Counting Kings: some experimental investigations

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My thanks

My thanks especially to

- Jon Borwein, David Bailey, and the other organizers for inviting me
- ICERM

Counting Kings: some problems

An Enumeration problem

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– enumeration vs generating functions vs asymptotic enumeration

Experimental Mathematics

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Why is this a good question to introduce undergraduates to experimental mathematics?

Opportunities for:

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Simulation.

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Using experimental mathematics to give them reason to learn other fields.

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Tim Gowers programs in BLOG.

Back to counting configurations of kings:
We'll let $f(m,n)$ denote the number of configurations of kings on a two dimensional $m \times n$ board.
It's easy to see that for each m,

$$\eta_m = \lim_{n\to\infty} f(m,n)^{\frac{1}{n}}$$

and

$$\eta = \lim_{m,n\to\infty} f(m,n)^{\frac{1}{mn}}$$

exist. The η's are measures of entropy.
One goal is to estimate η efficiently.

Topics

- A transfer matrix approach to the problem in 2 dimensions

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- Working with the generating functions
- Markov chain based approaches in 2 and in higher dimensions

The Transfer Matrix

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If columns i and j can be adjacent, $a_{ij} = 1$
I columns i and j have attacking Kings, $a_{ij} = 0$
Then configurations of Kings on an $m \times n$ board correspond to walks on the graph having adjacency matrix A_m with entries a_{ij}.

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- The number of walks in G from v_i to v_j of length 2 is equal to the number of vertices v_k for which $v_i v_k$ is an edge and $v_k v_j$ is an edge. This is equal to

$$\sum_k a_{ik} a_{kj} = (A^2)_{ij}$$

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- Similarly, the number of walks from v_i to v_j of length n is $(A^n)_{ij}$.
- The number of walks from v_i to v_j meeting n vertices (including the initial and final vertices, counting repetitions) is $(A^{n-1})_{ij}$.

The matrices A_m satisfy a lovely recurrence:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

and

$$A_k = \begin{pmatrix} A_{k-1} & {\scriptstyle A_{k-2}} \\ {\scriptstyle A_{k-2}} & \end{pmatrix}$$

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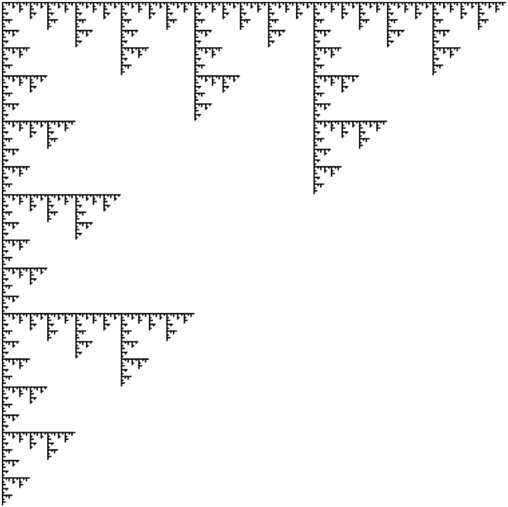
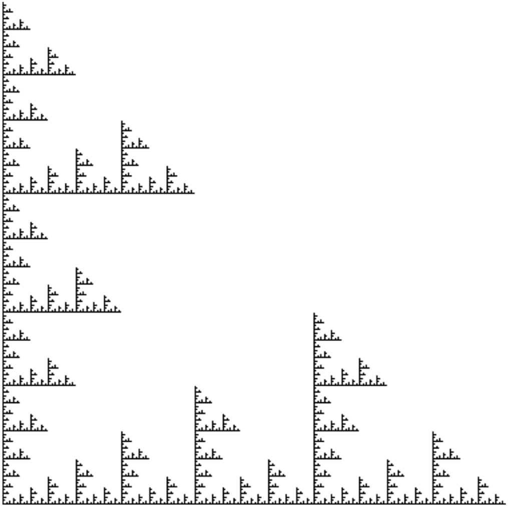


(Curiously, this picture looks quite different to me when rotated!)



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k	λ_k	$\lambda_k^{\frac{1}{k}}$
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These values, along with some other computations, allowed us to bound the value of η between

$$1.3426439 < \eta < 1.3426444$$

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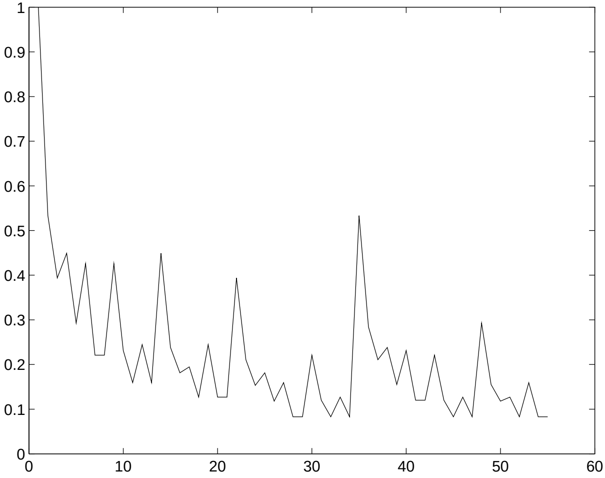
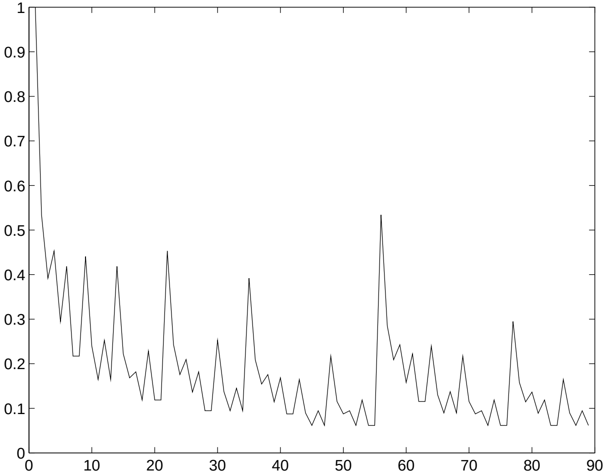
Plot the coordinates of $\boldsymbol{v}_m$ as a list of points.

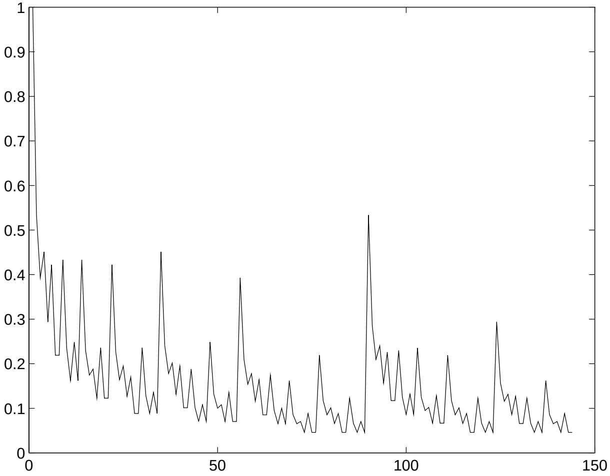
v_8



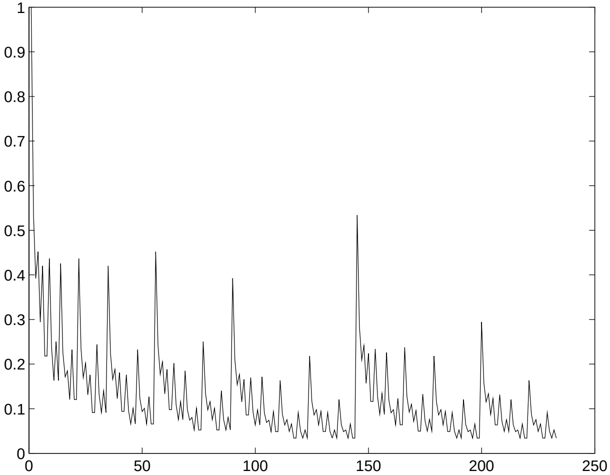
$|bmv_9$



v_{10}



v_{11}



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A big problem here seems to be that we don't have the right notation for the problem, and possibly we don't know even which space we should be working in!

“What if” questions

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Let

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$$F_{m,n}(1) = f(m,n)$$

and so

$$\lim_n F_{m,n}(1)^{1/n} = \eta.$$

What can we say about

$$\lim_n F_{m,n}(x)^{1/n}?$$

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The students discovered (interpreting things appropriately)

Theorem

$$\lim_n F_{m,n}(x)^{\frac{1}{n}} = 1 + x - x^2 + 2x^3 - 5x^4 + 14x^5 - 42x^6 + \dots$$

$$= \frac{1 + \sqrt{1 + 4x}}{2}.$$

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The students proved analogous theorems for $m = 2$ and $m = 3$, and we are working on determining the coeffcients of

$$\lim_{m,n} F_{m,n}(x)^{\frac{1}{mn}}.$$

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Are there other techniques that could work in 2 dimensions which might generalize better to higher dimensions?

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Theorem

Given a rooted tree, take a random walk from the root to a leaf, choosing uniformly from among the children at each node. Let X_j be the number of children seen at the j^{th} vertex. Let $X = X_1 X_2 \ldots X_l$ be the product of these values: then

$$E(X) = \#\ \textit{leaves in the tree}$$

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This is a surprising, surprisingly trivial, surprisingly powerful theorem!

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Let $X = X_1 X_2 \ldots X_n$: then $E(X) = f(m, n)$.
This is an exact formula! Unfortunately, computing $E(X)$ is not easy: but we can sample.

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Let d_j be the degree corresponding to the jth permissible column, and $Y_j(n)$ be the number of copies of the jth column which we have seen.
Then

$$\log X = \sum_{j=1}^{f(m,1)} Y_j \log d_j$$

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If we can estimate $\mathrm{Var}(\log X)$ either exactly or through sampling, we can estimate $E(X)$.

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Can a better understanding of how fast the two dimensional version converges give us information on how many times we have to sample the three dimensional version?

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Many ways to experiment.