

Computational complexity of problems in 3-dimensional topology

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slides at: <http://dunfield.info/preprints/>

[Novikov 1962] For $n \geq 5$, there does not exist an algorithm which solves:

ISSPHERE: Given a triangulated M^n is it homeomorphic to S^n ?

Thm (Geometrization + many results)
There is an algorithm to decide if two compact 3-mflds are homeomorphic.

Today: How hard are these 3-manifold questions? How quickly can we solve them?

Decision Problems: Yes or no answer.

SORTED: Given a list of integers, is it sorted?

SAT: Given $p_1, \dots, p_n \in \mathbb{F}_2[x_1, \dots, x_k]$ is there $\mathbf{x} \in \mathbb{F}_2^k$ with $p_i(\mathbf{x}) = 0$ for all i ?

UNKNOTTED: Given a planar diagram for K in S^3 is K the unknot?

INVERTIBLE: Given $A \in M_n(\mathbb{Z})$ does it have an inverse in $M_n(\mathbb{Z})$?

P: Decision problems which can be solved in polynomial time in the input size.

SORTED: $O(\text{length of list})$

INVERTIBLE: $O(n^{3.5} \log(\text{largest entry})^{1.1})$

NP: Yes answers have proofs that can be checked in polynomial time.

SAT: Given $\mathbf{x} \in \mathbb{F}_2^k$, can check all $p_i(\mathbf{x}) = 0$ in linear time.

UNKNOTTED: A diagram of the unknot with c crossings can be unknotted in $O(c^{11})$ Reidemeister moves. [Lackenby 2013]

coNP: No answers can be checked in polynomial time.

UNKNOTTED: Yes, assuming the GRH [Kuperberg 2011].

Conj: **UNKNOTTED** is in **P**.

KNOTGENUS: Given a triangulation T , a knot $K \subset T^{(1)}$, and a $g \in \mathbb{Z}_{\geq 0}$, does K bound an orientable surface of genus $\leq g$?

[Agol-Hass-W.Thurston 2006]

KNOTGENUS is **NP**-complete.

Conj (AHT) If $b_1(T) = 0$, then **KNOTGENUS** is in **coNP**.

[AHT] **KNOTAREA** is **NP**-complete.

[Dunfield-Hirani 2011] **KNOTAREA** is in **P** when $b_1(T) = 0$.

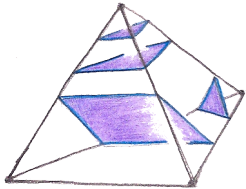
Is **KNOTGENUS** in **P** when $b_1 = 0$?

Is the homeomorphism problem for 3-manifolds in **NP**?

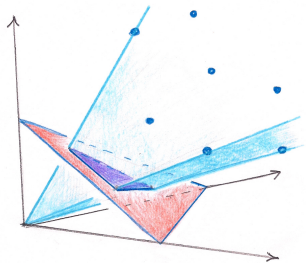
What about deciding hyperbolicity? or being an L-space?

Computing Khovanov homology and \widehat{HFK} are in **EXPTIME**. Just computing the Jones polynomial is **#P**-hard, but the Alexander polynomial can be computed in poly time.

Normal surfaces meet each tetrahedra in a standard way:



and correspond to lattice points in a finite polyhedral cone in \mathbb{R}^{7t} where $t = \#T$:



[Haken 1961] There is a minimal genus surface bounding K in normal form whose vector is fundamental (e.g. on a vertex ray). Hence **KNOTGENUS** is decidable.

[Hass-Lagarias-Pippenger 1999]

A fund surface has coordinates $O(\exp t^2)$.

[AHT 2006] KNOTGENUS is in **NP**.

Certificate: A vector x in \mathbb{Z}^{7t} with entries with a most $O(t^2)$ digits.

Check:

- (1) That x represents a normal surface S .
- (2) That $\chi(S) \leq 1 - 2g$.
- (3) That S connected and orientable.
- (4) That ∂S is as advertised.

All can be done in time polynomial in t but need a very clever idea for (3) and (4).

[Kuperberg 2011] Assuming GRH, **UNKNOTTING** is in **coNP**.

Certificate: $\rho: \pi_1(S^3 - K) \rightarrow \text{SL}_2\mathbb{F}_p$
where $\log p$ is $O(\text{poly}(\text{crossings}))$.

Check: The following imply π_1 is not cyclic and so K is knotted.

- (1) Relators for π_1 hold, so ρ is a rep.
- (2) A pair of generators have noncommuting images.

Proof that such a rep exists uses algebraic geometry/number theory and:

[Kronheimer-Mrowka 2004]

When $K \subset S^3$ is nontrivial, there is a rep $\pi_1(S^3 - K) \rightarrow \text{SU}_2$ with nonabelian image.

"In theory, there is no difference between theory and practice. But, in practice, there is."

-Jan L. A. van de Snepscheut

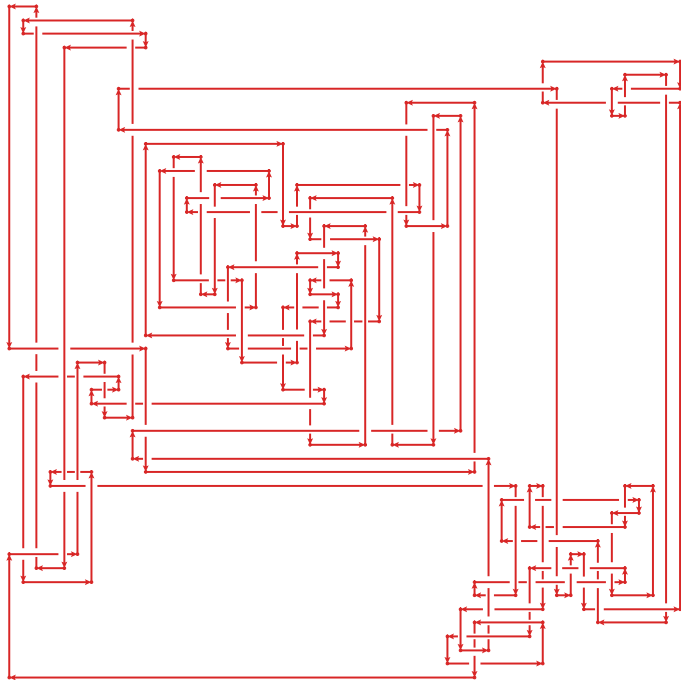
Mystery: In practice, many 3-mfld questions are easier than the best theoretical bounds indicate.

Q: How big a knot can we compute the genus for?

Q: Where do we even get big knots from?

There are more 100 crossing prime knots than there are atoms in the Earth!

Here's a sneak peak of joint work with Malik Obeidin, based on one natural model of random knot.



Personal best:

crossings: 126

genus: 27

fibered: No

time: 7 minutes

hyperbolic volume:

223.6132847441086613

tetrahedra: 243

