BUT !!!

```
> read kronprogs:
>
RF:=(q,T)->add(n*kron(n,12)*q^((n^2-1)/24),n=1..T):
> prodmake(RF(q,20),q,20);
```

$$(1-q)^5\left(-q^2+1\right)^3\left(-q^3+1\right)^5\left(-q^4+1\right)^3\left(-q^5+1\right)^5\left(-q^6+1\right)^3\left(-q\right.$$

$$\left(-q^{11}+1\right)^5\left(-q^{12}+1\right)^3\left(-q^{13}+1\right)^5\left(-q^{14}+1\right)^3\left(-q^{15}+1\right)^5\left(-q^{16}+\right.$$

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$$\sum_{n=1}^{\infty} n\left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}} = \prod_{n=1}^{\infty}(1-q^{2n-1})^5(1-q^{2n})^3 \textit{WRONG}!!!!$$

RAMANUJAN

$$\sum_{n=1}^{\infty} n\left(\frac{n}{12}\right) q^{\frac{n^2-1}{24}} = \prod_{n=1}^{\infty}(1-q^{2n-1})^5(1-q^{2n})^3$$

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Verifying Zagier's Strange Identity

```
> F:=(q,T)->add(aqprod(q,q,n),n=0..T):
>
RF:=(q,T)->add(n*kron(12,n)*q^((n^2-1)/24),n=1..T):
> prodmake(RF(q,20),q,10);
```

$$(1-q)^5\left(-q^2+1\right)^{17}\left(-q^3+1\right)^{75}\left(-q^4+1\right)^{346}\left(-q^5+1\right)^{1733}\left(-q^6+1\right.$$
$$\left(-q^7+1\right)^{46635}\left(-q^8+1\right)^{250128}\left(-q^9+1\right)^{1364555}$$

```
> simplify(RF(r*exp(2*Pi*I/3),451)):
> r3:=unapply(%,r):  evalf(r3(999/1000));
```

$$5.573727414 - 0.8710680617\,\mathrm{I}$$

```
> simplify(F(exp(2*Pi*I/3),2));
```

$$\frac{11}{2} - \frac{1}{2}\,\mathrm{I}\,3^{(1/2)}$$

```
> evalf(%);
```

$$5.500000000 - 0.8660254040\,\mathrm{I}$$

COMPUTING $\xi(n)$

$$e^{t/24}\sum_{n=0}^{\infty}(1-e^{t})\cdots(1-e^{nt})=\sum_{n=0}^{\infty}\frac{T_n}{n!}\left(\frac{-t}{24}\right)^{n},$$

where T_n are the Glaisher T-numbers [A002439] and which are given explicitly by

$$T_n=6\frac{(-144)^n}{n+1}\left[B_{2n+2}\left(\frac{1}{12}\right)-B_{2n+2}\left(\frac{5}{12}\right)\right],$$

where $B_n(x)$ denotes the n-th Bernoulli polynomial.

$$\xi(n) = \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^{n+k} \binom{-1/24}{n-m} \frac{s_1(m,k)}{m!24^k} T_k$$

CONGRUENCES

```
> read "fishcofs.m":
> FG:=add(fishcofs[j]*q^j,j=0..500):
> findcong(FG,500);
```

[3, 5, 5]

[4, 5, 5]

[6, 7, 7]

[8, 11, 11]

[9, 11, 11]

[10, 11, 11]

[16, 17, 17]

[17, 19, 19]

[18, 19, 19]

[18, 23, 23]

[19, 23, 23]

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$$[17,19,19]$$
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$$[18,23,23]$$
$$[19,23,23]$$

ANDREWS-SELLERS CONGRUENCES [January 21, 2014]

$$\begin{aligned}
\xi(5n+3) \equiv \xi(5n+4) &\equiv 0 \pmod{5},\\
\xi(7n+6) &\equiv 0 \pmod{7},\\
\xi(11n+8) \equiv \xi(11n+9) \equiv \xi(11n+10) &\equiv 0 \pmod{11},\\
\xi(17n+16) &\equiv 0 \pmod{17}, \quad \text{and}\\
\xi(19n+17) \equiv \xi(19n+18) &\equiv 0 \pmod{19}.
\end{aligned}$$

MISSING CONGRUENCES

$$\xi(23n+18) \equiv \xi(23n+19) \equiv \cdots \equiv \xi(23n+22) \equiv 0 \pmod{23}.$$

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ANDREWS-SELLERS THEOREM

There are analogous congruences for all primes p that are quadratic nonresidues mod 23. Define

$$S(p) = \{j : 0 \le j \le p-1 \text{ such that } \tfrac{1}{2}n(3n-1) \equiv j \pmod{p} \text{ for some} \tag{1}$$

and

$$T(p) = \{k : 0 \le k \le p-1 \text{ such that } k \text{ is larger than every element of } S \tag{2}$$

We state their main result.

Theorem (Andrews and Sellers)

If p is a prime and $i \in T(p)$ (as defined in (2)), then for all $n \ge 0$,

$$\xi(pn+i) \equiv 0 \pmod{p}.$$

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GARTHWAITE AND RHOADES [March 4, 2104]

Congruences for Fishburn numbers g_n and primitive Fishburn numbers p_n

p	Fn	Single	Pair	Triple	S(p)
5	g_n	4,3	2,1		2-0
	p_n	4	3,2,1	(4-3,3-2,2-1)	
7	g_n	6	5,4,3,2		5,3-0
	p_n		6,5,4,3	(6-3,2,1)	
11	g_n	10, 9, 8	7, 6, 5	(7-5, 4, 3)	7,5,4,2-0
	p_n			(10,9,8)	
13	g_n		12, 11, 10	(12-8,9-7,8-6)	12,9,7-5,2-0
	p_n			(10/8,8/6,6-5), (12-11,11-10,10-9)	
17	g_n	16	15, 14, 13, 12, 11, 10	(15-10,9,8)	15,12,9-5,...
	p_n			(16-15,15-14,14-13)	

EXAMPLES

$$\xi(5n+2) - 2\,\xi(5n+1) \equiv 0 \pmod{5},$$
$$\xi(11n+7) - 3\,\xi(11n+4) + 2\,\xi(11n+3) \equiv 0 \pmod{11}.$$

Define

$S(p,s) = \{j : 0 \le j \le p-1$ such that $\frac{1}{2}n(3n-1) \equiv j - s \pmod{p}$ for s

and

$T(p,s) = \{k : 0 \le k \le p-1$ such that k is larger than every element of

(3)

Theorem (G)

Suppose $p \ge 5$ is prime, and $0 \le s \le p-1$. If $m \in T(p,s)$ (as defined in (3)), then for all $n \ge 0$,

$$\sum_{j=0}^{s} \binom{s}{j} (-1)^j \xi(pn+m-j) \equiv 0 \pmod{p}.$$

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[July 14, 2014]

Reply Reply All Forward

Congruences for Fishburn numbers

Armin Straub [astraub@illinois.edu]

To: Garvan,Francis G

Attachments: fishburncongruences.pdf (222 KB)

Monday, July 14, 2014 12:14 PM

- You replied on 7/16/2014 12:46 PM.

Dear Frank:

In the hope that you might be interested, I have attached for you a preprint
on congruences for Fishburn numbers modulo prime powers, based on the
beautiful results of Andrews, Sellers and yourself. Any comments or
suggestions are, of course, most appreciated.

STRAUB HIGHER POWER CONGRUENCES

Theorem (Straub)

Suppose $p \geq 5$ is prime, and $0 \leq s \leq p-1$. If $m \in T(p,s)$ (as defined in (3)), then for all $n \geq 0$ and $\lambda \geq 1$,

$$\sum_{j=0}^{s} \binom{s}{j} (-1)^j \xi(p^\lambda n + m - j) \equiv 0 \pmod{p^\lambda}.$$

ANDREWS-SELLERS PROOF Define

$$F(q,N)=\sum_{n=0}^{N}(q;q)_n,$$

and the p-dissection

$$F(q,N)=\sum_{i=0}^{p-1}q^iA_p(N,i,q^p).$$

We consider the coefficients of the polynomials

$$A_p(pn-1,i,1-q)=\sum_{k\geq 0}\alpha(p,n,i,k)q^k.$$

The Andrews-Sellers Theorem depends crucially on

Lemma (Andrews and Sellers)

If $i\notin S(p)$, then

$$\alpha(p,n,i,k)=0,$$

for $0\leq k\leq n-1$.

- $$F(\zeta e^t) = \sum_{n=0}^{\infty} \frac{b_n(\zeta)(-t)^n}{n!} = e^{-t/24} \sum_{n=0}^{\infty} \frac{c_n(\zeta)}{n!} \left(\frac{-t}{24}\right)^n,$$

  where

  $$c_n(\zeta) = \frac{(-1)^n N^{2n+1}}{2n+2} \sum_{m=1}^{N/2} \left(\frac{12}{m}\right) \zeta^{\frac{1}{24}(m^2-1)} B_{2n+2}\left(\frac{m}{N}\right) \qquad (N = 12p)$$

- $$\left(\frac{d}{dt}\right)^n f(qe^t)\Big|_{t=0} = \left(q\frac{d}{dq}\right)^n f(q).$$

- $$F(\zeta e^t) = \sum_{n=0}^{\infty} \frac{b_n(\zeta)(-t)^n}{n!} = e^{-t/24} \sum_{n=0}^{\infty} \frac{c_n(\zeta)}{n!} \left(\frac{-t}{24}\right)^n,$$

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- $$\left(\frac{d}{dt}\right)^n f(qe^t)\Big|_{t=0} = \left(q\frac{d}{dq}\right)^n f(q).$$

- $$\left(q\frac{d}{dq}\right)^N F(q,n) = \sum_{j=0}^{N}\sum_{i=0}^{p-1} C_{N,i,j}(p)q^{i+jp}A_p^{(j)}(n,i,q^p),$$

  where

  $$C_{N+1,i,j}(p) = (i+jp)C_{N,i,j}(p)+pC_{N,i,j-1}(p), \qquad C_{0,0,0}(p)=1.$$

- $$\begin{aligned}(-1)^\nu b_\nu(\zeta) &= \left(q\frac{d}{dq}\right)^\nu F(q)\Big|_{q=\zeta} \\ &= \left(q\frac{d}{dq}\right)^\nu F(q,(\nu+1)p-1)\Big|_{q=\zeta} \\ &= \sum_{j=0}^{\nu}\sum_{i=0}^{p-1} C_{\nu,i,j}(p)\zeta^i A_p^{(j)}(p(\nu+1)-1,i,1).\end{aligned}$$

  which implies by induction that

  $$A_p^{(\nu)}(p(\nu+1)-1,i,1) = 0, \quad \text{when } i \notin S(p).$$

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  where

  $$C_{N+1,i,j}(p) = (i+jp)C_{N,i,j}(p) + pC_{N,i,j-1}(p), \qquad C_{0,0,0}(p) = 1.$$

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- Thus

$$F(1-q,pn-1) \equiv \sum_{\substack{i=0\\ i\in S(p)}}^{p-1} (1-q)^i A_p(pn-1,i,1-q^p) + O(q^{pn}) \pmod{p}.$$

Result follows by letting $n \to \infty$.

EXTENSION OF ANDREWS-SELLERS

Define

$$S^*(p) = S(p) \setminus \{i_0\},$$

where $24i_0 \equiv -1 \pmod{p}$, and

$$T^*(p) = \{k\,:\, 0 \le k \le p-1 \text{ such that } k \text{ is larger than every element of } S^*(p)\}\,. \tag{4}$$

Theorem (G)

If p is a prime and $i \in T^*(p)$ (as defined in (4)), then for all $n \ge 0$,

$$\xi(pn+i) \equiv 0 \pmod{p}.$$

For $p \geq 5$ prime define $\bar{\xi}_p(n)$ by

$$\sum_{n=0}^{\infty} \bar{\xi}_p(n) q^n = (1-q)^{\lfloor \frac{p}{24} \rfloor} F((1-q)^p). \tag{5}$$

Theorem (G)

Suppose $p \geq 5$ is prime and $24 i_0 \equiv -1 \pmod{p}$ where $1 \leq i_0 \leq p-1$. Then

$$\alpha(p, n, i_0, k) = p \left(\frac{12}{p}\right) \bar{\xi}_p(k),$$

for $0 \leq k \leq n-1$.

PROOF

$$\sum_{j=0}^{n} C(n,i,j,p) A_p^{(j)}(p(j+1)-1,i,1) = \frac{(-1)^n}{24^n} \sum_{j=0}^{n} \binom{n}{j} \gamma(j,i),$$

where

$$\gamma(j,i) = \frac{(-1)^j N^{2j+1}}{2j+2} \sum_{\substack{m=1 \\ (m^2-1)/24 \equiv i \pmod{p}}}^{N/2} \left(\frac{12}{m}\right) B_{2j+2}\left(\frac{m}{N}\right).$$

Note

$$i_0 = \frac{p^2-1}{24} - \left\lfloor \frac{p}{24} \right\rfloor p.$$

PROBLEM: Solve

$$\sum_{\ell=0}^{n} C(n,i_0,\ell,p) A_1(\ell,m) = (-1)^n z^n \sum_{k=0}^{n} \binom{n}{k} X(k),$$

where $i_0 = (p^2-1)z - mp$.

```
> i0:=(p^2-1)*z - m*p:
> EQN:=n->add(C(n,i0,L,p)*_A1(L,m),L=0..n)
=(-1)^n*z^n*add(binomial(n,k)*_X(k),k=0..n):
> EQNS8:=seq(EQN(n),n=0..8):
> SOL8:=solve(EQNS8,seq(_A1(n,m),n=0..8)):
>
G:=(k,n)->coeff(subs(m=0,subs(SOL8,_A1(n,m))),_X(k),1);
> [seq(expand(G(0,n)),n=0..3)];
```

$$[1, -pz, p^2z^2 + pz, -p^3z^3 - 3\,p^2z^2 - 2\,pz]$$

```
>
seq(seq(subs(p=1,coeff(expand(G(0,n)),z,k))*(-1)^n,k=1..n)
```

$$1, 1, 1, 2, 3, 1, 6, 11, 6, 1, 24, 50, 35, 10, 1, 120, 274, 225,$$

the on-line encyclopedia of integer sequences®

founded in 1964 by N. J. A. Sloane

Invitation: celebrating 50 years of OEIS, 250000 sequences, and Sloane's 75th, there will be a conference at DIMACS, Rutgers,

1, 1, 1, 2, 3, 1, 6, 11, 6, 1, 24, 50, 35, 10, 1, 120, 274, : Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: seq:1,1,1,2,3,1,6,11,6,1,24,50,35,10,1,120,274,225,85,15,1

Displaying 1-2 of 2 results found.

Sort: relevance | references | number | modified | created   Format: long | short | data

A130534 Triangle T(n,k), 0<=k<=n, read by rows, giving coefficients of the polynomial (x+1)(x+2)...(x+n), expanded in increasing powers of x. T(n,k) is also the unsigned Stirling number |s(n+1,k+1)|, denoting the number of permutations on n+1 elements that contain exactly k+1 cycles.

1, 1, 1, 2, 3, 1, 6, 11, 6, 1, 24, 50, 35, 10, 1, 120, 274, 225, 85, 15,
1, 720, 1764, 1624, 735, 175, 21, 1, 5040, 13068, 13132, 6769, 1960, 322, 28, 1
40320, 109584, 118124, 67284, 22449, 4536, 546, 36, 1, 362880, 1026576, 1172700
723680, 269325, 63273 (list; table; graph; refs; listen; history; text; internal format)

OFFSET 0,4

Eventually we find

$$A_1(n,m) = (-1)^n \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} s_1(n,j,m) p^{j-2k} X(k) z^j,$$

where $s_1(n,j,m)$ are defined by

$$\sum_{j=0}^{n} s_1(n,j,m) x^j = (x-m)(x-m+1)\cdots(x-m+n-1).$$

Also

$$\sum_{n=j}^{\infty} s_1(n,j,k) \frac{x^n}{n!} = (1-x)^k \frac{(-\log(1-x))^j}{j!}.$$

With $z = \frac{1}{24}$ and $m = \left\lfloor \frac{p}{24} \right\rfloor$ we find

$$A_p^{(n)}(p(n+1)-1, i_0, 1)$$
$$= (-1)^n \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} s_1(n, j, \lfloor p/24 \rfloor) \frac{p^{j+1}}{24^j}$$
$$\left(\frac{12}{p}\right) \frac{(-1)^k 12^{2k+1}}{2k+2} \left(B_{2k+2}\left(\frac{1}{12}\right) - B_{2k+2}\left(\frac{5}{12}\right)\right)$$
$$= (-1)^n \left(\frac{12}{p}\right) \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} s_1(n, j, \lfloor p/24 \rfloor) \frac{p^{j+1}}{24^j} T_k.$$

Since

$$\alpha(p, n+1, i_0, n) = \frac{(-1)^n}{n!} A_p^{(n)}(p(n+1)-1, i_0, 1)$$

we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \alpha(p, n+1, i_0, n) x^n \\
&= p\left(\frac{12}{p}\right) \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} s_1(n, j, \lfloor p/24 \rfloor) \frac{p^j}{24^j} T_k, \\
&= p\left(\frac{12}{p}\right) \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left(\sum_{n=j}^{\infty} s_1(n, j, \lfloor p/24 \rfloor) \frac{x^n}{n!} \right) \frac{p^j}{24^j} \binom{j}{k} T_k
\end{aligned}$$

Using

$$F(\exp(t)) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}T_k\right)\frac{(-t)^n}{24^n n!}$$

we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\alpha(p, n+1, i_0, n)q^n \\
&= p\left(\frac{12}{p}\right)(1-q)^{\lfloor p/24\rfloor}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}T_k\right)\frac{(-p\log(1-q))^j}{24^j j!} \\
&= p\left(\frac{12}{p}\right)(1-q)^{\lfloor p/24\rfloor}F(\exp(p\log(1-q)) \\
&= p\left(\frac{12}{p}\right)(1-q)^{\lfloor p/24\rfloor}F((1-q)^p).
\end{aligned}$$

GENERALIZATION

Define the r-Fishburn numbers $\xi_r(n)$ by

$$\sum_{n=0}^{\infty} \xi_r(n) q^n = F((1-q)^r).$$

Theorem (G)

Suppose $p \geq 5$ is prime, r is a nonzero integer relatively prime to p and $0 \leq s \leq p-1$. If the Legendre symbol $\left(\frac{-24(1+s)\overline{r}+1}{p}\right) = -1$ or 0, then then for all $n \geq 0$,

$$\sum_{j=0}^{s} \binom{s}{j} (-1)^j \xi_r(pn+m-j) \equiv 0 \pmod{p}.$$

EXAMPLE

$$\xi_{-1}(5n+4) \equiv 0 \pmod{5}$$

Conjectured by Andrews and Sellers

$$\begin{aligned}
&\xi_{-1}(99) = \\
&-19726727709670518469561834415842952406230 \\
&19827345310353861754981718438182070492710 \\
&25010500128674511772931367082845430849607 \\
&6953421653995 \\
&\approx -1.97 \times 10^{135}
\end{aligned}$$

c.f.

$$N \approx 10^{84}$$

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$$\begin{aligned}
&\xi_{-1}(99) = \\
&-197267277096705184695618344158429524062 30\\
&1982734531035386175498171843818207049 2710\\
&250105001286745117729313670828454308 49607\\
&6953421653995\\
&\approx -1.97 \times 10^{135}
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REFERENCES

- G. E. Andrews, J. Jiménez-Urroz, and K. Ono, *q-series identities and values of certain L-functions*, Duke Math. J. **108** (2001), 395–419.
- G. E. Andrews and J. A. Sellers, *Congruences for the Fishburn numbers*, preprint (arXiv:1401.5345).
- K. Bringmann, Y. Li and R. Rhoades, *Asymptotics for the number of row Fishburn matrices*, J. Combin. Theory Ser. A, to appear.
- S. A. Garthwaite and R. C. Rhoades, Private communication, March 4, 2014.
- P. Levande, *Fishburn diagrams, Fishburn numbers and their refined generating functions*, J. Combin. Theory Ser. A **120** (2013), 194–217.
- D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Topology **40** (2001), 945–960.
- D. Zagier, *Quantum modular forms*, in "Quanta of Maths,"

THANK YOU