Rectangular Diagrams and Jones' Conjecture II: Legendrian Graphs, bypasses, and simplifying discs

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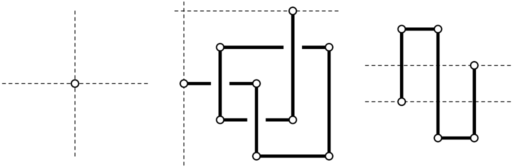
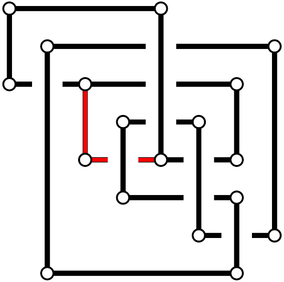
Rectangular paths



Bypasses

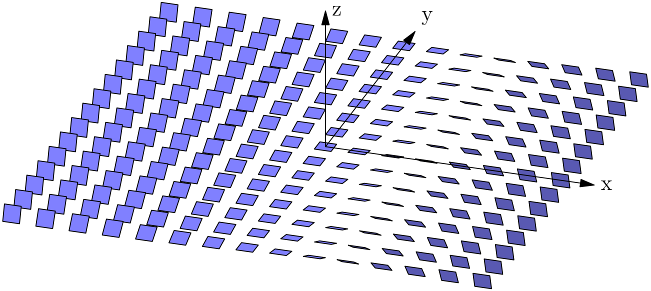


Rectangular diagram with rectangular path attached = Θ−diagram.

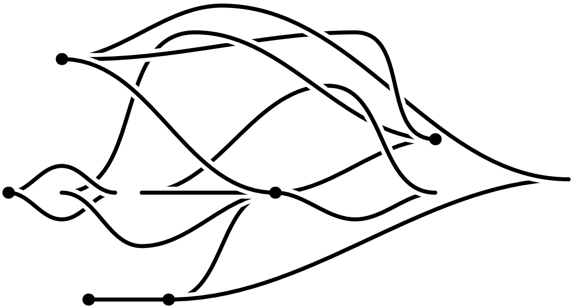
Standard Contact Structure

The *standard contact structure* $\xi_{\rm std}$ in $\mathbb{R}^3$ is a plane distribution defined by the kernel of a 1-form

$$\alpha_{\rm std} = dz + x dy.$$

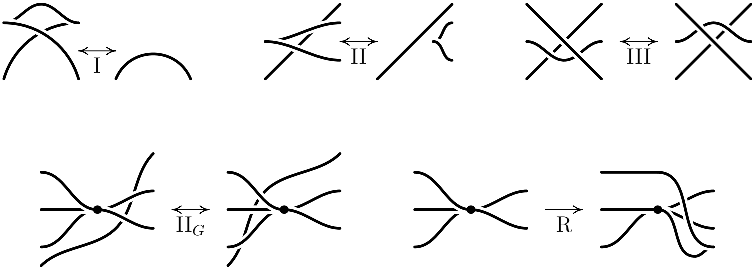


Fronts



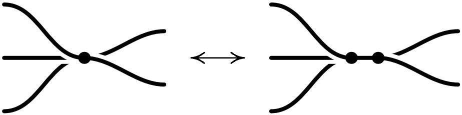
Front Projection of a Legendrian graph.

Front moves

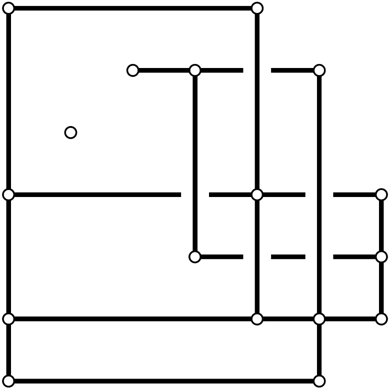


Theorem. (Baader and Ishikawa, 2009) Two generic fronts represent Legendrian isotopic Legendrian graphs iff they are related by moves which are illustrated on the picture.

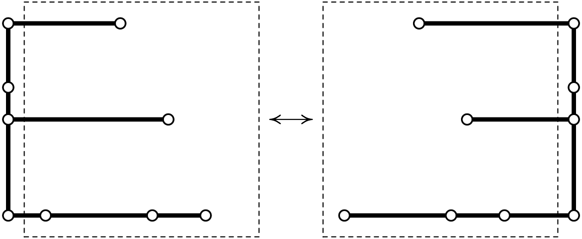
Non-isotopy move: Blow-up and edge contraction



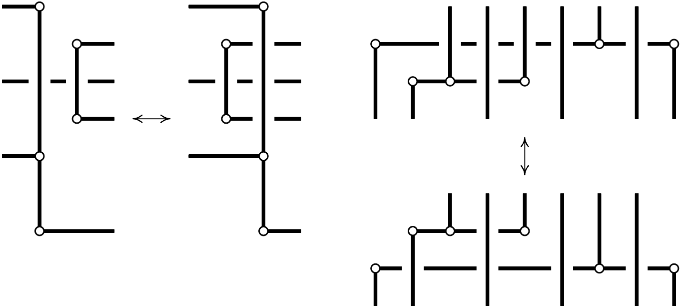
Generalized Rectangular Diagrams



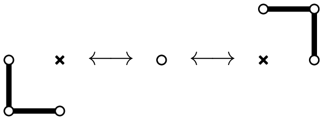
Elementary moves: cyclic permutation



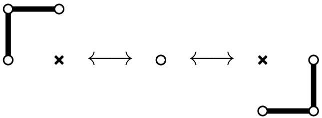
Elementary moves: commutation



Elementary moves: (de)stabilization

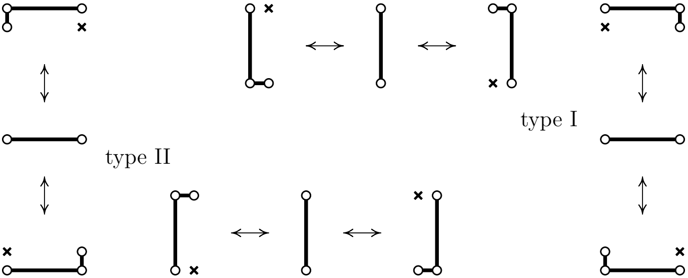


type I

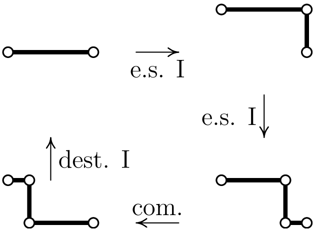


type II

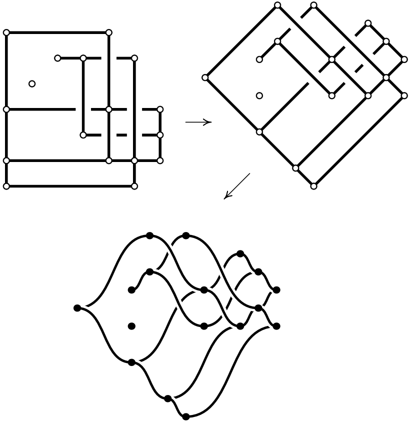
Elementary moves: end shift



Elementary moves: inverting an end shift



From rectangular diagrams to Legendrian graphs



From rectangular diagrams to Legendrian graphs

Theorem. The map $R \mapsto G_R$ induces a bijection between classes of generalized rectangular diagrams modulo elementary moves of type I and Legendrian graphs modulo Legendrian isotopy and edge contraction / blow-up.

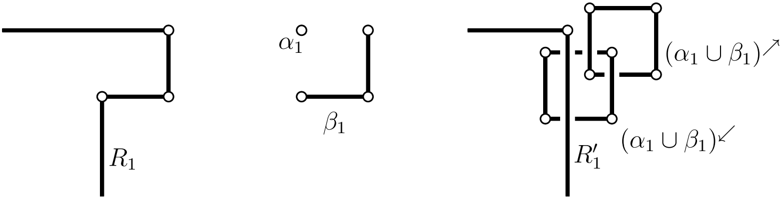
Bypasses

By *a bypass* for a rectangular diagram R we call an ordered pair (α, β) of rectangular paths having common ends such that β is a subset of R, and there exists an embedded two-dimensional disc $D \subset \mathbb{R}^3$ satisfying the following:

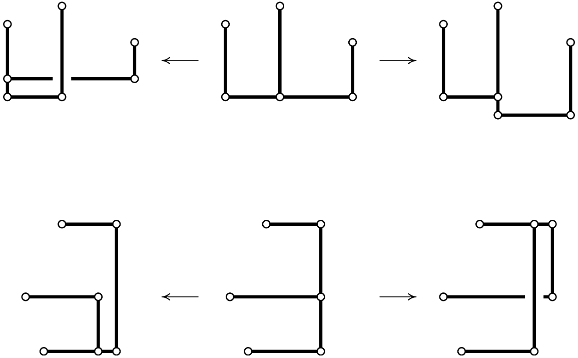
- the disc boundary ∂D coincides with $\widetilde{\alpha} \cup \widetilde{\beta}$;
- the intersection $D \cap \widetilde{R}$ coincides with $\widetilde{\beta}$;
- in the link defined by the rectangular diagram $(R \setminus \beta) \cup \alpha \cup (\alpha \cup \beta)^{\nearrow} \cup (\alpha \cup \beta)^{\swarrow}$, the components presented by $(R \setminus \beta) \cup \alpha$ are unlinked with the two others.

A bypass (α, β) is called *elementary* if we have $\mathrm{tb}(\alpha \cup \beta) = 1$. The value $\mathrm{tb}(\alpha \cup \beta)$ will be called *the weight* of the bypass (α, β).

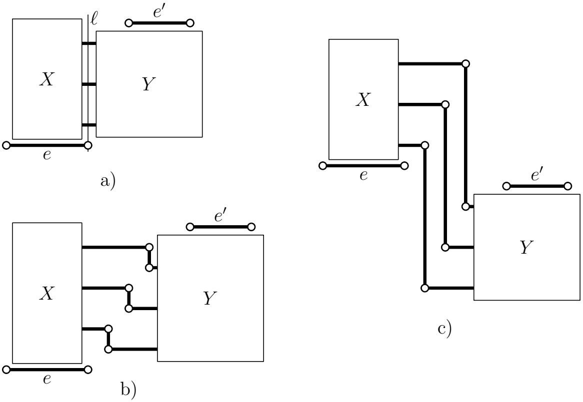
Elementary bypass



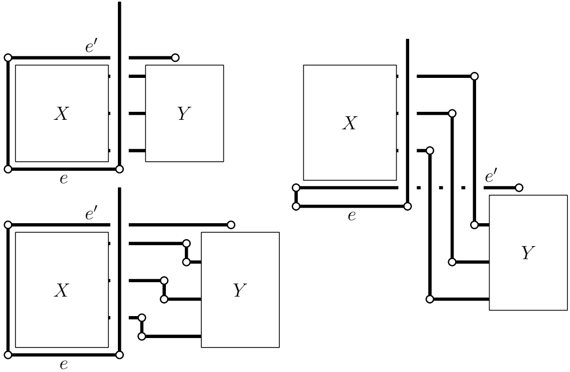
Applying an end shift to the Θ−diagram



An attached path is avoiding the diagram during Legendrian move



An attached path is avoiding the diagram during Legendrian move



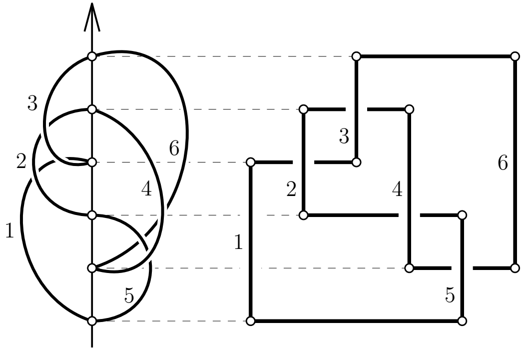
Bypasses

Proposition. Let R and R' be Legendrian equivalent rectangular diagrams such that α is a bypass of weight b for R. Then there exists a bypass α' of weight b such that $\Theta-$diagrams $R \cup \alpha$ and $R' \cup \alpha'$ are Legendrian equivalent.

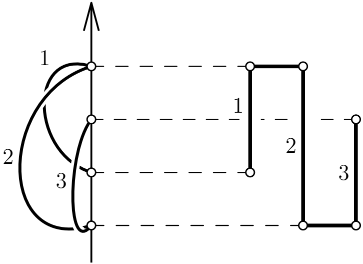
Plan

- With the rectangular diagram R and the rectangular paths α and β we associate geometrical objects $\hat{R}, \hat{\alpha}, \hat{\beta}$ in $\mathbb{R}^3$ that are called arc presentations.
- We span the trivial knot $\widehat{\alpha \cup \beta}$ by a disc D whose open book foliation obey certain restrictions.
- Then we apply induction. For the induction step we modify $\hat{R} \cup D$ in a certain way so that the disc D gets simpler. As a result, type N destabilizations may occur on the path β, type L destabilizations on α, and commutations as well as cyclic permutations may occur everywhere in the Θ-diagram $R \cup \alpha$.

Arc-presentation



Arc-presentation



Exercise

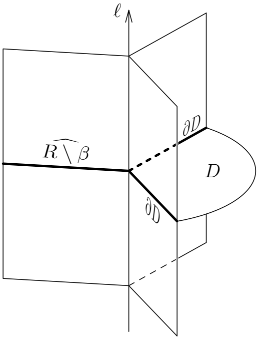
$\hat{R}$ is isotopic to $\tilde{R}$.

Restrictions on the disc

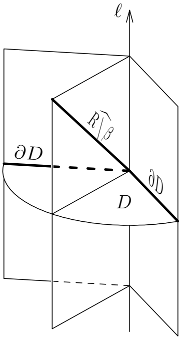
If (α, β) is a bypass for R then there exists a disc D such that

- the boundary ∂D coincides with $\widehat{\alpha \cup \beta}$, and the interior of D is disjoint from $\hat{R} \cup \hat{\alpha}$;
- D is a smooth image of a polygon whose vertices map exactly to vertices of $\widehat{\alpha \cup \beta}$;
- D intersects the binding of open book transversely finitely many times;
- for any common endpoint of $\hat{\alpha}$ and $\hat{\beta}$ the arc of $\widehat{R \setminus \beta}$ coming from the endpoint lies outside the $\theta-$interval occupied by D;

Suitable disc: the 4th restriction



correct

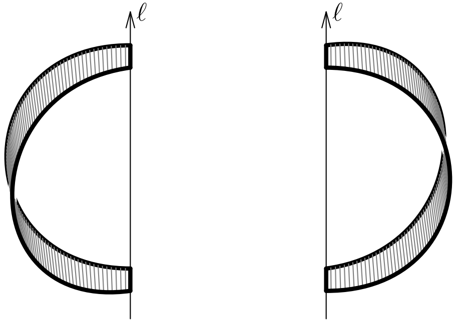


incorrect

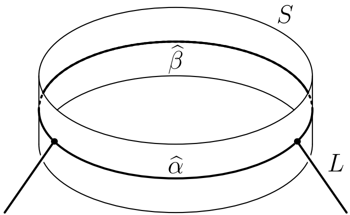
Suitable disc

- the open book foliation defined on D outside the binding has only simple saddle singularities inside D and have no regular closed fibers;
- at intersection points of ∂D with the binding the coorientation of D is induced by the vector $\frac{\partial}{\partial z}$;
- there is exactly one positive (respectively, negative) half-saddle of the foliation at every arc of α (respectively, β);
- all saddles and half-saddles lie in distinct pages.

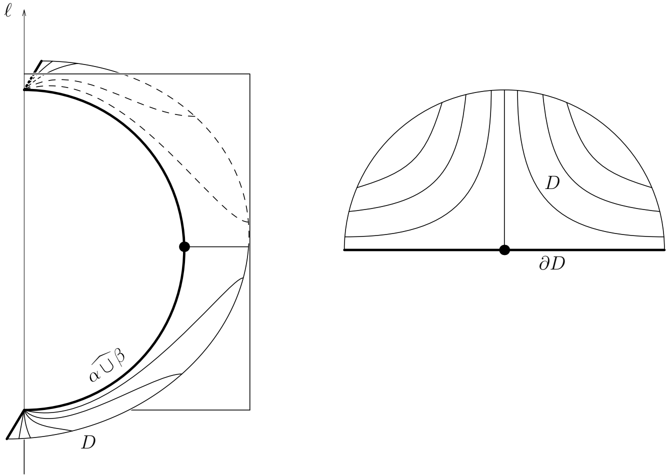
Strips forming the band $\widehat{\alpha^{\nearrow\swarrow}} \cup \widehat{\beta^{\nwarrow\searrow}}$



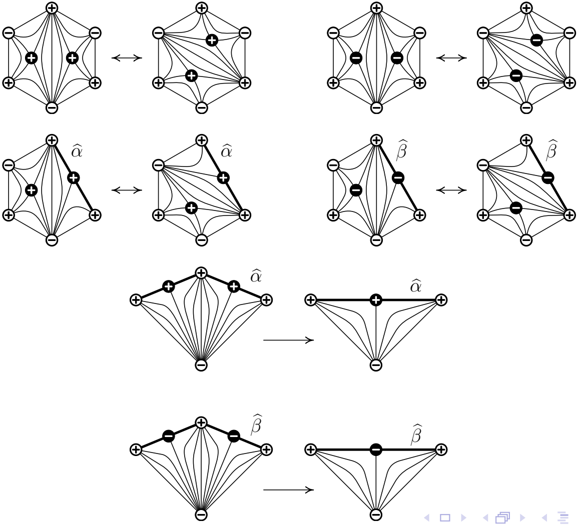
Band $\widehat{\alpha^{\nearrow\swarrow}} \cup \widehat{\beta^{\nwarrow\searrow}}$ viewing topologically



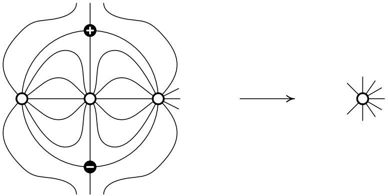
A disc D near the boundary



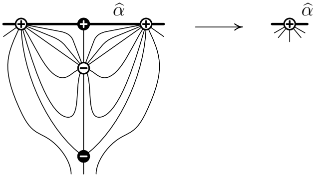
Simplifying the charactaristic foliation: rearranging saddles

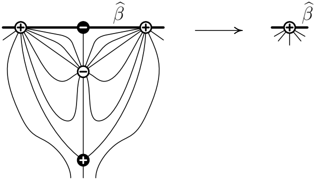


Simplifying the charactaristic foliation: removing a 2-valent vertex

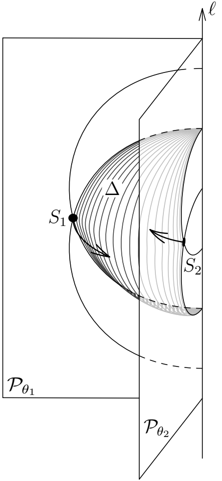


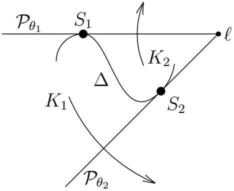
Simplifying the charactaristic foliation: removing a 2-valent vertex



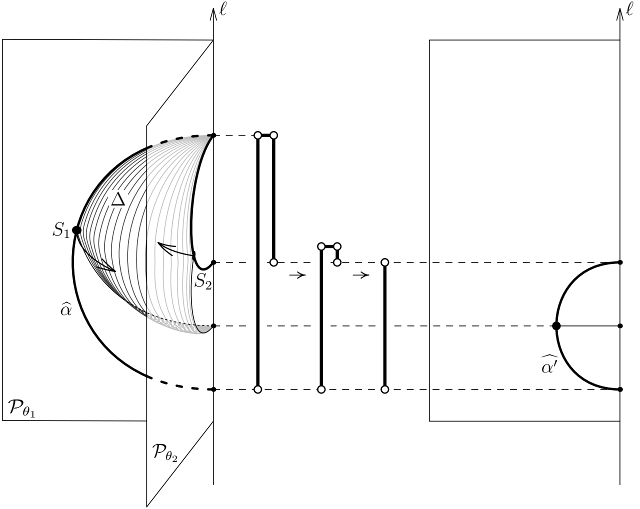


Rearranging of saddles

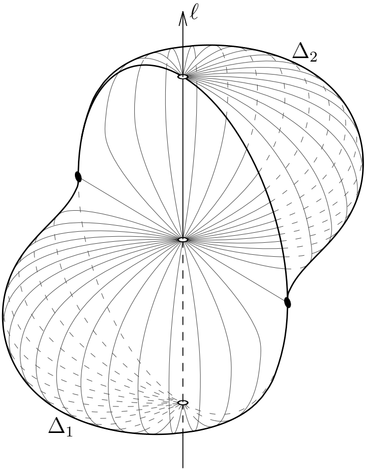




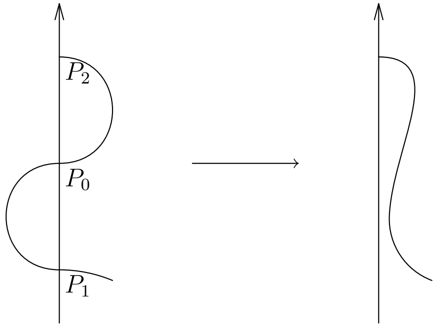
Rearranging of saddles at boundary



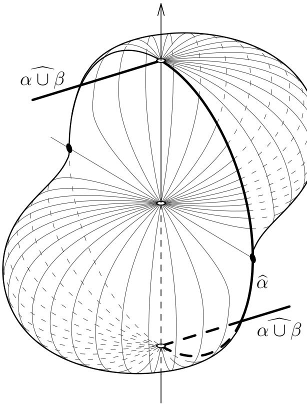
Wrinkle



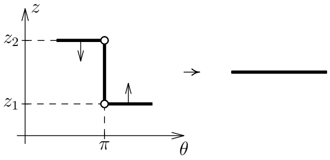
Removing a wrinkle



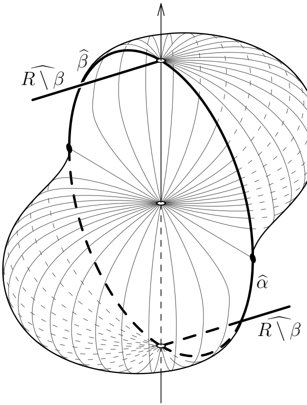
Wrinkle near the boundary



Removing a wrinkle near the boundary yields a destabilization



The disc consisting of a single wrinkle



The disc consisting of a single wrinkle