# Efficient Experimental Mathematics for Lattice Path Combinatorics


Alin Bostan


based on joint works with

M. Bousquet-Mélou, F. Chyzak, M. van Hoeij, M. Kauers,

S. Melczer, L. Pech, K. Raschel, B. Salvy

Challenges in 21st Century Experimental Mathematical Computation

ICERM, July 23, 2014

# Why Lattice Paths?

Applications in many areas of science

- discrete mathematics (permutations, trees, words, urns, . . . )
- statistical physics (Ising model, . . . )
- probability theory (branching processes, games of chance, . . . )
- operations research (queueing theory, . . . )

Journal of Statistical Planning and Inference 140 (2010) 2237–2254

ELSEVIER

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

A history and a survey of lattice path enumeration

Katherine Humphreys

*Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, FL 33431, USA*

ARTICLE INFO

Available online 21 January 2010

*Keywords:*
Lattice path
Reflection principle
Method of images

ABSTRACT

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

# Context: enumeration of lattice walks

▷ *Nearest-neighbor walks in the quarter plane* $\mathbb{N}^2$; admissible steps

$$\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ $\mathfrak{S}$-*walks* = walks in $\mathbb{N}^2$ starting at $(0, 0)$ and using steps in $\mathfrak{S}$.

## Context: enumeration of lattice walks

- *Nearest-neighbor walks in the quarter plane* $\mathbb{N}^2$; admissible steps

$$\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

- $\mathfrak{S}$-*walks* = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in $\mathfrak{S}$.
- $f_{\mathfrak{S}}(n; i, j)$ = number of $\mathfrak{S}$-walks of length $n$ ending at $(i, j)$.

# Context: enumeration of lattice walks

▷ *Nearest-neighbor walks in the quarter plane* $\mathbb{N}^2$; admissible steps

$$\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ $\mathfrak{S}$-*walks* = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in $\mathfrak{S}$.

▷ $f_{\mathfrak{S}}(n; i, j)$ = number of $\mathfrak{S}$-walks of length $n$ ending at $(i, j)$.

▷ Complete generating function

$$F_{\mathfrak{S}}(t; x, y) = \sum_{n=0}^{\infty} \Big( \sum_{i,j=0}^{\infty} f_{\mathfrak{S}}(n; i, j) x^i y^j \Big) t^n \ \in \mathbb{Q}[x, y][[t]].$$

# Context: enumeration of lattice walks

- *Nearest-neighbor walks in the quarter plane* $\mathbb{N}^2$; admissible steps

$$\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

- $\mathfrak{S}$-*walks* = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in $\mathfrak{S}$.
- $f_{\mathfrak{S}}(n; i, j)$ = number of $\mathfrak{S}$-walks of length $n$ ending at $(i, j)$.
- Complete generating function

$$F_{\mathfrak{S}}(t; x, y) = \sum_{n=0}^{\infty} \Big( \sum_{i,j=0}^{\infty} f_{\mathfrak{S}}(n; i, j) x^i y^j \Big) t^n \ \in \mathbb{Q}[x, y][[t]].$$

**Questions:** Given $\mathfrak{S}$, what can be said about $F_{\mathfrak{S}}(t; x, y)$?
*Structure*? (e.g. *Is it algebraic*?) *Explicit form*? *Asymptotics*?

## Context: enumeration of lattice walks

▷ *Nearest-neighbor walks in the quarter plane* $\mathbb{N}^2$; admissible steps

$$\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ $\mathfrak{S}$-*walks* = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in $\mathfrak{S}$.

▷ $f_{\mathfrak{S}}(n; i, j)$ = number of $\mathfrak{S}$-walks of length $n$ ending at $(i, j)$.

▷ Complete generating function

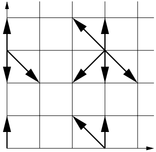
$$F_{\mathfrak{S}}(t; x, y) = \sum_{n=0}^{\infty} \Big( \sum_{i,j=0}^{\infty} f_{\mathfrak{S}}(n; i, j) x^i y^j \Big) t^n \ \in \mathbb{Q}[x, y][[t]].$$

**Questions:** Given $\mathfrak{S}$, what can be said about $F_{\mathfrak{S}}(t; x, y)$?
*Structure*? (e.g. *Is it algebraic*?) *Explicit form*? *Asymptotics*?

$F_{\mathfrak{S}}(t; 0, 0) \rightsquigarrow$ counts $\mathfrak{S}$-walks returning to the origin (excursions);
$F_{\mathfrak{S}}(t; 1, 1) \rightsquigarrow$ counts $\mathfrak{S}$-walks with prescribed length;
$F_{\mathfrak{S}}(t; 1, 0) \rightsquigarrow$ counts $\mathfrak{S}$-walks ending on the horizontal axis.

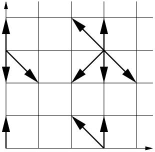
# Small step sets in the quarter plane

$\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$

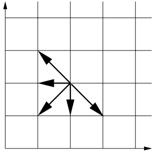


There are $2^8$ such sets.

## Small step sets in the quarter plane

$\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$  There are $2^8$ such sets.

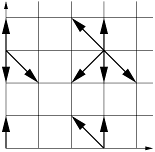
Some of these $2^8 = 256$ step sets are:



trivial,

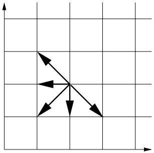
## Small step sets in the quarter plane

$\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0,0)\}$

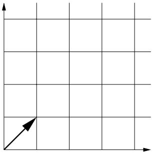


There are $2^8$ such sets.

Some of these $2^8 = 256$ step sets are:

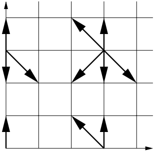


trivial,

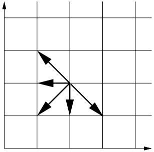


simple,

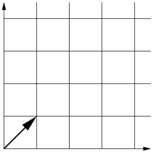
# Small step sets in the quarter plane

$\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$  There are $2^8$ such sets.

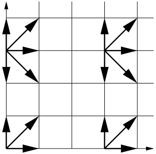
Some of these $2^8 = 256$ step sets are:



trivial,



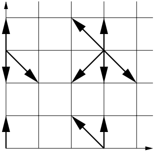
simple,



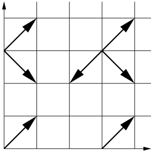
intrinsic to the half plane,

# Small step sets in the quarter plane

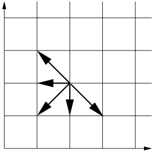
$\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$



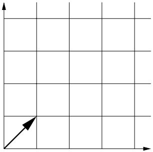
There are $2^8$ such sets.



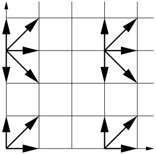
Some of these $2^8 = 256$ step sets are:



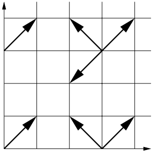
trivial,



simple,



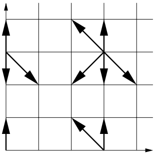
intrinsic to the half plane,



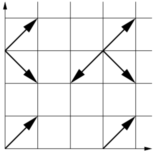
symmetrical.

## Small step sets in the quarter plane

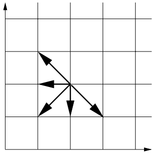
$\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$



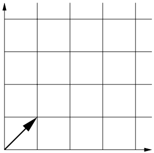
There are $2^8$ such sets.



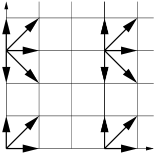
Some of these $2^8 = 256$ step sets are:



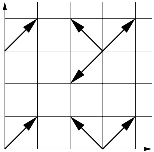
trivial,



simple,



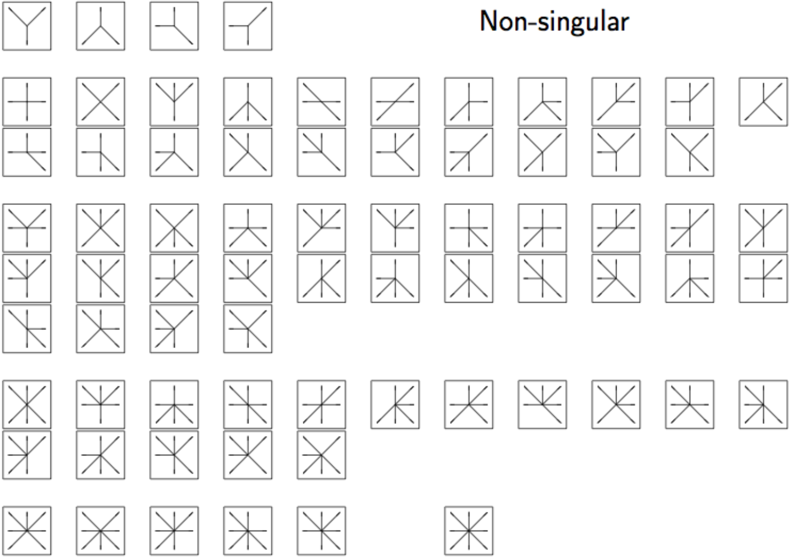
intrinsic to the half plane,



symmetrical.

One is left with 79 interesting distinct models.

Non-singular

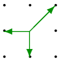


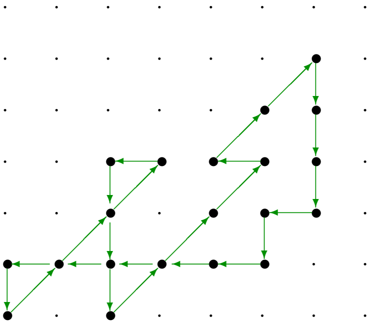
Singular

# Two important cases: Kreweras and Gessel walks

$\mathfrak{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\qquad F_{\mathfrak{S}}(t; x, y) \equiv K(t; x, y)$

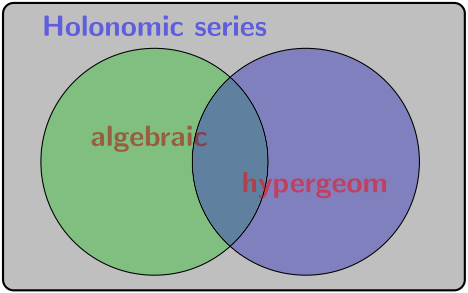
$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\qquad F_{\mathfrak{S}}(t; x, y) \equiv G(t; x, y)$



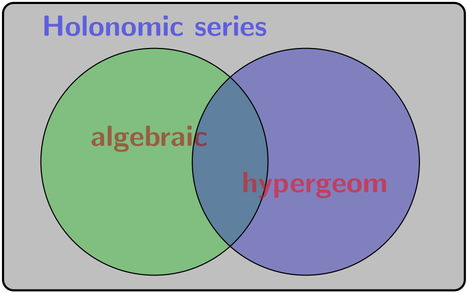


Example: A Kreweras excursion.

# Important classes of univariate power series

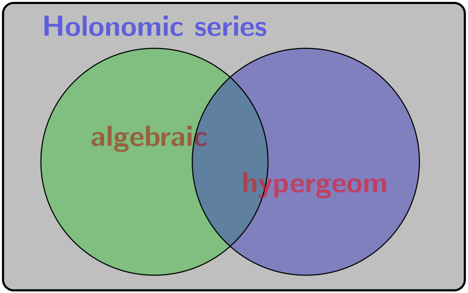


# Important classes of univariate power series




*Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$.

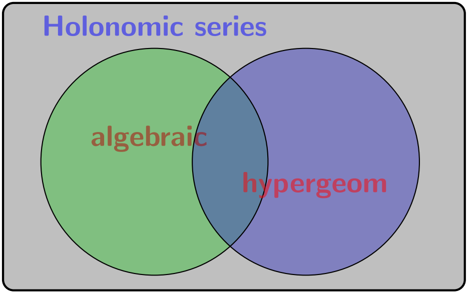
# Important classes of univariate power series




*Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$.

*Holonomic*: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

# Important classes of univariate power series



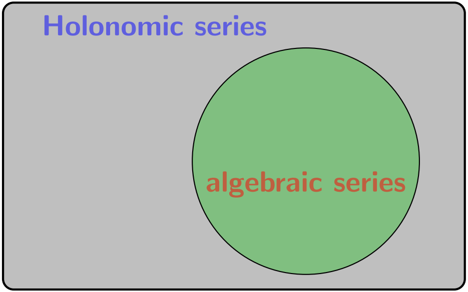

*Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$.

*Holonomic*: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

*Hypergeometric*: $S(t) = \sum_n s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.

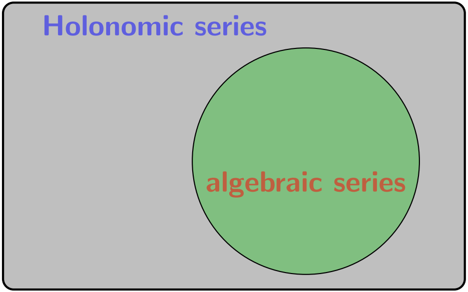
$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1). $$

# Important classes of multivariate power series




$S \in \mathbb{Q}[[x, y, t]]$ is *algebraic* if it is the root of a $P \in \mathbb{Q}[x, y, t, T]$.

# Important classes of multivariate power series




$S \in \mathbb{Q}[[x, y, t]]$ is *algebraic* if it is the root of a $P \in \mathbb{Q}[x, y, t, T]$.

$S \in \mathbb{Q}[[x, y, t]]$ is *holonomic* if the set of all partial derivatives of $S$ spans a finite-dimensional vector space over $\mathbb{Q}(x, y, t)$.

# Main results (I): algebraicity of Gessel walks

*Theorem* [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)}\,t^{3n}.$$

# Main results (I): algebraicity of Gessel walks

*Theorem* [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

*Theorem* [Gessel's conjecture; Kauers, Koutschan, Zeilberger 2009]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

## Main results (I): algebraicity of Gessel walks

*Theorem* [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)}\,t^{3n}.$$

*Theorem* [Gessel's conjecture; Kauers, Koutschan, Zeilberger 2009]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

**Question:** What about $K(t;x,y)$ and $G(t;x,y)$?

## Main results (I): algebraicity of Gessel walks

*Theorem* [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)}\,t^{3n}.$$

*Theorem* [Gessel's conjecture; Kauers, Koutschan, Zeilberger 2009]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

**Question:** What about $K(t;x,y)$ and $G(t;x,y)$?

*Theorem* [Gessel'86, Bousquet-Mélou'05] $K(t;x,y)$ is algebraic.

# Main results (I): algebraicity of Gessel walks

*Theorem* [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

*Theorem* [Gessel's conjecture; Kauers, Koutschan, Zeilberger 2009]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

**Question:** What about $K(t;x,y)$ and $G(t;x,y)$?

*Theorem* [Gessel'86, Bousquet-Mélou'05] $K(t;x,y)$ is algebraic.

▷ $G(t;x,y)$ had been conjectured to be non-holonomic.

## Main results (I): algebraicity of Gessel walks

*Theorem* [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

*Theorem* [Gessel's conjecture; Kauers, Koutschan, Zeilberger 2009]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

**Question:** What about $K(t;x,y)$ and $G(t;x,y)$?

*Theorem* [Gessel'86, Bousquet-Mélou'05] $K(t;x,y)$ is algebraic.

▷ $G(t;x,y)$ had been conjectured to be non-holonomic.

*Theorem* [B. & Kauers'10] $G(t;x,y)$ is holonomic, even algebraic.

▷ Computer-driven discovery and proof; no human proof yet.

## Main results (I): algebraicity of Gessel walks

*Theorem* [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix} \,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)}\,t^{3n}.$$

*Theorem* [Gessel's conjecture; Kauers, Koutschan, Zeilberger 2009]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix} \,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

**Question:** What about $K(t;x,y)$ and $G(t;x,y)$?

*Theorem* [Gessel'86, Bousquet-Mélou'05] $K(t;x,y)$ is algebraic.

▷ $G(t;x,y)$ had been conjectured to be non-holonomic.

*Theorem* [B. & Kauers'10] $G(t;x,y)$ is holonomic, even algebraic.

▷ Computer-driven discovery and proof; ~~no human proof yet~~.

## Main results (I): algebraicity of Gessel walks

*Theorem* [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

*Theorem* [Gessel's conjecture; Kauers, Koutschan, Zeilberger 2009]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

**Question:** What about $K(t;x,y)$ and $G(t;x,y)$?

*Theorem* [Gessel'86, Bousquet-Mélou'05] $K(t;x,y)$ is algebraic.

▷ $G(t;x,y)$ had been conjectured to be non-holonomic.

*Theorem* [B. & Kauers'10] $G(t;x,y)$ is holonomic, even algebraic.

▷ Fresh news: recent human proof [B., Kurkova, Raschel'14].

# Main results (I): algebraicity of Gessel walks

*Theorem* [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

*Theorem* [Gessel's conjecture; Kauers, Koutschan, Zeilberger 2009]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

**Question:** What about $K(t;x,y)$ and $G(t;x,y)$?

*Theorem* [Gessel'86, Bousquet-Mélou'05] $K(t;x,y)$ is algebraic.

▷ $G(t;x,y)$ had been conjectured to be non-holonomic.

*Theorem* [B. & Kauers'10] $G(t;x,y)$ is holonomic, even algebraic.

▷ Guess'n'Prove method, using Hermite-Padé approximants.

## Main results (II): Explicit form for $G(t; x, y)$

*Theorem* Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2$$
$$-xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0,$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

▷ Computer-driven discovery and proof; no human proof yet.

## Main results (II): Explicit form for $G(t; x, y)$

*Theorem* Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2$$
$$-xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0,$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

▷ Proof uses guessed minimal polynomials for $G(t; x, 0)$ & $G(t; 0, y)$

# Main results (III): Experimental classification of walks with holonomic $F_{\mathfrak{S}}(t;1,1)$ [B. & Kauers 2009]

| OEIS Tag | Sample step set | Equation sizes | | | OEIS Tag | Sample step set | Equation sizes | | |
|---|---|---|---|---|---|---|---|---|---|
| A000012 |  | 1, 0 | 1, 1 | 1, 1 | A000079 |  | 1, 0 | 1, 1 | 1, 1 |
| A001405 |  | 2, 1 | 2, 3 | 2, 2 | A000244 |  | 1, 0 | 1, 1 | 1, 1 |
| A001006 |  | 2, 1 | 2, 3 | 2, 2 | A005773 |  | 2, 1 | 2, 3 | 2, 2 |
| A126087 |  | 3, 1 | 2, 5 | 2, 2 | A151255 |  | 6, 8 | 4, 16 | – |
| A151265 |  | 6, 4 | 4, 9 | 6, 8 | A151266 |  | 7, 10 | 5, 16 | – |
| A151278 |  | 7, 4 | 4, 12 | 6, 8 | A151281 |  | 3, 1 | 2, 5 | 2, 2 |
| A005558 |  | 2, 3 | 3, 5 | – | A005566 |  | 2, 2 | 3, 4 | – |
| A018224 |  | 2, 3 | 3, 5 | – | A060899 |  | 2, 1 | 2, 3 | 2, 2 |
| A060900 |  | 2, 3 | 3, 5 | 8, 9 | A128386 |  | 3, 1 | 2, 5 | 2, 2 |
| A129637 |  | 3, 1 | 2, 5 | 2, 2 | A151261 |  | 5, 8 | 4, 15 | – |
| A151282 |  | 3, 1 | 2, 5 | 2, 2 | A151291 |  | 6, 10 | 5, 15 | – |
| A151275 |  | 9, 18 | 5, 24 | – | A151287 |  | 7, 11 | 5, 19 | – |
| A151292 |  | 3, 1 | 2, 5 | 2, 2 | A151302 |  | 9, 18 | 5, 24 | – |
| A151307 |  | 8, 15 | 5, 20 | – | A151318 |  | 2, 1 | 2, 3 | 2, 2 |
| A129400 |  | 2, 1 | 2, 3 | 2, 2 | A151297 |  | 7, 11 | 5, 18 | – |
| A151312 |  | 4, 5 | 3, 8 | – | A151323 |  | 2, 1 | 2, 3 | 4, 4 |
| A151326 |  | 7, 14 | 5, 18 | – | A151314 |  | 9, 18 | 5, 24 | – |
| A151329 |  | 9, 18 | 5, 24 | – | A151331 |  | 3, 4 | 3, 6 | – |

Equation sizes = {order, degree}(rec, diffeq, algeq).

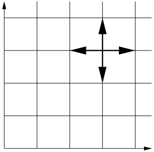
▷ Computer-driven; confirmed by human proofs in [Bousquet-Mélou & Mishna, 2010].

# Experimental classification of walks with holonomic $F_{\mathfrak{S}}(t;1,1)$

| OEIS Tag | Steps | Equation sizes | | | Asymptotics | OEIS Tag | Steps | Equation sizes | | | Asymptotics |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A000012 |  | 1, 0 | 1, 1 | 1, 1 | $1$ | A000079 |  | 1, 0 | 1, 1 | 1, 1 | $2^n$ |
| A001405 |  | 2, 1 | 2, 3 | 2, 2 | $\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n}{\sqrt{n}}$ | A000244 |  | 1, 0 | 1, 1 | 1, 1 | $3^n$ |
| A001006 |  | 2, 1 | 2, 3 | 2, 2 | $\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{n^{3/2}}$ | A005773 |  | 2, 1 | 2, 3 | 2, 2 | $\frac{\sqrt{3}}{\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$ |
| A126087 |  | 3, 1 | 2, 5 | 2, 2 | $\frac{12\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^{3n/2}}{n^{3/2}}$ | A151255 |  | 6, 8 | 4, 16 | – | $\frac{24\sqrt{2}}{\pi}\frac{2^{3n/2}}{n^2}$ |
| A151265 |  | 6, 4 | 4, 9 | 6, 8 | $\frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$ | A151266 |  | 7, 10 | 5, 16 | – | $\frac{\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$ |
| A151278 |  | 7, 4 | 4, 12 | 6, 8 | $\frac{3\sqrt{3}}{\sqrt{2}\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$ | A151281 |  | 3, 1 | 2, 5 | 2, 2 | $\frac{1}{2}3^n$ |
| A005558 |  | 2, 3 | 3, 5 | – | $\frac{8}{\pi}\frac{4^n}{n^2}$ | A005566 |  | 2, 2 | 3, 4 | – | $\frac{4}{\pi}\frac{4^n}{n}$ |
| A018224 |  | 2, 3 | 3, 5 | – | $\frac{2}{\pi}\frac{4^n}{n}$ | A060899 |  | 2, 1 | 2, 3 | 2, 2 | $\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$ |
| A060900 |  | 2, 3 | 3, 5 | 8, 9 | $\frac{4\sqrt{3}}{3\Gamma(\frac{1}{3})}\frac{4^n}{n^{2/3}}$ | A128386 |  | 3, 1 | 2, 5 | 2, 2 | $\frac{6\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n 3^{n/2}}{n^{3/2}}$ |
| A129637 |  | 3, 1 | 2, 5 | 2, 2 | $\frac{1}{2}4^n$ | A151261 |  | 5, 8 | 4, 15 | – | $\frac{12\sqrt{3}}{\pi}\frac{2^n 3^{n/2}}{n^2}$ |
| A151282 |  | 3, 1 | 2, 5 | 2, 2 | $\frac{A^2B^{3/2}}{2^{3/4}\Gamma(\frac{1}{2})}\frac{B^n}{n^{3/2}}$ | A151291 |  | 6, 10 | 5, 15 | – | $\frac{4}{3\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$ |
| A151275 |  | 9, 18 | 5, 24 | – | $\frac{12\sqrt{30}}{\pi}\frac{(\sqrt{24})^n}{n^2}$ | A151287 |  | 7, 11 | 5, 19 | – | $\frac{\sqrt{8}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$ |
| A151292 |  | 3, 1 | 2, 5 | 2, 2 | $\frac{\sqrt[4]{3}C^2D^{3/2}}{8\Gamma(\frac{1}{2})}\frac{D^n}{n^{3/2}}$ | A151302 |  | 9, 18 | 5, 24 | – | $\frac{\sqrt{5}}{3\sqrt{2}\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$ |
| A151307 |  | 8, 15 | 5, 20 | – | $\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$ | A151318 |  | 2, 1 | 2, 3 | 2, 2 | $\frac{\sqrt{5/2}}{\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$ |
| A129400 |  | 2, 1 | 2, 3 | 2, 2 | $\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{6^n}{n^{3/2}}$ | A151297 |  | 7, 11 | 5, 18 | – | $\frac{\sqrt{3}C^{7/2}}{2\pi}\frac{(2C)^n}{n^2}$ |
| A151312 |  | 4, 5 | 3, 8 | – | $\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$ | A151323 |  | 2, 1 | 2, 3 | 4, 4 | $\frac{\sqrt{2}\,3^{3/4}}{\Gamma(\frac{1}{4})}\frac{6^n}{n^{3/4}}$ |
| A151326 |  | 7, 14 | 5, 18 | – | $\frac{2\sqrt{3}}{3\Gamma(\frac{1}{2})}\frac{6^n}{\sqrt{n}}$ | A151314 |  | 9, 18 | 5, 24 | – | $\frac{EF^{7/2}}{5\sqrt{95}\pi}\frac{(2F)^n}{n^2}$ |
| A151329 |  | 9, 18 | 5, 24 | – | $\frac{\sqrt{7/3}}{3\Gamma(\frac{1}{2})}\frac{7^n}{\sqrt{n}}$ | A151331 |  | 3, 4 | 3, 6 | – | $\frac{8}{3\pi}\frac{8^n}{n}$ |

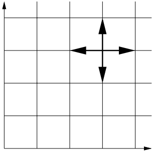
▷ Computer-driven; some human proofs of asymptotics by [Fayolle & Raschel, 2012].

# The group of a walk: an example



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$

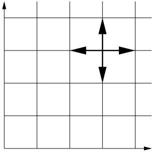
# The group of a walk: an example



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \qquad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

# The group of a walk: an example



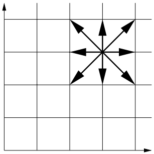
The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

and thus under any element of the group

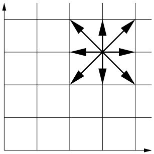
$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

# The group of a walk: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

## The group of a walk: the general case

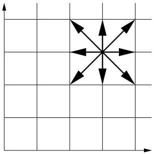


The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

is left invariant under

$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$

## The group of a walk: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

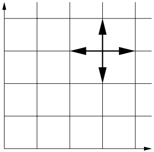
is left invariant under

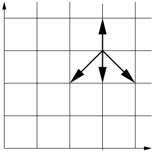
$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$

and thus under any element of the group

$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$

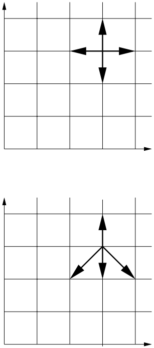
# Examples of groups of walks

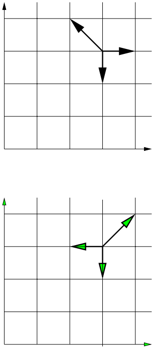




Order 4,

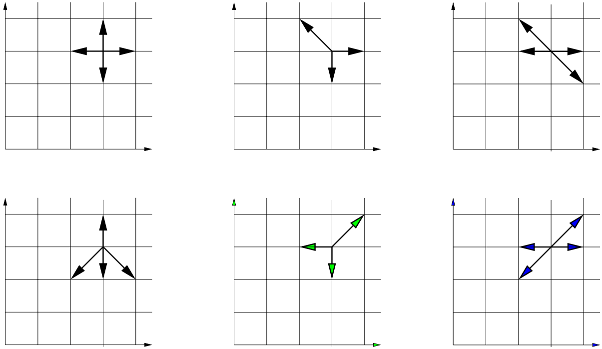
# Examples of groups of walks





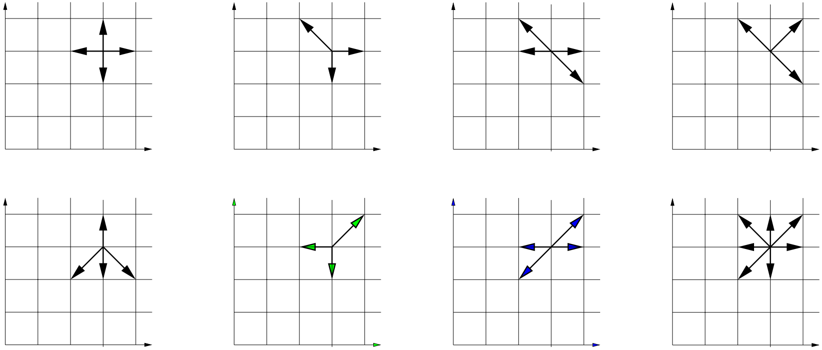
Order 4, order 6,

# Examples of groups of walks



Order 4, order 6, order 8,

# Examples of groups of walks



Order 4, order 6, order 8, order $\infty$.

# The 79 cases: finite and infinite groups

79 step sets

## The 79 cases: finite and infinite groups

79 step sets

→ 23 admit a finite group

[Mishna'07]

→ 56 have an infinite group

[Bousquet-Mélou-Mishna'10]

# The 79 cases: finite and infinite groups

79 step sets

→ 23 admit a finite group
[Mishna'07]

→ all are holonomic

→ 19 transcendental
[Gessel-Zeilberger'92]
[Bousquet-Mélou'02]

→ 4 are algebraic
(3 Kreweras-type + Gessel)
[BMM'10] + [B.-Kauers'10]

→ 56 have an infinite group
[Bousquet-Mélou-Mishna'10]

## The 79 cases: finite and infinite groups

79 step sets

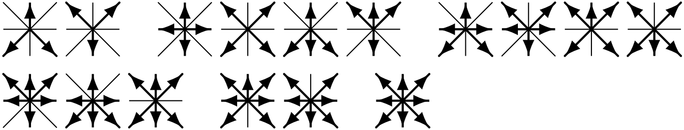
→ 23 admit a finite group [Mishna'07]

→ all are holonomic
- 19 transcendental [Gessel-Zeilberger'92] [Bousquet-Mélou'02]
- 4 are algebraic (3 Kreweras-type + Gessel) [BMM'10] + [B.-Kauers'10]

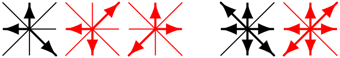
→ 56 have an infinite group [Bousquet-Mélou-Mishna'10] ⟶ all non-holonomic
- [Mishna-Rechnitzer'07] and [Melczer-Mishna'13] for 5 singular cases
- [Kurkova-Raschel'13] and [B.-Raschel-Salvy'13] for all others

# The 23 cases with a finite group

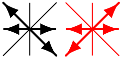
(i) 16 with a *vertical symmetry*, and group isomorphic to $D_2$



(ii) 5 with a *diagonal* or an *anti-diagonal symmetry*, and group isomorphic to $D_3$



(iii) 2 with group isomorphic to $D_4$



(i): vertical symmetry; (ii)+(iii): zero drift $\sum_{s\in\mathfrak{S}} s$

In red, cases with algebraic generating functions

# Main results (IV): explicit expressions for the 19 holonomic transcendental cases

*Theorem* [B.-Chyzak-van Hoeij-Kauers-Pech, 2014]

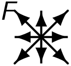
Let $\mathfrak{S}$ be one of the 19 step sets with finite group $\mathcal{G}_{\mathfrak{S}}$, and such that the generating series $F = F_{\mathfrak{S}}(t; x, y)$ is not algebraic.
Then $F$ is expressible using iterated integrals of ${}_2F_1$ expressions.

# Main results (IV): explicit expressions for the 19 holonomic transcendental cases

*Theorem* [B.-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the 19 step sets with finite group $\mathcal{G}_{\mathfrak{S}}$, and such that the generating series $F = F_{\mathfrak{S}}(t; x, y)$ is not algebraic.
Then $F$ is expressible using iterated integrals of ${}_2F_1$ expressions.

*Example* (King walks in the quarter plane, A025595)



$$F(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

# Main results (IV): explicit expressions for the 19 holonomic transcendental cases

*Theorem* [B.-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the 19 step sets with finite group $\mathcal{G}_{\mathfrak{S}}$, and such that the generating series $F = F_{\mathfrak{S}}(t; x, y)$ is not algebraic.
Then $F$ is expressible using iterated integrals of ${}_2F_1$ expressions.

*Example* (King walks in the quarter plane, A025595)

$$F(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses creative telescoping, ODE factorization, ODE solving.

# Main results (V): algorithmic proof of non-holonomy for the 51 non-singular cases with infinite group

*Theorem* [B.-Rachel-Salvy, 2013]

Let $\mathfrak{S}$ be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{S}}$. Then $F_{\mathfrak{S}}(t;0,0)$, and in particular $F_{\mathfrak{S}}(t;x,y)$, are non-holonomic.

# Main results (V): algorithmic proof of non-holonomy for the 51 non-singular cases with infinite group

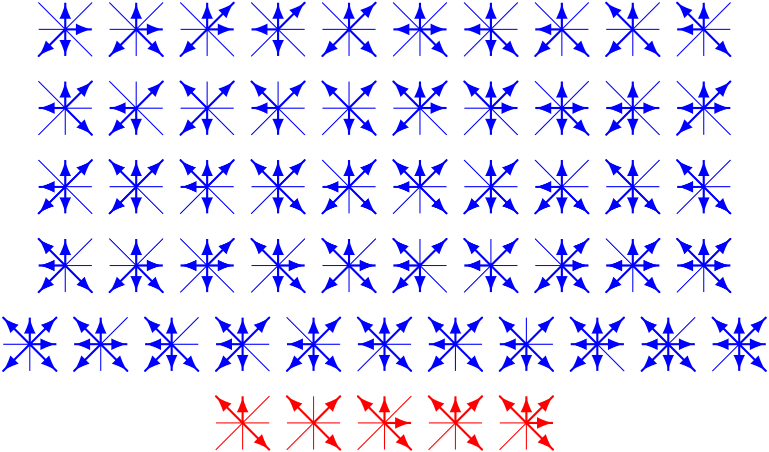
*Theorem* [B.-Rachel-Salvy, 2013]

Let $\mathfrak{S}$ be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{S}}$. Then $F_{\mathfrak{S}}(t;0,0)$, and in particular $F_{\mathfrak{S}}(t;x,y)$, are non-holonomic.

▷ *Algorithmic, computer-driven, proof*. Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.
▷ Based on *two ingredients*: asymptotics + irrationality.

▷ [Kurkova-Raschel 2013] Human proof that $F_{\mathfrak{S}}(t;x,y)$ is non-holonomic. No human proof yet for $F_{\mathfrak{S}}(t;0,0)$ *non-holonomic*.

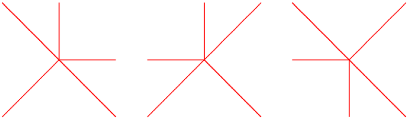
# The 56 cases with infinite group



In blue, non-singular cases, solved by [B., Raschel & Salvy, 2013]
In red, singular cases, solved by [Melczer & Mishna 2013]

# Example: the scarecrows

[B., Raschel & Salvy, 2013]: $F_{\mathfrak{S}}(t;0,0)$ is not holonomic for



For the 1st and the 3rd, the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

is $\sim K \cdot 5^n \cdot n^{\alpha}$, with $\alpha = 1 + \pi/\arccos(1/4) = 3.383396\ldots$

The *irrationality* of $\alpha$ prevents $F_{\mathfrak{S}}(t;0,0)$ from being holonomic.

## Summary – classification of 2D non-singular walks

*The Main Theorem* Let $\mathfrak{S}$ be one of the 74 non-singular step sets. The following assertions are equivalent:

(1) The full generating series $F_{\mathfrak{S}}(t; x, y)$ is holonomic

(2) the excursions generating series $F_{\mathfrak{S}}(t; 0, 0)$ is holonomic

(3) the excursions seq. $[t^n]\, F_{\mathfrak{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^{\alpha}$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

## Summary – classification of 2D non-singular walks

*The Main Theorem* Let $\mathfrak{S}$ be one of the 74 non-singular step sets. The following assertions are equivalent:

(1) The full generating series $F_{\mathfrak{S}}(t; x, y)$ is holonomic

(2) the excursions generating series $F_{\mathfrak{S}}(t; 0, 0)$ is holonomic

(3) the excursions seq. $[t^n]\, F_{\mathfrak{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^{\alpha}$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^{\star} \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

Moreover, under (1)–(5), $F_{\mathfrak{S}}(t; x, y)$ is algebraic if and only if the step set $\mathfrak{S}$ has positive covariance $\sum_{(i,j)\in\mathfrak{S}} ij - \sum_{(i,j)\in\mathfrak{S}} i \cdot \sum_{(i,j)\in\mathfrak{S}} j > 0$.

## Summary – classification of 2D non-singular walks

*The Main Theorem* Let $\mathfrak{S}$ be one of the 74 non-singular step sets. The following assertions are equivalent:

(1) The full generating series $F_{\mathfrak{S}}(t; x, y)$ is holonomic

(2) the excursions generating series $F_{\mathfrak{S}}(t; 0, 0)$ is holonomic

(3) the excursions seq. $[t^n]\, F_{\mathfrak{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

Moreover, under (1)–(5), $F_{\mathfrak{S}}(t; x, y)$ is algebraic if and only if the step set $\mathfrak{S}$ has positive covariance $\sum_{(i,j)\in\mathfrak{S}} ij - \sum_{(i,j)\in\mathfrak{S}} i \cdot \sum_{(i,j)\in\mathfrak{S}} j > 0$.

In this case, $F_{\mathfrak{S}}(t; x, y)$ is expressible using nested radicals. If not, $F_{\mathfrak{S}}(t; x, y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

# Main methods

(1) for proving holonomy
  (1a) Guess'n'Prove
  (1b) Diagonals of rational functions

(2) for proving non-holonomy
  (2a) Infinite number of singularities, or lacunary
  (2b) Asymptotics

# Main methods

(1) for proving holonomy

- (1a) Guess'n'Prove — *Hermite-Padé approximants*
- (1b) Diagonals of rational functions — *Creative telescoping*

(2) for proving non-holonomy

- (2a) Infinite number of singularities, or lacunary
- (2b) Asymptotics

▷ All methods are *algorithmic*.

# Guess'n'Prove

## Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) *Generate data*

(S2) *Conjecture*

(S3) *Prove*

# Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) *Generate data*
compute a high order expansion of the series $F_{\mathfrak{S}}(t; x, y)$;

(S2) *Conjecture*
guess candidates for minimal polynomials of $F_{\mathfrak{S}}(t; x, 0)$ and $F_{\mathfrak{S}}(t; 0, y)$, using Hermite-Padé approximation;

(S3) *Prove*
rigorously certify the minimal polynomials, using (exact) polynomial computations.

# Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) *Generate data*
compute a high order expansion of the series $F_{\mathfrak{S}}(t; x, y)$;

(S2) *Conjecture*
guess candidates for minimal polynomials of $F_{\mathfrak{S}}(t; x, 0)$ and $F_{\mathfrak{S}}(t; 0, y)$, using Hermite-Padé approximation;

(S3) *Prove*
rigorously certify the minimal polynomials, using (exact) polynomial computations.

\+ Efficient Computer Algebra

## Step (S1): high order series expansions

$f_{\mathfrak{S}}(n; i, j)$ satisfies the recurrence with constant coefficients

$$f_{\mathfrak{S}}(n+1; i, j) = \sum_{(u,v)\in\mathfrak{S}} f_{\mathfrak{S}}(n; i-u, j-v) \quad \text{for} \quad n, i, j \geq 0$$

$+$ init. cond. $f_{\mathfrak{S}}(0; i, j) = \delta_{0,i,j}$ and $f_{\mathfrak{S}}(n; -1, j) = f_{\mathfrak{S}}(n; i, -1) = 0$.

# Step (S1): high order series expansions

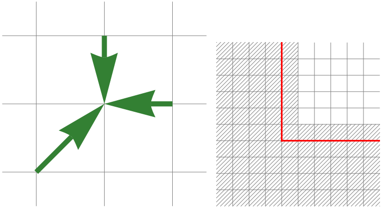
$f_{\mathfrak{S}}(n; i, j)$ satisfies the recurrence with constant coefficients

$$f_{\mathfrak{S}}(n+1; i, j) = \sum_{(u,v)\in\mathfrak{S}} f_{\mathfrak{S}}(n; i-u, j-v) \quad \text{for} \quad n, i, j \geq 0$$

$+$ init. cond. $f_{\mathfrak{S}}(0; i, j) = \delta_{0,i,j}$ and $f_{\mathfrak{S}}(n; -1, j) = f_{\mathfrak{S}}(n; i, -1) = 0$.

Example: for the Kreweras walks,

$$\begin{aligned} k(n+1; i, j) &= k(n; i+1, j) \\ &+ k(n; i, j+1) \\ &+ k(n; i-1, j-1) \end{aligned}$$



# Step (S1): high order series expansions

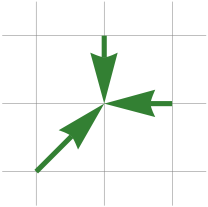
$f_{\mathfrak{S}}(n;i,j)$ satisfies the recurrence with constant coefficients

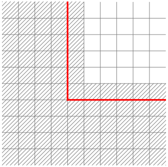
$$f_{\mathfrak{S}}(n+1;i,j) = \sum_{(u,v)\in\mathfrak{S}} f_{\mathfrak{S}}(n;i-u,j-v) \quad \text{for} \quad n,i,j \geq 0$$

+ init. cond. $f_{\mathfrak{S}}(0;i,j) = \delta_{0,i,j}$ and $f_{\mathfrak{S}}(n;-1,j) = f_{\mathfrak{S}}(n;i,-1) = 0$.

Example: for the Kreweras walks,

$$\begin{aligned} k(n+1;i,j) = k(n;i+1,j) \\ + k(n;i,j+1) \\ + k(n;i-1,j-1) \end{aligned}$$





▷ Recurrence is used to compute $F_{\mathfrak{S}}(t;x,y) \bmod t^N$ for large $N$.

$$\begin{aligned} K(t;x,y) &= 1 + xyt + (x^2y^2 + y + x)t^2 + (x^3y^3 + 2xy^2 + 2x^2y + 2)t^3 \\ &+ (x^4y^4 + 3x^2y^3 + 3x^3y^2 + 2y^2 + 6xy + 2x^2)t^4 \\ &+ (x^5y^5 + 4x^3y^4 + 4x^4y^3 + 5xy^3 + 12x^2y^2 + 5x^3y + 8y + 8x)t^5 + \cdots \end{aligned}$$

## Step (S2): guessing equations for $F_{\mathfrak{S}}(t;x,y)$, a first idea

In terms of generating series, the recurrence on $k(n;i,j)$ reads

$$\boxed{\begin{aligned}&\big(xy-(x+y+x^2y^2)t\big)K(t;x,y)\\&\qquad=xy-xt\,K(t;x,0)-yt\,K(t;0,y)\end{aligned}}\qquad\text{(KerEq)}$$

▷ A similar kernel equation holds for $F_{\mathfrak{S}}(t;x,y)$, for any $\mathfrak{S}$-walk.

*Corollary.* $F_{\mathfrak{S}}(t;x,y)$ is algebraic (resp. holonomic) if and only if $F_{\mathfrak{S}}(t;x,0)$ and $F_{\mathfrak{S}}(t;0,y)$ are both algebraic (resp. holonomic).

▷ Crucial simplification: equations for $G(t;x,y)$ are huge ($\approx$30Gb)

## Step (S2): guessing equations for $F_{\mathfrak{S}}(t; x, 0)$ & $F_{\mathfrak{S}}(t; 0, y)$

*Task 1:* Given the first $N$ terms of $S = F_{\mathfrak{S}}(t; x, 0) \in \mathbb{Q}[x][[t]]$, search for a *differential equation* satisfied by $S$ at precision $N$:

$$c_r(x, t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x, t) \cdot \frac{\partial S}{\partial t} + c_0(x, t) \cdot S = 0 \mod t^N.$$

## Step (S2): guessing equations for $F_{\mathfrak{S}}(t;x,0)$ & $F_{\mathfrak{S}}(t;0,y)$

*Task 1:* Given the first $N$ terms of $S = F_{\mathfrak{S}}(t;x,0) \in \mathbb{Q}[x][[t]]$, search for a *differential equation* satisfied by $S$ at precision $N$:

$$c_r(x,t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x,t) \cdot \frac{\partial S}{\partial t} + c_0(x,t) \cdot S = 0 \mod t^N.$$

*Task 2:* Search for an *algebraic equation* $\mathcal{P}_{x,0}(S) = 0 \mod t^N$.

## Step (S2): guessing equations for $F_{\mathfrak{S}}(t; x, 0)$ & $F_{\mathfrak{S}}(t; 0, y)$

*Task 1:* Given the first $N$ terms of $S = F_{\mathfrak{S}}(t; x, 0) \in \mathbb{Q}[x][[t]]$, search for a *differential equation* satisfied by $S$ at precision $N$:

$$c_r(x,t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x,t) \cdot \frac{\partial S}{\partial t} + c_0(x,t) \cdot S = 0 \mod t^N.$$

*Task 2:* Search for an *algebraic equation* $\mathcal{P}_{x,0}(S) = 0 \mod t^N$.

- Both tasks amount to linear algebra in size $N$ over $\mathbb{Q}(x)$.
- In practice, we use modular Hermite-Padé approximation (Beckermann-Labahn algorithm) combined with (rational) evaluation-interpolation and rational number reconstruction.
- Fast (FFT-based) arithmetic in $\mathbb{F}_p[t]$ and $\mathbb{F}_p[t]\langle \frac{t}{\partial t} \rangle$.

## Step (S2): guessing equations for $G(t; x, 0)$ and $G(t; 0, y)$

Using $N = 1200$ terms of $G(t; x, y)$, we guessed candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[x, t, T]$ of degree $(32, 43, 24)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[y, t, T]$ of degree $(40, 44, 24)$, coefficients of 23 digits

such that

$$\mathcal{P}_{x,0}(x, t, G(t; x, 0)) = \mathcal{P}_{0,y}(y, t, G(t; 0, y)) = 0 \mod t^{1200}.$$

[†] of order 11, and bidegree $(96, 78)$ for $G(t; x, 0)$, and $(68, 28)$ for $G(t; 0, y)$

## Step (S2): guessing equations for $G(t;x,0)$ and $G(t;0,y)$

Using $N = 1200$ terms of $G(t;x,y)$, we guessed candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[x, t, T]$ of degree $(32, 43, 24)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[y, t, T]$ of degree $(40, 44, 24)$, coefficients of 23 digits

such that

$$\mathcal{P}_{x,0}(x, t, G(t;x,0)) = \mathcal{P}_{0,y}(y, t, G(t;0,y)) = 0 \mod t^{1200}.$$

▷ We actually first guessed *differential equations*[†], then computed their *p-curvatures* to empirically certify them. This led to suspect the algebraicity of $G(t;x,0)$ and $G(t;0,y)$, using a conjecture of Grothendieck (on differential equations modulo $p$) as an oracle.

[†] of order 11, and bidegree $(96, 78)$ for $G(t;x,0)$, and $(68, 28)$ for $G(t;0,y)$

## Step (S2): guessing equations for $G(t;x,0)$ and $G(t;0,y)$

Using $N = 1200$ terms of $G(t;x,y)$, we guessed candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[x,t,T]$ of degree $(32,43,24)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[y,t,T]$ of degree $(40,44,24)$, coefficients of 23 digits

such that

$$\mathcal{P}_{x,0}(x,t,G(t;x,0)) = \mathcal{P}_{0,y}(y,t,G(t;0,y)) = 0 \mod t^{1200}.$$

▷ We actually first guessed *differential equations*†, then computed their *p-curvatures* to empirically certify them. This led to suspect the algebraicity of $G(t;x,0)$ and $G(t;0,y)$, using a conjecture of Grothendieck (on differential equations modulo $p$) as an oracle.

▷ Guessing $\mathcal{P}_{x,0}$ by *undetermined coefficients* would require solving a dense linear system of size $\approx 100\,000$, and $\approx 1000$ digits entries!

† of order 11, and bidegree $(96,78)$ for $G(t;x,0)$, and $(68,28)$ for $G(t;0,y)$

## Step (S3): warm-up – Gessel excursions are algebraic

*Thm.* $g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n$ *is algebraic*.

## Step (S3): warm-up – Gessel excursions are algebraic

*Thm.* $g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n$ *is algebraic*.

Proof: *Guess* a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then *prove* that $P$ admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

## Step (S3): warm-up – Gessel excursions are algebraic

*Thm.* $g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n$ *is algebraic*.

Proof: *Guess* a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then *prove* that $P$ admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Such a $P$ can be *guessed* from the first 100 terms of $g(t)$.

## Step (S3): warm-up – Gessel excursions are algebraic

*Thm.* $g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n$ *is algebraic*.

Proof: *Guess* a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then *prove* that $P$ admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Such a $P$ can be *guessed* from the first 100 terms of $g(t)$.
2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of $P$.

## Step (S3): warm-up – Gessel excursions are algebraic

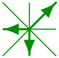
*Thm.* $g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n$ *is algebraic*.

Proof: *Guess* a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then *prove* that $P$ admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Such a $P$ can be *guessed* from the first 100 terms of $g(t)$.
2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of $P$.
3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is holonomic, and so is $(r_n)$:

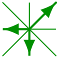
$$(n+2)(3n+5) r_{n+1} - 4(6n+5)(2n+1) r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

## Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$
in the kernel equation

$$\underbrace{(xy - (x + y + x^2y^2)t)}_{\overset{!}{=}\,0} K(t; x, y) = xy - xt K(t; x, 0) - yt K(t; 0, y)$$

# Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$
in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\stackrel{!}{=}0} K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

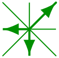
## Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$
in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\overset{!}{=}0} K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the *reduced kernel equation*

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

# Step (S3): rigorous proof for Kreweras walks 

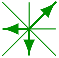
1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\overset{!}{=}0}K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the *reduced kernel equation*

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. $U = K(t;x,0)$ is the *unique solution* in $\mathbb{Q}[[x,t]]$ of (RKerEq).

# Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x - t - \sqrt{x^2 - 2tx + t^2(1 - 4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

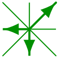
$$\underbrace{(xy - (x + y + x^2y^2)t)}_{\stackrel{!}{=}\,0} K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the *reduced kernel equation*

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. $U = K(t;x,0)$ is the *unique solution* in $\mathbb{Q}[[x,t]]$ of (RKerEq).

3. The guessed candidate $\mathcal{P}_{x,0}$ has one solution $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

# Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy - (x+y+x^2y^2)t)}_{\stackrel{!}{=}0} K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

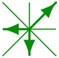
shows that $U = K(t;x,0)$ satisfies the *reduced kernel equation*

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. $U = K(t;x,0)$ is the *unique solution* in $\mathbb{Q}[[x,t]]$ of (RKerEq).

3. The guessed candidate $\mathcal{P}_{x,0}$ has one solution $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

4. Resultant computations + verification of initial terms
   $\Longrightarrow \quad U = H(t,x)$ also satisfies (RKerEq).

# Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x - t - \sqrt{x^2 - 2tx + t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy - (x + y + x^2y^2)t)}_{\overset{!}{=}\, 0} K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the *reduced kernel equation*

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. $U = K(t;x,0)$ is the *unique solution* in $\mathbb{Q}[[x,t]]$ of (RKerEq).

3. The guessed candidate $\mathcal{P}_{x,0}$ has one solution $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

4. Resultant computations + verification of initial terms
   $\implies$ $U = H(t,x)$ also satisfies (RKerEq).

5. *Uniqueness*: $H(t,x) = K(t;x,0) \implies K(t;x,0)$ is algebraic!

## Algebraicity of Kreweras walks: our Maple proof in a nutshell

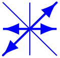
```
[bostan@inria ~]$ maple
    |\^/|     Maple 17 (APPLE UNIVERSAL OSX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2013
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.

# HIGH ORDER EXPANSION (S1)
> st,bu:=time(),kernelopts(bytesused):
> f:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
  end:
> S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..80),t,80):

# GUESSING (S2)
> libname:=".",libname:gfun:-version();
                                        3.62
> gfun:-seriestoalgeq(S,Fx(t)):
> P:=collect(numer(subs(Fx(t)=T,%[1])),T):

# RIGOROUS PROOF (S3)
> ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
> pol := unapply(P,T,t,x):
> p1 := resultant(pol(z-T,t,x),ker(t*z,t,x),z):
> p2 := subs(T=x*T,resultant(numer(pol(T/z,t,z)),ker(z,t,x),z)):
> normal(primpart(p1,T)/primpart(p2,T));
                                          1

# time (in sec) and memory consumption (in Mb)
> trunc(time()-st),trunc((kernelopts(bytesused)-bu)/1000^2);
                                       7, 618
```

# Step (S3): rigorous proof for Gessel walks 

Same philosophy, but several complications:

- stepset diagonal symmetry is lost: $G(t; x, y) \neq G(t; y, x)$;
- $G(t; 0, 0)$ occurs in (KerEq);
- equations are $\approx 5\,000$ times bigger.

$\longrightarrow$ replace (RKerEq) by a *system* of 2 reduced kernel equations.

$\longrightarrow$ fast algorithms needed (e.g., [B.-Flajolet-Salvy-Schost'06] for computations with algebraic series).

ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Symbolic Computation 41 (2006) 1–29

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Fast computation of special resultants

Alin Bostan[a,*], Philippe Flajolet[a], Bruno Salvy[a], Éric Schost[b]

[a] Algorithms Project, Inria Rocquencourt, 78153 Le Chesnay, France

[b] LIX, École polytechnique, 91128 Palaiseau, France

Received 3 September 2003; accepted 9 July 2005

Available online 25 October 2005

# Summary

☺ 2D classification of $F(t;0,0)$ and $F(t;x,y)$ is fully completed

☺ robust algorithmic methods:

- Guess'n'Prove approach based on modern CA algorithms
- Creative Telescoping for integration of rational functions

☹ Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(t;x,y) \approx$ 30Gb.

☺ Remarkable properties discovered experimentally. E.g.:
all algebraic cases have solvable Galois groups

$$G(t;1,1) = -\frac{3}{6t} + \frac{\sqrt{3}}{6t}\sqrt{U(t) + \sqrt{\frac{16t(2t+3)+2}{(1-4t)^2 U(t)} - U(t)^2 + 3}}$$

where $U(t) = \sqrt{1 + 4t^{1/3}(4t+1)^{1/3}/(4t-1)^{4/3}}$.

# Summary

☺ 2D classification of $F(t;0,0)$ and $F(t;x,y)$ is fully completed

☺ robust algorithmic methods:

- Guess'n'Prove approach based on modern CA algorithms
- Creative Telescoping for integration of rational functions

☺ Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(t;x,y) \approx$ 30Gb.

☺ Remarkable properties discovered experimentally. E.g.:
all algebraic cases have solvable Galois groups

$$G(t;1,1) = -\frac{3}{6t} + \frac{\sqrt{3}}{6t}\sqrt{U(t) + \sqrt{\frac{16t(2t+3)+2}{(1-4t)^2U(t)} - U(t)^2 + 3}}$$

where $\quad U(t) = \sqrt{1 + 4t^{1/3}(4t+1)^{1/3}/(4t-1)^{4/3}}.$

☹ lack of "purely human" proofs for many results. E.g.:
non-holonomy of $F(t;0,0)$ and ${}_2F_1$s expressions for $F(t;x,y)$

☹ still missing a unified proof of: finite group ↔ holonomic

☹ open: is $F(t;1,1)$ non-holonomic in the 51 non-singular cases with infinite group?

# Extensions

▷ 3D walks [B., Bousquet-Mélou, Kauers & Melczer, 2014]
Many distinct cases: 11 074 225 (instead of 79)

- We focus on the 20 804 cases with at most 6 steps.
- [B. & Kauers, 2009] conjectured 35 holonomic. Now proved.
- 116 new holonomic cases: guessed, then proved.
- New phenomenon (empirically discovered, no proof yet): cases (e.g. 3D Kreweras) with finite group, but non-holonomic GF (?!)

▷ Longer 2D steps [B., Bousquet-Mélou & Melczer, 2014]

- 680 step sets with one large step, 643 proved non holonomic, 32 of 37 have differential equations guessed.
- 5910 step sets with two large steps, 5754 proved non holonomic, 69 of 156 have differential equations guessed.

# Bibliography


- *Automatic classification of restricted lattice walks*, with M. Kauers. Proc. FPSAC, 2009.
- *The complete generating function for Gessel walks is algebraic*, with M. Kauers. Proc. Amer. Math. Soc., 2010.
- *Explicit formula for the generating series of diagonal 3D Rook paths*, with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
- *Non-D-finite excursions in the quarter plane*, with K. Raschel and B. Salvy. Journal of Combinatorial Theory A, 2013.
- *A human proof of Gessel's lattice path conjecture*, with I. Kurkova, K. Raschel, arXiv:1309.1023, 2014.
- *Explicit Differentiably Finite Generating Functions of Walks with Small Steps in the Quarter Plane*, with F. Chyzak, M. Kauers, M. van Hoeij and L. Pech, 2014.
- *On 3-dimensional lattice walks confined to the positive octant*, with M. Bousquet-Mélou, M. Kauers and S. Melczer, 2014.

Thanks for your attention!