Garside structure and Dehornoy ordering of braid groups for topologist (mini-course I)

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Combinatorial Link Homology Theories, Braids, and Contact Geometry
Aug, 2014

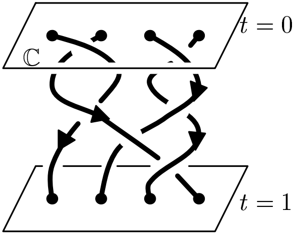
Contents

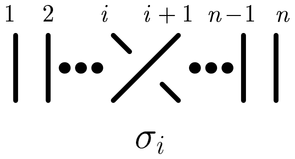
Introduction

Braid group

The n-strand braid group

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & |i-j| = 1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| > 1 \end{array} \right\rangle.$$



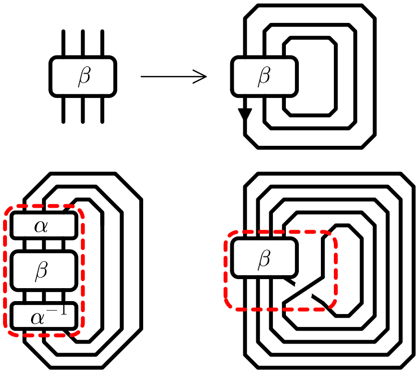


An element of B_n is represented by n-strings (braid) in $\mathbb{C} \times [0,1]$.
We have a natural map $\pi : B_n \to S_n$, and the pure braid group P_n is the kernel of π.

Braid group in topology (I) relation to knot theory

Alexander-Markov Theorem

$$\{\text{Braids}\}/\text{conjugation,stabilization}) \overset{1:1}{\longleftrightarrow} \{ \text{ Oriented links in } S^3\}$$



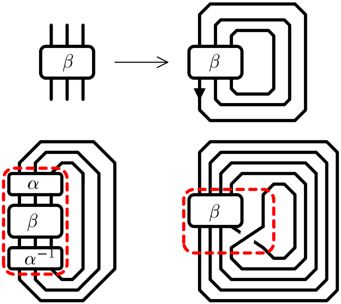
Braid group in topology (I) relation to knot theory

Transverse Markov Theorem (Orevkov-Shevchishin, Wrinkle '02)

{Braids}/conjugation, positive stabilization)

$1:1 \updownarrow$

{Transverse links in standard contact S^3}



Braid group in topology (II) relation to MCG

$D_n = \{z \in \mathbb{C} \mid |z| \leq n+1\} - \{1, \ldots, n\}$: n-punctured disc

$$\begin{aligned} B_n &\cong MCG(D_n) \\ &= \{\text{Mapping class group of } D_n\} \\ &= \{f : D_n \overset{Homeo}{\longrightarrow} D_n \mid f|_{\partial D_n} = id\}/\{Isotopy\} \end{aligned}$$

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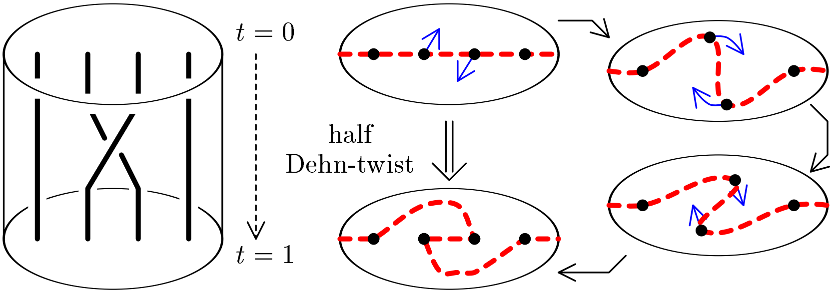
- $\sigma_i \leftrightarrow$ Half Dehn-twist swapping i and $(i+1)$.



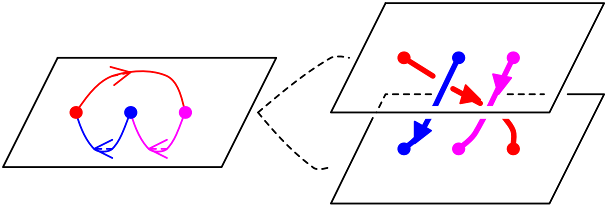
Braid group in topology (III) configuration space

The ordered/unordered configuration space of n-points in $\mathbb{C}$:

$$C_n(\mathbb{C}) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}, \quad UC_n(\mathbb{C}) = C_n(\mathbb{C})/S_n$$

Based loops in $UC_n(\mathbb{C})$ are naturally regarded as braids so

$$\begin{cases} \Omega UC_n(\mathbb{C}) = \{\text{Space of braids}\} \\ \pi_1(C_n(\mathbb{C})) = P_n,\ \pi_1(UC_n(\mathbb{C})) = B_n. \end{cases}$$



Braid group in topology (III) configuration space

A natural projection

$$p : C_n(\mathbb{C}) \to C_{n-1}(\mathbb{C}), \quad p(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1})$$

is a fibration with fiber $\mathbb{C} - \{(n-1) \text{ points}\}$, with section

$$s : C_{n-1}(\mathbb{C}) \to C_n(\mathbb{C}), \quad s(z_1, \ldots, z_{n-1}, \max\{|z_i|\} + 1).$$

This shows

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Theorem (Atrin '47, Fox-Neuwirth '62, Fadell-Neuwirth '62)

1. $C_n(\mathbb{C}) = K(P_n, 1)$, $UC_n(\mathbb{C}) = K(B_n, 1)$
2. The cohomological dimension of B_n and P_n are finite, and both B_n and P_n are torsion-free.
3. $P_n = F_{n-1} \rtimes P_{n-1} = (F_{n-1} \rtimes (F_{n-2} \rtimes (F_{n-3} \cdots (F_2 \rtimes F_1)) \cdots)$.

Braid group in topology (IV) Hyperplane arrangement

$C_n(\mathbb{C})$ is regarded as the complement of an hyperplane arrangement called the braid arrangement:
For $1 \leq i < j \leq n$, let

$$H_{i,j} = \mathrm{Ker}(z_i - z_j) \subset \mathbb{C}^n, \ \mathcal{A} = \{H_{i,j}\}_{1 \leq i < j \leq n}$$

Then

$$C_n(\mathbb{C}) = M(\mathcal{A}) = \mathbb{C}^n - \bigcup_{1 \leq i < j \leq n} H_{i,j},$$

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Then

$$C_n(\mathbb{C}) = M(\mathcal{A}) = \mathbb{C}^n - \bigcup_{1 \leq i < j \leq n} H_{i,j},$$

1. Reflections with respect to $H_{i,j}$'s forms the symmetric group S_n.
   - Close and natural connection between root systems, Coxeter groups and Artin groups.
   - Source of combinatorics of braids.
2. A well-known method to construct cellular decomposition of $M(\mathcal{A})$ (Salvetti complex) gives a presentation of B_n.

Part I: Garside theory for braid groups

Word and conjugacy problem

Word/Conjugacy Problem

For given braids α, β (as a word over $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$)

- Determine whether $\alpha = \beta$ or not.
- Determine whether α and β are conjugate or not.
  (and, find γ such that $\gamma\beta\gamma^{-1} = \alpha$.)

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Since group is suited for computations (encoding), our ultimate goal is:

Algebraic link Problem

For two links (represented as closed braids),

- Determine whether they are the same or not
- Determine basic properties (prime/split/satellite/hyperbolic,etc...)

Word/conjugacy problem is the first step towards this problem.

What is Garside theory ?

Garside theory (Garside structure) is a machinery of:

1. Producing the normal form of a group.
   - Easy to calculate (and suited for computor)
   - Idea and its meaning sounds natural.
2. Giving several nice structures of the group (automatic, lattice...)
3. Allowing us to solve other problems (conjugacy, extracting roots, etc...)

In particular, for the case of braid groups:

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In particular, for the case of braid groups:

Motto

Garside structure yields "the best" normal form – it reflects

- Dynamics (Nielsen-Thurston classification)
- Topology (infinite cyclic coverings)
- Algebra (quantum/homological representation)
- Dehornoy's ordering

I-1: Toy model: Garside structure on $\mathbb{Z}^2$

Toy model: Garside structure on $\mathbb{Z}^2$

$G = \mathbb{Z}^2 = \langle x, y \rangle$: Free abelian group of rank two
$P = \{x^a y^b \mid a, b \geq 0\}$: set of "positive" elements
$\Delta = xy = yx$: Garside element

Toy model: Garside structure on $\mathbb{Z}^2$

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$\Delta = xy = yx$: Garside element

Key features:

- P is a submonoid: $\alpha, \beta \in P \Rightarrow \alpha\beta \in P$.
- For any $\alpha \in G$, $\Delta^n z \in P$ for sufficiently large n.
- For $\alpha = x^a y^b, \beta = x^{a'} y^{b'} \in G$, define

$$\begin{aligned} \alpha \preccurlyeq \beta \quad &\Longleftrightarrow \quad a \leq a' \text{ and } b \leq b' \\ &\Longleftrightarrow \quad \alpha^{-1}\beta \in P. \end{aligned}$$

  Then $x, y \preccurlyeq \Delta$.
- $[1, \Delta] \stackrel{Def}{=} \{\beta \in G \mid 1 \preccurlyeq \beta \preccurlyeq \Delta\} = \{1, x, y, \Delta\}$.

Normal form for $\mathbb{Z}^2$

Definition

For $\beta \in G$, the normal form of β is a word over $\{x, y, \Delta^{\pm 1}\}$

$$N(\beta) = \Delta^p s_1 s_2 \cdots s_r \quad (p \in \mathbb{Z}, s_i \in \{x, y, \Delta\})$$

such that

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$$N(\beta) = \Delta^p s_1 s_2 \cdots s_r \quad (p \in \mathbb{Z}, s_i \in \{x, y, \Delta\})$$

such that

1. $\Delta^{-p}\beta \in P$.
2. s_i is the $\preccurlyeq$-largest element among $\{x, y, \Delta\}$ satisfying

$$s_i \preccurlyeq (s_{i-1}^{-1} \cdots s_1^{-1} \Delta^{-p})\beta$$

(So normal form of $\beta = x^a y^b$ is actually written as:

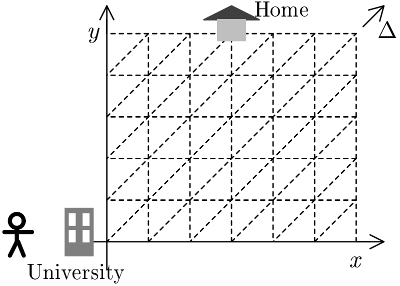
$$N(\beta) = \begin{cases} \Delta^a y^{b-a} & b \geq a \\ \Delta^b x^{a-b} & a \geq b \end{cases}$$

What is the meaning of normal form ?

Idea

Normal form = path in the Cayley graph which approaches the destination in the "fastest" way at any intermediate time.

Q: How to go back to home from university ?



We are tired, so we want to go back to home as early as possible...

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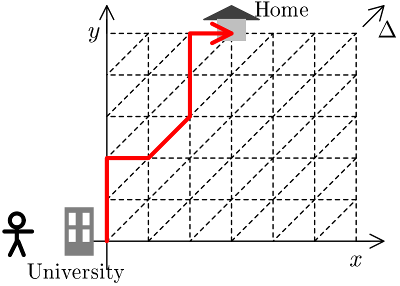

$$yyx\Delta yyx$$

This path is not effective (geodesic) – we can do several short-cuts.

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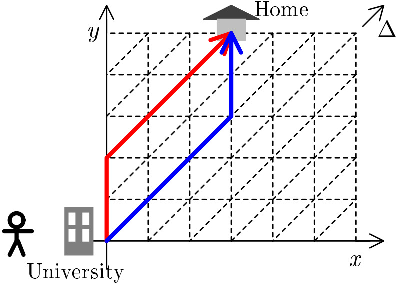

$$yy\Delta\Delta\Delta$$
v.s.
$$\Delta\Delta\Delta yy$$

These paths are both geodesic (so the total arrival time is the same) but...

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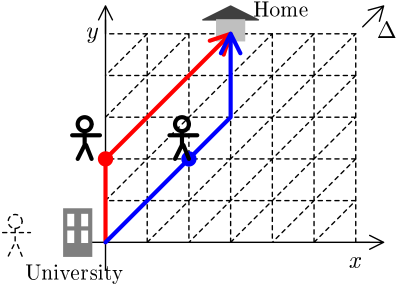

$yy\Delta\Delta\Delta$

v.s.

$\underline{\Delta\Delta\Delta yy}$

Normal form

After 2minutes, normal form path lies closer than other path.

How to computing normal forms ?

Strategy to get normal form

1. By considering $\Delta^n\beta$ for sufficiently large n, we assume $\beta \in P$.
2. Starting at the final destination, we do:
   - let us look sub-path $s_i s_{i+1}$: check whether this sub-path is "nice" of not (whether this sub-path is a normal form or not)
   - If this sub-path is not nice (i.e. we are going by a roundabout route) replace this sub-path $s_i s_{i+1}$ with better one (tighten locally).

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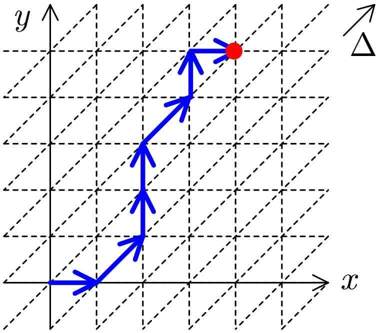
Crucial fact

By resolving local roundabouts, we will eventually get globally nice path, the normal form.

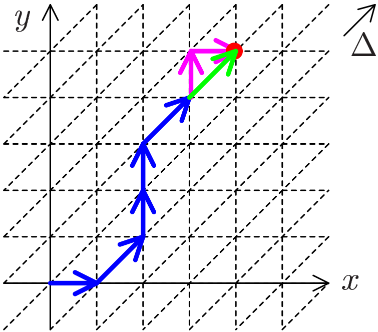
(cf. Length of geodesic connecting two point x, y in Riemannian manifold $\neq$ distance of x and y)

Computing normal forms: example



$$x\Delta yy\Delta yx$$

Computing normal forms: example

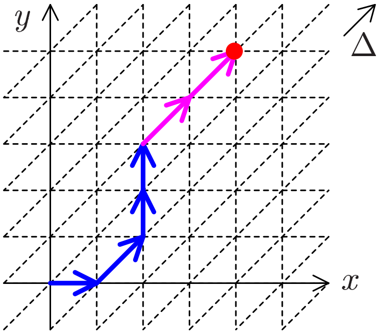


$$x\Delta yy\Delta yx$$

$$\Downarrow$$

$$x\Delta yy\Delta\Delta$$

Computing normal forms: example



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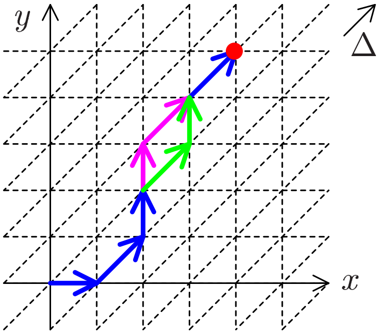
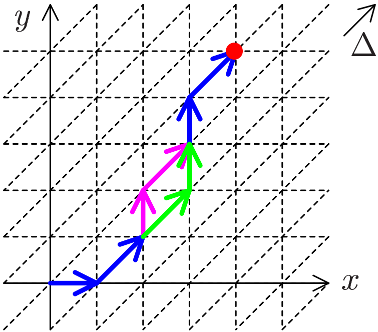


$$x\Delta y y \Delta \Delta$$

$$\Downarrow$$

$$x\Delta y \Delta y \Delta$$

Computing normal forms: example

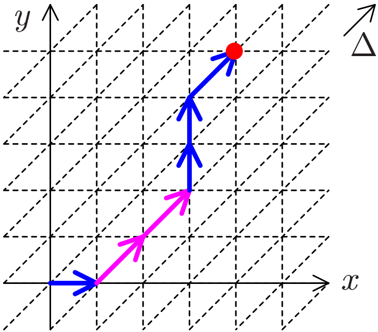


$$x\Delta y\Delta y\Delta$$

$$\Downarrow$$

$$x\Delta\Delta yyy\Delta$$

Computing normal forms: example



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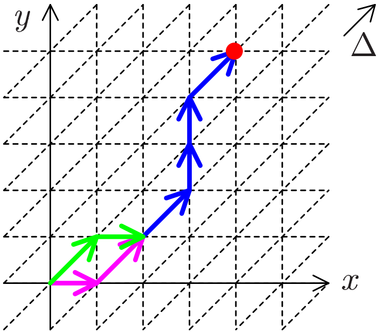
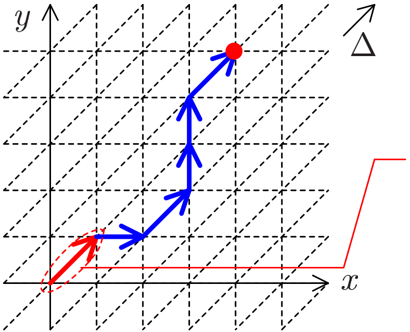


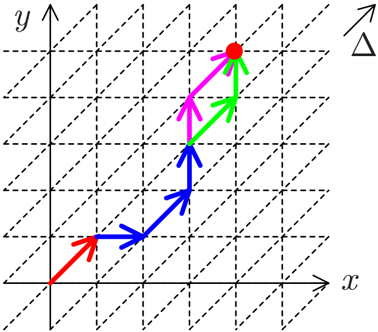
$$x\Delta\Delta yy\Delta$$
$$\Downarrow$$
$$\Delta x\Delta yy\Delta$$

Computing normal forms: example



The best choice
of the first path
||
The first letter of
the normal form

Computing normal forms: example

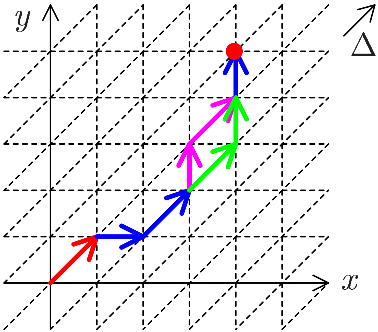


$$\Delta x \Delta y y \Delta$$

$$\Downarrow$$

$$\Delta x \Delta y \Delta y$$

Computing normal forms: example



$$\Delta x \Delta y \Delta y$$

$$\Downarrow$$

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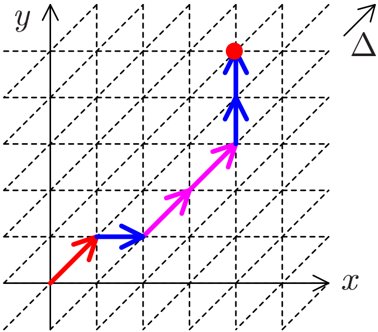
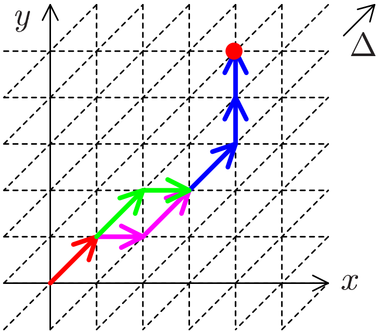
Computing normal forms: example



$$\Delta x \Delta \Delta y y$$

Computing normal forms: example



$$\Delta\Delta x\Delta yy$$

$$\Downarrow$$

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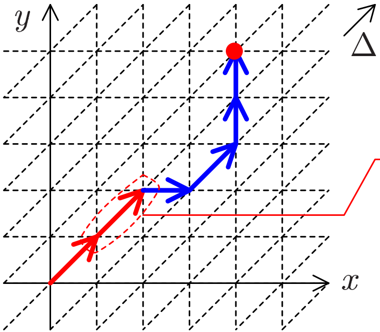
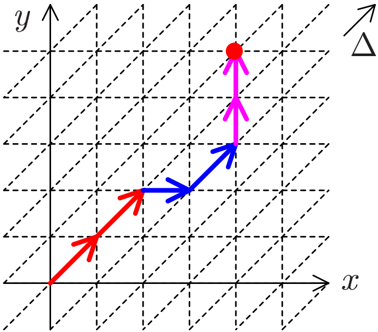
Computing normal forms: example



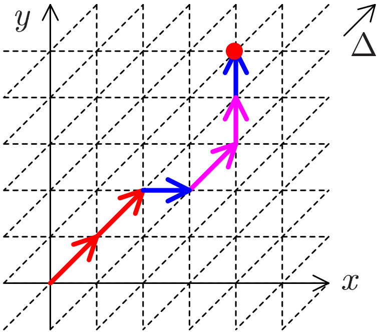
The best choice
of the second path
‖
The second letter of
the normal form

Computing normal forms: example



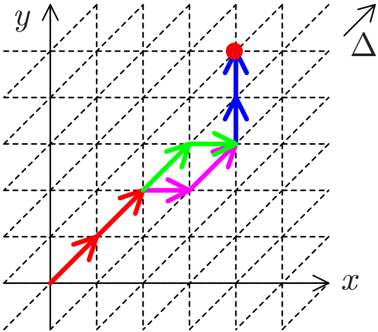
$$\Delta\Delta x \Delta yy$$

Computing normal forms: example



$$\Delta\Delta x{\color{magenta}\Delta y}y$$

Computing normal forms: example

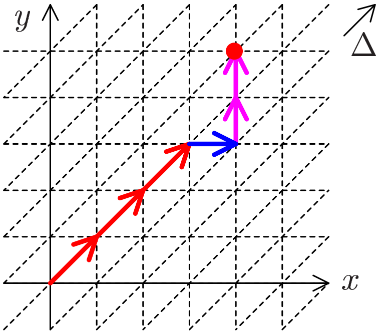


$$\Delta\Delta{\color{magenta}x\Delta}yy$$

$$\Downarrow$$

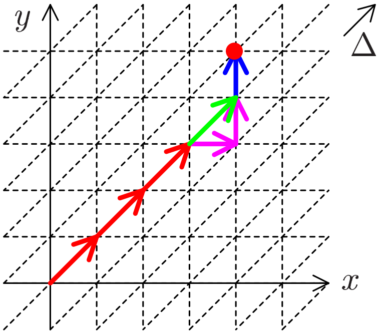
$$\Delta\Delta{\color{green}\Delta x}yy$$

Computing normal forms: example



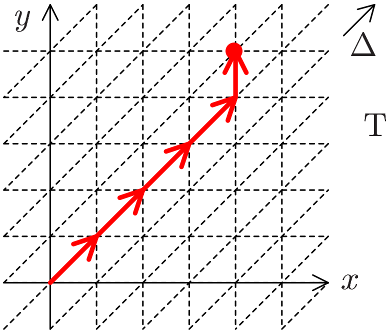
$$\Delta\Delta\Delta x{\color{magenta}yy}$$

Computing normal forms: example



$$\Delta\Delta\Delta xyy$$
$$\Downarrow$$
$$\Delta\Delta\Delta\Delta y$$

Computing normal forms: example



The normal form is:

$\Delta\Delta\Delta\Delta y$

Computing normal forms: conclusion

How fast can we compute the normal form ?
Previous argument says:

Conclusion

For $\beta \in G$ of length ℓ (as a word over $\{x, y, \Delta\}$), after performing $\frac{\ell(\ell-1)}{2} = \mathcal{O}(\ell^2)$ times of "local tightening" (replacing local roundabout route with the best one), we are able to get a normal form of β.

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Moreover, note that in the process of local tightening, we just need to look at the path of length two. This says

Conclusion'

To compute normal form, we only need finite data (of which path is better).

I-2: Classical Garside structure

General idea of Garside structure

We want to generalize idea and method for "toy model" for more general and complicated group G – what we need ?
In toy model, we have used:

General idea of Garside structure

We want to generalize idea and method for "toy model" for more general and complicated group G – what we need ?
In toy model, we have used:

1. Submonoid P consisting of "positive elements":
   P consists of positive words over certain generating sets $\{x, y, \ldots,\}$ of G.
   - The notion of positive elements yields a subword ordering $\preccurlyeq$:

$$\alpha \preccurlyeq \beta \overset{Def}{\iff} \alpha^{-1}\beta \in P.$$

2. Special element Δ:
   - For any $\beta \in G$, $\Delta^n \beta \in P$ for sufficiently large n.
   - $x, y, \ldots \preccurlyeq \Delta$.

By giving "good" Δ and P, one can generalize the toy model idea.

The classical Garside structure of braid

$$
\begin{aligned}
B_n^+ &= \{\text{Product of } \sigma_1, \ldots, \sigma_{n-1}\} : \text{Positive braid monoid} \\
\Delta &= (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)(\sigma_1) : \text{Garside element}
\end{aligned}
$$

The classical Garside structure of braid

$$B_n^+ = \{\text{Product of } \sigma_1, \ldots, \sigma_{n-1}\} : \text{Positive braid monoid}$$
$$\Delta = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)(\sigma_1) : \text{Garside element}$$

Definition-Proposition

Define the relation $\preccurlyeq$ of B_n by $x \preccurlyeq y \iff x^{-1}y \in B_n^+$. Then $\preccurlyeq$ is a lattice ordering:

- $\preccurlyeq$ admits the greatest common divisor
$$x \wedge y = \max_{\preccurlyeq}\{z \in B_n \mid z \preccurlyeq x, y\}$$
- $\preccurlyeq$ admits the least common multiple
$$x \vee y = \min_{\preccurlyeq}\{z \in B_n \mid x, y \preccurlyeq z\}$$
- $\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \preccurlyeq \Delta$.

Why we choose such Δ and B_n^+ ?

We want to define the normal form

$$N(\beta) = \Delta^p s_1 \cdots s_r$$

as we have done in the case $\mathbb{Z}^2$ (toy model): So we first need

$$\Delta^{-p}\beta \in B_n^+$$

and s_1 should be:

the $\preccurlyeq$ -maximal element satisfying $s_1 \preccurlyeq \Delta^{-p}\beta$ ($\in B_n^+$)

$\Rightarrow$ We need to know such $\preccurlyeq$-maximal element always exists
$\Rightarrow$ Lattice structure naturally appear.

Simple braids

- $\preccurlyeq$ is a "subword" ordering: $\sigma_2\sigma_3 \preccurlyeq \sigma_2\sigma_3 \underbrace{\sigma_1\sigma_2^3}_{\text{Positive braids}}$
- $\Delta = \sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{n-1}$.

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- $\Delta = \sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{n-1}$.

definition

A simple braid is a braid that satisfies $1 \preccurlyeq x \preccurlyeq \Delta$.

Note:

$$\begin{aligned} B_n^+ &= \{\text{Product of } \sigma_1, \ldots, \sigma_{n-1}\} \\ &= \{\text{Product of simple braids}\} \end{aligned}$$

Proposition

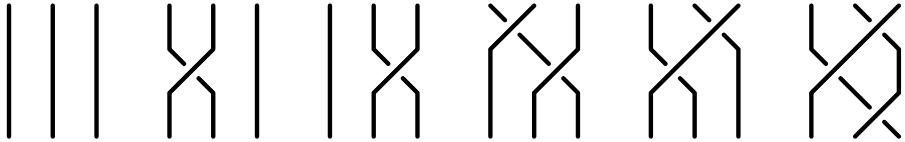
$$[1, \Delta] \overset{Def}{=} \{ \text{ simple braids } \} \overset{1:1}{\longleftrightarrow} S_n$$

(so simple braids are often called premutation braids)

Example: B_3 case

$\Delta = (\sigma_1\sigma_2)\sigma_1 = \sigma_2\sigma_1\sigma_2$, so

$$[1, \Delta] = \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \Delta\}$$



Simple braids: each strand positively crosses with other strands at most once.

Normal form

Theorem-Definition (Garside, Elrifai-Morton, Thurston)

A braid $\beta \in B_n$ admits the normal form

$$N(\beta) = \Delta^p x_1 x_2 \cdots x_r \quad (p \in \mathbb{Z},\ x_i \in [1, \Delta])$$

where

1. $\Delta^{-p}\beta \in B_n^+$.
2. $x_i = \Delta \wedge (x_{i-1}^{-1} \cdots x_1^{-1} \Delta^{-p} \beta)$.

By absorbing first few Δ terms in $x_1, \ldots$, $N(\beta)$ is uniquely written as

$$N(\beta) = \Delta^p x_1 x_2 \cdots x_r \quad (p \in \mathbb{Z},\ x_i \neq \Delta).$$

We define the infimum, supremum of β by

$$\inf(\beta) = p,\ \sup(\beta) = p + r.$$

How to compute normal form ?

As in the toy model case, a word is a normal form if and only if it is locally a normal form:

Theorem (Elrifai-Morton, Thurston)

A word

$$N'(\beta) = \Delta^p x_1 x_2 \cdots x_r \quad (p \in \mathbb{Z}, x_i \in [1, \Delta])$$

is a normal form if and only if

$$(x_i x_{i+1}) \wedge \Delta = x_i \text{ for all } i$$

(i.e., $x_i x_{i+1}$ is also a normal form)

How to compute normal form ?

The strategy for computing normal form applies to the braid group case:

Strategy to get normal form

1. Express β as a word of the form

$$\beta = \Delta^p x_1 \cdots x_r \quad (p \in \mathbb{Z}, x_i \in [1, \Delta])$$

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$$\beta = \Delta^p x_1 \cdots x_r \quad (p \in \mathbb{Z}, x_i \in [1, \Delta])$$

- $\Delta^2 = (\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$ is the full-twist braid (as an element of $MCG(D_n)$, it is the Dehn twist along ∂D_n), which is a generator of the center of B_n, so

$$\cdots \sigma_i^{-1} \cdots = \cdots \Delta^{-2}\Delta^2\sigma_i^{-1} \cdots = \Delta^{-2} \cdots \underbrace{(\Delta^2\sigma_i^{-1})}_{\text{Positive braid}} \cdots$$

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2. Apply local tightening repeatedly: for $i = r, \ldots, 1$ rewrite each sub-path $x_i x_{i+1}$ so that it is a normal form

$$x_i x_{i+1} = x'_i x'_{i+1}, \quad x'_i = (x_i x_{i+1}) \wedge \Delta$$

Simple example

Let us compute the normal form of a 3-braid $\beta = (\sigma_2^{-1})(\sigma_1\sigma_2)(\sigma_2)(\sigma_1\sigma_2)$.

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$$\beta = \Delta^{-1}(\sigma_1\sigma_2)(\sigma_1\sigma_2)(\sigma_2)(\sigma_1\sigma_2)$$

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$$\beta = \Delta^{-1}(\sigma_1\sigma_2)(\sigma_1\sigma_2)(\sigma_2)(\sigma_1\sigma_2)$$

2. Apply local tightenings for

$$\beta' = (\sigma_1\sigma_2)(\sigma_1\sigma_2)(\sigma_2)(\sigma_1\sigma_2)$$

to get normal forms

Simple example: local tightening

$$\beta' = (\sigma_1\sigma_2)(\sigma_1\sigma_2)\underline{(\sigma_2)(\sigma_1\sigma_2)}$$

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$$\beta' = (\sigma_1\sigma_2)(\sigma_1\sigma_2)\underline{(\sigma_2)(\sigma_1\sigma_2)}$$

$(\sigma_2)(\sigma_1\sigma_2) \wedge \Delta = \Delta$, so

$$\beta' = (\sigma_1\sigma_2)\underline{(\sigma_1\sigma_2)(\Delta)}.$$

Simple example: local tightening

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$(\sigma_1\sigma_2)(\Delta) \wedge \Delta = \Delta$, and $(\sigma_1\sigma_2)(\Delta) = (\Delta)(\sigma_2\sigma_1)$, so

$$\beta' = \underline{(\sigma_1\sigma_2)(\Delta)}(\sigma_2\sigma_1)$$

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Simple example: local tightening

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$(\sigma_1\sigma_2)(\sigma_1\sigma_2) \wedge \Delta = \Delta$, and $(\sigma_1\sigma_2)(\sigma_1\sigma_2) = (\Delta)(\sigma_2)$, so

$$\beta' = \Delta\Delta\sigma_2.$$

Simple example: local tightening

$$\beta' = (\sigma_1\sigma_2)(\sigma_1\sigma_2)\underline{(\sigma_2)(\sigma_1\sigma_2)}$$

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$$\beta' = \Delta\Delta\sigma_2.$$

Hence $\beta = \Delta^{-1}\beta' = \Delta^{-1}\Delta\Delta\sigma_2$ and its normal form is

$$N(\beta) = (\Delta)(\sigma_2)$$

Meaning of normal form condition

What is the meaning of condition $(x_i x_{i+1}) \wedge \Delta = x_i$?

Proposition

For $x \in [1, \Delta]$, define the starting set $S(x)$ by

$$S(x) = \{\sigma_i \mid x = \sigma_i \cdot (\text{positive braid}) \ \ (i.e.\ \sigma_i \preccurlyeq x)\}$$

and the finishing set $F(x)$ by

$$F(x) = \{\sigma_i \mid x = (\text{positive braid}) \cdot \sigma_i\}$$

Then for simple braids x and y,

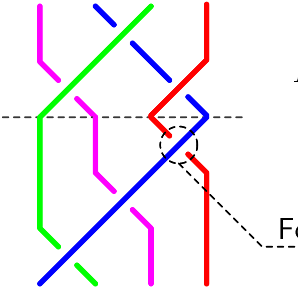
$$xy \wedge \Delta = x \iff F(x) \supset S(y)$$

Meaning of normal form condition

The situation $F(x) \supset S(y)$ prevents to absorb crossings in y into x:

(Recall that:

simple braid $\iff$ each pair of strand crosses at most by once



$$F(\sigma_2\sigma_1\sigma_3) = \{\sigma_1, \sigma_3\}$$
$$\bigcup$$
$$S(\sigma_1\sigma_2\sigma_3) = \{\sigma_1\}$$

Forces to have second crossings between two strands

Geodesic property

Lemma

$x^{-1}\Delta$ and $x^{\Delta} = \Delta x \Delta^{-1}$ are simple if x is simple.

Geodesic property

Lemma

$x^{-1}\Delta$ and $x^{\Delta} = \Delta x \Delta^{-1}$ are simple if x is simple.

Rewrite a normal form $N(\beta) = \Delta^p x_1 \cdots x_r$ as

$$W(\beta) = \begin{cases} \Delta^p x_1 \cdots x_r & (p > 0) \\ (\Delta^{-1}x_1)^{\Delta^{p+1}} (\Delta^{-1}x_2)^{\Delta^{p+2}} \cdots (\Delta^{-1}x_{-p}) x_{-p+1} \cdots x_r & \\ & (p < 0, p + r > 0) \\ (\Delta^{-1}x_1)^{\Delta^{p+1}} \cdots (\Delta^{-1}x_r)^{\Delta^{p+r}} \Delta^{-p-r} & (p + r < 0) \end{cases}$$

Theorem (Charney)

$W(\beta)$ is a geodesic word. So the length of β (with respect to simple braids $[1, \Delta]$ is

$$\ell_{[1,\Delta]}(\beta) = \max\{\sup(\beta), 0\} - \min\{\inf(\beta), 0\}.$$

Normal form produces automatic structure

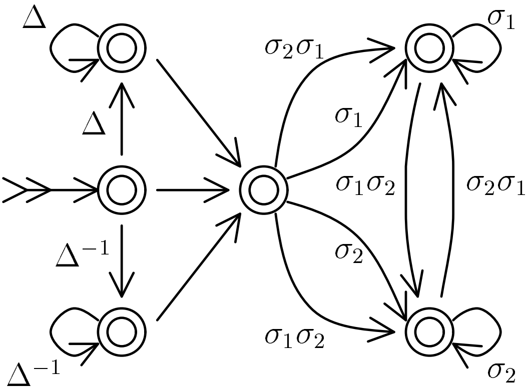
The characterizing property of normal form is "local"
(we only need to see consecutive factor $x_i x_{i+1}$)

Theorem (Thurston, Charney, Dehornoy)

The normal forms of B_n provides a geodesic automatic structure. In particular,

$$\{\text{Set of normal forms}\} \overset{1:1}{\longleftrightarrow} \{\text{Path of certain graph (automata)}\}$$

Example: Automata for B_3



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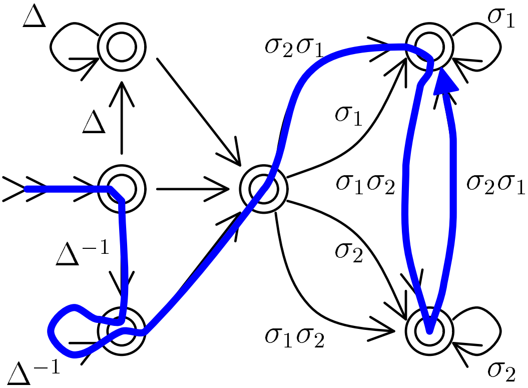


Normal form

$$N(\beta) = \Delta^{-1}\Delta^{-1}(\sigma_2\sigma_1)(\sigma_1\sigma_2)(\sigma_2\sigma_1)$$

Conjugacy problem (I)

Using normal form technique, we can solve the conjugacy (search) problem.

Basic strategy

For given $\alpha \in B_n$, try to determine the set of "simplest" normal forms among its conjugacy class, called ... summit set.

$$S(\alpha) = \left\{ \beta \;\middle|\; \begin{array}{c} \beta \text{ is conjugate to } \alpha, \text{ with the "simplest"} N(\beta) \\ + \text{"Additional requirements"} \end{array} \right\}$$

Then,

$$S(\alpha) = S(\alpha') \iff \alpha \text{ and } \beta \text{ are conjugate}$$

Conjugacy problem (II)

By cycling and decycling operation, we may find simpler normal form among the conjugacy class of given braid β:

$$N(\beta) = \Delta^p x_1 \cdots x_{r-1} x_r = \Delta^p x_1 \Delta^{-p} \Delta^p x_2 \cdots x_r$$

decycling

$$x_r \Delta^p x_1 \cdots x_{r-1}$$

$$\|$$

$$\Delta^p (\Delta^{-p} x_r \Delta^p) x_1 \cdots x_{r-1}$$

cycling

$$\Delta^p x_2 \cdots x_{r-1} \Delta^p x_1 \Delta^{-p}$$

$$\Delta^{p'} x'_1 \cdots x'_{r'} : \text{ simpler normal form}$$

It may happen $p' > p$ or $r' < r$

Conjugacy problem (II)

Theorem (Garside, ElRifai-Morton, Gebhardt, González-Meneses)

Let $\alpha \in Bn$.

1. By applying cycling and decylings finitely many times, we can find one element in $S(\alpha)$.
2. Staring from one element $\beta \in S(\alpha)$, by repeatedly computing the conjugate of β by simple elements, we can find all elements of $S(\alpha)$:

In particular, we have an algorithm to solve the conjugacy decision and problem (determine $\alpha \sim_{conj} \alpha'$) and the conjugacy search problem (find β such that $\alpha = \beta^{-1}\alpha'\beta$).

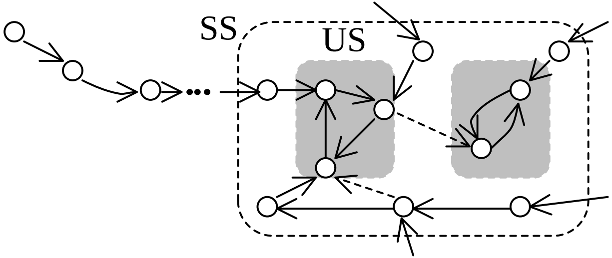
Conjugacy problem (II) example of "... (summit) set"

The super summit set

$$SS(\alpha) = \left\{ \beta \;\middle|\; \begin{array}{l} \beta \text{ is conjugate to } \alpha \text{ with} \\ \text{maximal inf, minimum sup} \end{array} \right\}$$

The ultra summit set

$$US(\alpha) = \{\beta \in SS(\alpha) \mid \text{ closed under cycling operation } \}$$



Conjugacy problem (III)

Using idea of summit set, we can solve the conjugacy problem (but in time $\mathcal{O}(e^{\text{length}})$, in general):

- Computing a normal form is easy (done in polynomial time).
- Starting from α, finding one element of $S(\alpha)$ is (conjecturally) done in polynomial time.
- Size of $S(\alpha)$ might be quite huge – the size of $S(\alpha)$ might be $\mathcal{O}(e^{\text{length}})$ (So computing whole $S(\alpha)$ might require exponential times...)

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Problem

Find polynomial time algorithm for conjugacy problem of braids.

Problem

Understand the structure of summit sets.

I-3: Dual Garside structure

Dual Garside structure

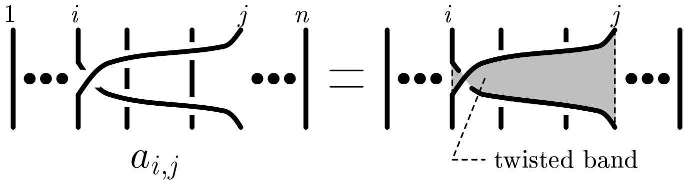
The braid group has another Garside structure called dual Garside structure, by consdiering different P (the set of positive elements) and δ (Garside element)

Definition

For $1 \leq i < j \leq n$, let

$$a_{i,j} = (\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1})^{-1}\sigma_i(\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1})$$

The generating set $\Sigma^* = \{a_{i,j}\}_{1 \leq i < j \leq n}$ is called a *dual Garside generator* (Birman-Ko-Lee generator or band generator).



Dual Garside structure

$$B_n^{+*} = \{\text{Product of positive } a_{i,j}\} : \text{Dual positive monoid}$$
$$\delta = \sigma_1\sigma_2\cdots\sigma_{n-1} = a_{1,2}a_{2,3}\cdots a_{n-1,n} : \text{Dual Garside element}$$

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- $\preccurlyeq$ admits the greatest common divisor
$$x \wedge^* y = \max_{\preccurlyeq^*}\{z \in B_n \mid z \preccurlyeq^* x, y\}$$
- $\preccurlyeq$ admits the least common multiple
$$x \vee^* y = \min_{\preccurlyeq^*}\{z \in B_n \mid x, y \preccurlyeq^* z\}$$
- $a_{i,j} \preccurlyeq^* \delta$ for all $1 \leq i < j \leq n$.

Dual Garside structure

Definition

A dual simple braid is a braid that satisfies $1 \preccurlyeq^* x \preccurlyeq^* \delta$.

$$[1,\delta] = \{\beta \in B_n \mid 1 \preccurlyeq^* \beta \preccurlyeq^* \delta\} = \{\text{Dual simple braids}\}$$

Theorem-Definition (Birman-Ko-Lee)

A braid $\beta \in B_n$ admits the unique the normal form (dual Garside normal form)

$$N^*(\beta) = \delta^p d_1 d_2 \cdots d_r \quad (p \in \mathbb{Z},\ x_i \in [1,\delta])$$

which is characterized by

1. $p = \min\{n \in \mathbb{Z} \mid \delta^n \beta \in B_n^{+*}\}$
2. $x_i = \delta \wedge^* (d_{i-1}^{-1} \cdots d_1^{-1} \delta^{-p} \beta)$.

We define the dual supremum, dual infimum of β by

$$\sup{}^*(\beta) = p + r, \inf{}^*(\beta) = p$$

Dual Garside structure

A parallel argument applies for the dual Garside structure:

Theorem (Birman-Ko-Lee)

The dual normal form provides an automatic structure.

Theorem (Birman-Ko-Lee)

An appropriate modification of dual normal form provides a geodesic word with respect to the length $\ell_{[1,\delta]}$. In particular,

$$\ell_{[1,\delta]}(\beta) = \max\{\sup{}^*(\beta), 0\} - \min\{\inf{}^*(\beta), 0\}.$$

By the similar method, one can use dual normal form to solve the conjugacy problem.

Dual Garside structure

Example: 3-braid case

$\delta = a_{1,2}a_{2,3} = a_{2,3}a_{1,3} = a_{1,3}a_{1,2}$, so

$$[1,\delta] = \{1, a_{1,2}, a_{2,3}, a_{1,3}, \delta\}$$

Recall that: Classical simple elements $[1,\Delta] \overset{1:1}{\leftrightarrow}$ Permutations S_n

Dual Garside structure

Example: 3-braid case

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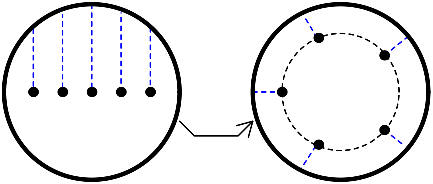
Recall that: Classical simple elements $[1,\Delta] \overset{1:1}{\leftrightarrow}$ Permutations S_n

What is the (combinatorial) meaning of dual simple elements ?

To treat dual Garside elements, it is convenient to n-punctured disc D_n with circular symmetry:



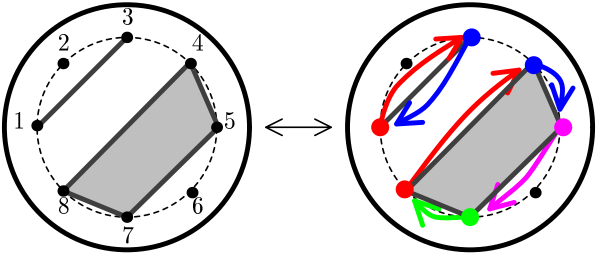
A geometric understanding of dual simple elements

Let us identify B_n with $MCG(D_n)$. Then,

Proposition (Bessis)

$$\{\text{Set of convex polygons in } D_n\} \xleftrightarrow{1:1} [1,\delta]$$

(Convex polygons is understood as non-crossing partition of n-points)



$$(a_{1,3})(a_{4,5}a_{5,7}a_{7,8})$$

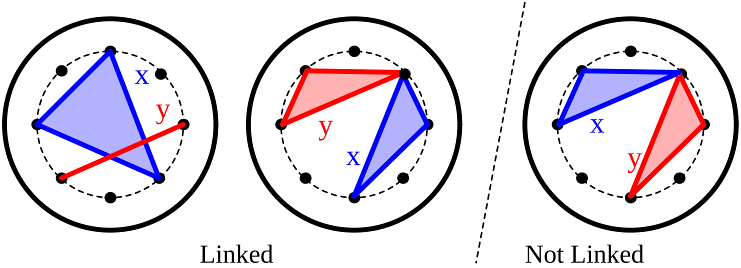
A geometric understanding of the normal form condition

Like classical Garside case, we have geometric useful interpretation of the normal form condition $\delta \wedge^* (xy) = x$.

Proposition

For $x, y \in [1, \delta]$,

$$\delta \wedge^* (xy) = x \iff \text{Corresponding convex polygons } x \text{ are "linked" to } y$$



Open problem

Open problem

Are there other "Garside structures" (i.e. the submonoid P and element Δ which allows us to develop a machinery for normal forms) for B_n ?

Open problem

Clarify the meaning of the word "dual":
Currently, we use the name "dual" Garside structure because of numerical correspondence of several data of the Garside structures (numbers of atoms, simple elements, ...) and there is no theoretical "duality" at all !

I-3: Application to topology (1)
Nielsen-Thurston classification

Nielsen-Thurston theory

According to the dynamics of $B_n \cong MCG(D_n)$, a braid β viewed as a homeomorphism, $\beta : D_n \to D_n$ is classified into one of the following three types: Periodic, reducible, pseudo-Anosov

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$\beta^n = \Delta^{2m}$ for some $n, m \in \mathbb{Z}$
(i.e., Powers of β = Dehn twists along ∂D_n)

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1: Periodic

$\beta^n = \Delta^{2m}$ for some $n, m \in \mathbb{Z}$
(i.e., Powers of β = Dehn twists along ∂D_n)

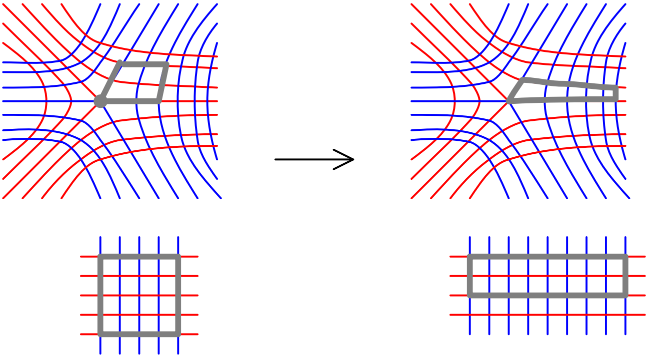
2: Reducible

$\beta(C) = C$ for some essential simple closed curves $C \subset D_n$
(A simple curve is essential $\iff$ C encloses more than one punctures and is not isotopic to ∂D_n)

Nielsen-Thurston theory

3: Pseudo-Anosov

β is a pseudo-Anosov homomorphism (locally, there are β is λ-expanding in one direction and λ-shrinking in transverse direction for some $\lambda > 1$
(This λ is called the dilatation)



Nielsen-Thurston theory

Knowing the Nielsen-Thurston type is important in dynamics, topology (and algebraic properties like centralizers), so

Problem

How to determine the Nielsen-Thurston type of β ?

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How to determine the Nielsen-Thurston type of β ?

Train-track method (graph encoding of surface automorphisms) provides a solution of this problem (Bestvina-Handel algorithm).

Now, Garside theory (normal form) provides alternative solution !

Nielsen-Thurston type via Garside theory

Recognizing a periodic braid is easy:

Theorem (Eilenberg, Kerékjártó)

A periodic n-braid is conjugate to $(\sigma_1\sigma_2\cdots\sigma_{n-1})^m$ or $(\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_1)^m$. In particular, if β is periodic, then β^n or $\beta^{(n-1)}$ is a power of Δ^2.

The problem is how to recognize a reducible braid.

Why recognizing reducible braid is not so easy ? Because, β may preserve very,very,very complicated "simple" (so it is not simple – rather complex !!!) closed curve.

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Idea

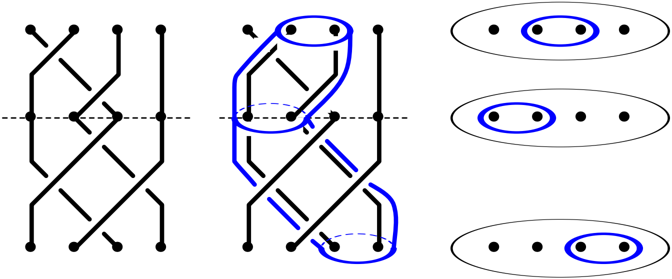
Assume β is reducible. If $N(\beta)$ is simple among its conjugacy class, then β preserves "simple" (not complicated, near "standard") simple closed curves.

Simple normal form $\Longleftrightarrow$ Preserving "simple" simple closed curve

Easy, but informative observation

Observation

For simple braids x, y, if xy is a normal form preserving "standard" round curve patterns, then x and y also preserves such a curve pattern.

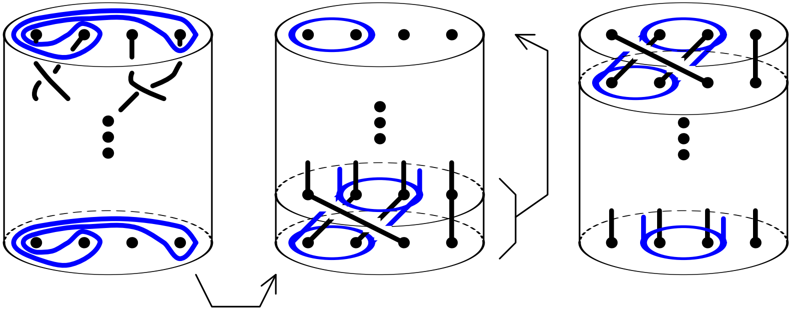


Nielsen-Thurston type via Garside theory

Theorem (Barnadete-Nitecki-Gutiérrez '95)

If β is reducible, then there exists $\alpha \in US(\beta) \subset SS(\beta)$ such that α preserves standard a round curve. Thus by computing $US(\beta)$ or $SS(\beta)$, we can determine whether β is reducible or not.

Proof: If β is reducible, by conjugating, β preserves standard round curve. By previous observation, (de)cycling of β has the same property.



Nielsen-Thurston type via Garside theory

Drawback

The theorem says at least one element in $US(\beta)$ is very nice (preserves round curves). But, computing all $US(\beta)$ may be hard (may require exponential time !)

Nielsen-Thurston type via Garside theory

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Reasonably-sounding result

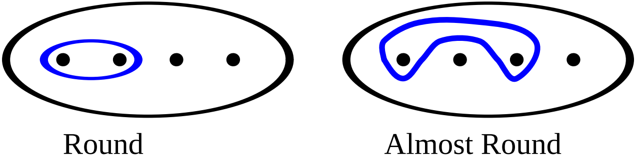
An element of $US(\beta)$ has the "simplest" normal form, so if β is reducible, elements of all $US(\beta)$ preserves the simplest, a standard round curve.

This is true under some assumptions (Lee-Lee '08), but is not true in general: (think appropriate simple element, for example)

Fast Nielsen-Thurston type via Garside theory

Theorem (González-Meneses, Wiest '11)

If β is reducible, then after taking m-th power β^m for some $m < n^6$, every element in $\alpha \in SC(\beta^m)$ preserves either standard round curves or, almost round curves. (Here $SC \subset US$ is a sliding circuit, a more refinement of the Ultra summit set)



Conclusion

Having simple normal form (simple in algebraic prospect) = Having simple reduction curve (simple in geomteric prospect),

Fast Nielsen-Thurston type via Garside theory

Moreover, by applying linear bounded conjugator property

Theorem (Mazur-Minsky '00, Tao '13)

If $x, y \in B_n$ are conjugate, then $x = wxw^{-1}$, where the length of $w \in B_n$ is at most Constant $C(n) \cdot$ (length of $x + y$)))

We have (theoretically fast) algorithm:

Theorem (Calvez '14)

By using Garside theory machinery, one can determine whether β is reducible or not in quadratic time.

Remark

Unfortunately, due to the lack of our knowledge of precise value of $C(n)$, the algorithm in thw above theorem is not practical at this moment.

Questions

At this moment, our argument recognizes periodic and reducible braids.

Problem

Can we recoginze/understand pseudo-Anosov braid (dilatation, their invariant train-track) from Garside theory ?

A reasonably-sounding idea is that if α is pseudo-Anosov and $\beta \in SS(\alpha)$, then the invariant train-track of β is simple in some sense.

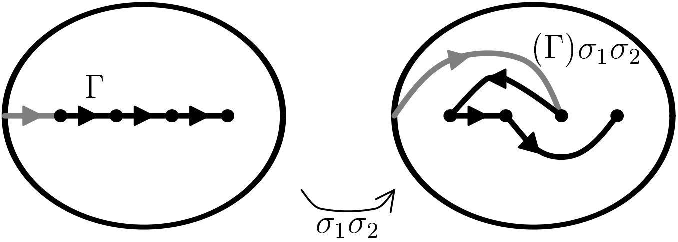
Remark

For a pseudo-Anosov braid β, then there exists $m < n^6$ such that the normal form of β^m has certain nice property called rigidity.

I-5: Application to topology (2): Curve diagram and linear representation

Curve diagram

Using identification $B_n \cong MCG(D_n)$, we can represent $\beta \in B_n$ by the (isotopy class of the) image of horizontal line Γ, called Curve Diagram.

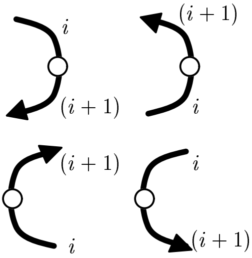


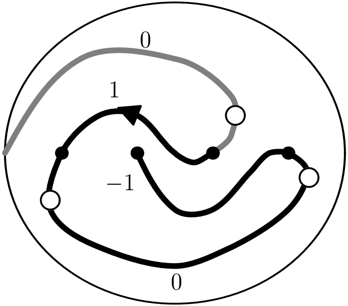
(We often distinguish the first segment e of Γ connecting the boundary and the first puncture, and define

$$\Gamma_\beta = (\Gamma - e)\beta$$

Labelling of Curve diagram I: winding number labelling

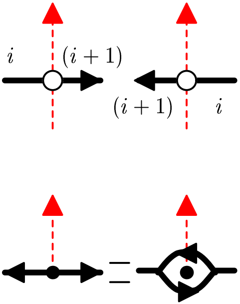
Make curve diagram so that it has minimum vertical tangencies, and assign labelling (winding number labelling) as follows: if we turn clockwise direction, add $+1$ and if we turn counter-clockwise direction, add -1

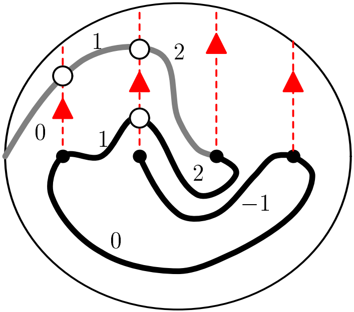




Labelling of Curve diagram II: wall-crossing number labelling

Make curve diagram so that it has minimum intersection with walls (vertical line from punctures) and that near the puncture it is horizontal. Assign labelling wall crossing labelling by signed counting of intersections with walls (here we escape puncture in counter-clockwise direction).





Labelling of Curve diagram and Garside theory

Theorem (Wiest '09)

1. max $\{$Winding number labelling on $\Gamma_\beta\} = \sup(\beta)$
2. min $\{$Winding number labelling on $\Gamma_\beta\} = \inf(\beta)$

(Classical Garside normal form measures "how many times the braid β winds real axis")

Labelling of Curve diagram and Garside theory

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Theorem (I-Wiest '12)

1. max {Wall crossing number labelling on Γ_β} $= \sup^*(\beta)$
2. min {Wall crossing number labelling on Γ_β} $= \inf^*(\beta)$

(Dual Garside normal form measures "how many times the image of the real axis across the walls")

Sketch of proof

Strategy:

- Multiply inverse of (dual) simple elements so that maximum labelling decreases

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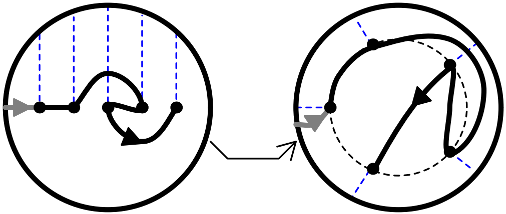
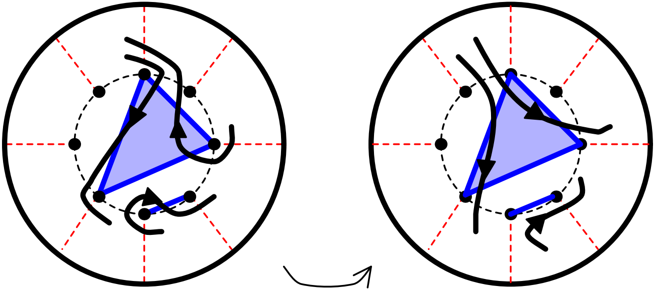
- This process provides an effective (fastest) way to make the braid trivial by using (dual) simple elements
  $\Rightarrow$ it is the meaning of normal form!

Here we give a proof for dual case: we isotope curve diagram and wall so that it has circular symmetry (wall-corssing labelling does not change).



Sketch of proof

The set of arcs in curve diagram with maximal wall-crossing labelling suggests which dual simple element is needed to simplify the diagram: the "convex hull" of maximally labelled arcs provides the most economical untangling dual simple element.



Lawrence-Krammer-Bigelow representation

C : Configration space of two points in D_n

$$C = \{(z_1, z_2) \in D_n^2 \mid z_1 \neq z_2\}/(z_1, z_2) \equiv z_2, z_1)$$

then $H_1(C; \mathbb{Z}) = \mathbb{Z}^n \oplus \mathbb{Z} = \bigoplus \langle x_i \rangle \oplus \langle t \rangle$, where

$$\begin{cases} x_i : \text{ meridian of hypersurface } \{z_1 = i\text{-th puncture}\} \\ t : \text{ meridian of hypersurface } \{z_1 = z_2\} \end{cases}$$

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Let $\pi : \widetilde{C} \to C$ be the $\mathbb{Z}^2$-cover associated with the kernel of

$$\alpha : \pi_1(C) \overset{Hurewicz}{\to} H_1(C; \mathbb{Z}) \to \mathbb{Z}^2 \cong \langle x \rangle \oplus \langle t \rangle \quad (x_i \mapsto x, t \mapsto t).$$

$H_2(\widetilde{C}; \mathbb{Z})$ is a free $\mathbb{Z}[x^{\pm 1}, t^{\pm 1}]$-module of rank $\binom{n}{2}$.

Lawrence-Krammer-Bigelow representation

The braid group $B_n = MCG(D_n)$ action on D_n induces an action on $\widetilde{C}$ (up to homotopy), so we get

$$\rho_{LKB} : B_n \to \mathrm{GL}(H_2(\widetilde{C}; \mathbb{Z}))$$

called the Lawrence-Krammer-Bigelow representation. By choosing appropriate basis $\{v_{i,j}\}_{1 \leq i < j \leq n}$ coming from topology, the LKB representation is given by

$$\rho_{LKB}(\sigma_i)(v_{j,k}) = \begin{cases} F_{j,k} & i \notin \{j-1, j, k-1, k\} \\ qF_{i,k} + (q^2 - q)F_{i,j} + (1-q)F_{j,k} & i = j-1 \\ F_{j+1,k} & i = j \neq k-1 \\ qF_{j,i} + (1-q)F_{j,k} + (q-q^2)tF_{i,k} & i = k-1 \neq j \\ F_{j,k+1} & i = k \\ -q^2 tF_{j,k} & i = j = k-1 \end{cases}$$

Lawrence-Krammer-Bigelow representation

Surprisingly, Lawrence-Krammer-Bigelow representation detects the normal forms.

Theorem (Krammer '02, I-Wiest '12)

For $\beta \in B_n$,

1. $\max\{\text{degree of } t \text{ in the matrix } \rho_{LKB}(\beta)\} = \sup(\beta)$.
2. $\min\{\text{degree of } t \text{ in the matrix } \rho_{LKB}(\beta)\} = \inf(\beta)$
3. $\max\{\text{degree of } q \text{ in the matrix } \rho_{LKB}(\beta)\} = 2\sup{}^*(\beta)$.
4. $\min\{\text{degree of } q \text{ in the matrix } \rho_{LKB}(\beta)\} = 2\inf{}^*(\beta)$

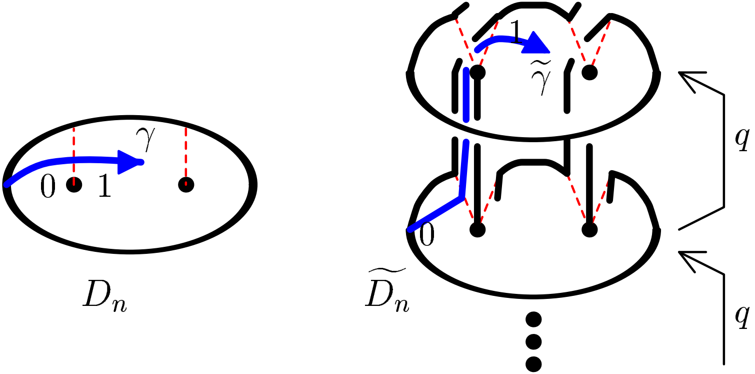
Corollary (Krammer, Bigelow '02)

The Lawrence-Krammer-Bigelow representation is faithful – so, the braid groups are linear.

Why LKB representation know the Garside structures ?

Compare the definition of $\alpha : \pi_1(\widetilde{C}) \to \langle x \rangle \oplus \langle t \rangle$ with the definition of labelling of curve diagram:

Labelling $=$ Position of the lift of the curve $\cong$ variables q and t.



Quantum representation

By theory of quantum group, for a $U_q(\mathfrak{g})$-module V, (quantum enveloping algebra of semi-simple lie algebra $\mathfrak{g}$), we have a linear representation called quantum representations

$$\rho_V : B_n \to \mathrm{GL}(V^{\otimes n})$$

that is a q-deformation of permutation

$$\phi_V : S_n \to \mathrm{GL}(V^{\otimes n}),$$

$$(i, i+1)(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes v_n$$

Quantum representations are important because they produces invariants of knto and 3-manifolds, called *Quantum invariants*.

Quantum representation and invariants

$$
\begin{array}{ccccc}
\{\text{Braids}\} & \xrightarrow{\textit{Closure}} & \{(\text{Oriented})\ \text{Links}\ \} & \xrightarrow{\textit{Surgery}} & \{\text{Closed 3-manifolds}\} \\
\downarrow {\scriptstyle \rho_V}\ \textit{Quantum representation} & & \downarrow \textit{Quantum invariant} & & \downarrow \textit{Quantum invariant} \\
GL(V^{\otimes n}) & \xrightarrow{\text{“}\textit{Trace}\text{''}} & \mathbb{C}[q, q^{-1}] & \xrightarrow[\text{Take linear sums}]{q=e^{\frac{2\pi\sqrt{-1}}{N}}} & \mathbb{C}
\end{array}
$$

Quantum representation and Garside theory

Using KZ-equation argument (realizing quantum representation as certain monodromy representation), one identifies "generic" quantum $\mathfrak{sl}_2$ representation with homological representation similar to Lawrence-Krammer-Bigelow representation (Kohno,I, Jackson-Kerler). Then, we have:

Theorem (I. '12)

For $\beta \in B_n$. the maximal and the minimal degree of weight variable in "Generic" quantum $\mathfrak{sl}_2$-representation is equal to the constant multiples of $\sup{}^*(\beta)$ and $\inf{}^*(\beta)$.

$\Rightarrow$ Quantum representation (quantum group) is also related to (dual) Garside structure.

Problems

Problem

Find a relationship between linear representations and the classical Garside structure: Conjecturally, it should be related to the quantum parameter q.

Problem

Find a relationship between quantum knots or3-manifold invariants (for example, Jones polynomial) and Garside theory.

Problem

Find a direct, more conceptual understanding between quantum representation and Garside structure.