Intersection of quadrics and Abelian varieties: An experimental study

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• Based on the results of F. Kötter, W.Barth, M.Adler, P. van Moerbeke, L.Heine, E. Horozov, V. Enolski
• Inspired by HK discretization of the Clebsch case (M. Petrera, A. Pfadler, Yu. Suris)
• Could not be made without assistance of V.Enolski, L. Garcia Naranjo.

Intersection of two quadrics in $\mathbb{P}^3$
An example: The classical Euler top on $so(3)$

$$\dot{M} = M \times aM\,, \qquad M = (M_1, M_2, M_3)^T \in \mathbb{C}^3, \quad a = \mathrm{diag}(a_1, a_2, a_3)$$

Constants of motion

$$\langle M, aM\rangle = \underbrace{a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 = I}_{\text{energy}}, \quad \langle M, M\rangle = \underbrace{M_1^2 + M_2^2 + M_3^2 = k^2}_{\text{momentum}}\,.$$

Complex invariant manifold $\mathcal{I}$ = intersection of 2 quadrics in $\mathbb{C}^3$
= an open subset of the elliptic curve
$E\ : \mu^2 = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c)$
A (local) parametrization of $\mathcal{I}$ in terms of $\lambda \in \mathbb{C}$ (Jacobi)

$$M_i(\lambda) = k\sqrt{\frac{(a_j - c)(a_k - c)}{(a_i - a_j)(a_i - a_k)}}\sqrt{\frac{\lambda - a_i}{\lambda - c}}\,, \quad (i, j, k) = (1, 2, 3) \qquad c = I/k^2$$

In the real case, when $a_1 < a_2 < a_3$, one has $c \in [a_1, a_3]$.

The Weierstrass form of E:

$$E \longleftrightarrow \{w^2 = 4(z-e_1)(z-e_2)(z-e_3), \quad e_1+e_2+e_3=0\},$$
$$\frac{\lambda - a_i}{\lambda - c} = z - e_i, \quad i = 1,2,3.$$

$$u = \int_\infty^z \frac{dx}{2\sqrt{(x-e_1)(x-e_2)(x-e_3)}} \iff z = \wp(u|\omega_1,\omega_3),\ w = \frac{d}{du}\wp(u|\omega_1,\omega_3),$$

$\omega_1, \omega_3, \omega_2 = -\omega_1 - \omega_3$ being half-periods of $\wp$ and $e_i = \wp(\omega_i)$.

- Rational parameterization of the elliptic curve in $\mathbb{C}^3$ in terms of Z, W:

$$M_i = \beta_i \frac{Z^2 - 2e_iZ + e_ie_j + e_ie_k - e_je_k}{W},$$
$$\beta_i = \frac{k\sqrt[6]{2(c-a_1)(c-a_2)(c-a_3)}}{\sqrt{(a_i-a_j)(a_i-a_k)}}, \qquad (i,j,k) = (1,2,3),$$
$$\tilde{E} = \left\{W^2 = 4(Z-e_1)(Z-e_2)(Z-e_3)\right\}.$$

Intersection of 4 quadrics in $\mathbb{P}^6(x_1 : \dots : x_6 : x_0)$ and Abelian varieties

A rank k quadric $Q(x) = \{\sum_{i,j=0}^{6} q_{ij} x_i x_j = 0\}$, k=rank q is a projective closure of an affine quadric $\bar{Q} \in \mathbb{C}^6(X_1, \dots, X_6)$:

$$\sum_{i,j=1}^{n} q_{ij} X_i X_j + \dots + \sum_{i=1}^{n} q_{i0} X_i = -q_{00}, \quad X_i = \frac{x_i}{x_0}$$

The space W of all quadrics in $\mathbb{P}^6$ admits stratification

$$W_6 \subset \dots \subset \underbrace{W_2}_{\text{corank 2 quadrics}} \subset \underbrace{W_1}_{\text{corank 1 quadrics}} \subset W$$

For given four quadrics $Q_0(x), \dots Q_3(x) \in \mathbb{P}^6$ consider their *linear system*

$$\Lambda_3 = \{t_0 Q_0(x) + \dots + t_3 Q_3(x) = 0\}, \quad (t_0 : t_1 : t_2 : t_3) = \mathbb{P}^3$$

with stratification

$$\tilde{W}_6 \subset \dots \subset \tilde{W}_2 \subset \tilde{W}_1 \subset \Lambda_3, \quad \tilde{W}_k = W_k \cap \Lambda_3,$$

The discriminant surface $\widetilde{W}_1 = \{ \left| t_0 q_0 + \dots + t_3 q_3 \right| = 0\} \subset \Lambda_3$ defines the set of degenerate quadrics. For a general Λ_3, codim $\tilde{W}_k = k$.
If this does not hold, i.e., dim $\tilde{W}_k > 3 - k$, Λ_3 is *singular*.

Theorem (Grifiths, Harris)

The basis variety $\mathcal{B} = Q_0 \cap Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^6$ of a linear system Λ_3 is singular if and only if Λ_3 is singular.

Let $\mathcal{I}_c$ be intersection of 4 quadrics $Q_0(x), \ldots, Q_3(x)$ of the form[1]

$$\sum_{i=1}^{3} \left(a_i x_i^2 + a_{i+3} x_{x+3}^2 + 2 b_{i,i+3} x_i x_{i+3} \right) = c\, x_0^2$$

and $\mathcal{S} \subset \mathcal{I}_c \cap \{x_0 = 0\}$ be its singular set at infinity (if any).

Theorem (Adler, van Moerbeke)

$\mathcal{I}_c \setminus \mathcal{S}$ is an open subset of a 2-dim. Abelian variety $\mathcal{A}$ iff the discriminant surface $\widetilde{W}_1$ contains
1) a union of lines (rational curves) of quadrics of rank ≤ 4, or
2) an irreducible non-degenerate elliptic curve $\mathcal{E}$ of quadrics of rank ≤ 4.
In the first case $\mathcal{A}$ is the Jacobian of a genus 2 curve,
in the second case $\mathcal{A}$ has polarization (1,2) and can be regarded as Prym(C, E) *of a double covering $C \to E$ (genus(C)=3).*

[1] This condition can be relaxed.

Prym(C,E) with polarization (1,2) is not a Jacobian variety. To get an algebraic description of Prym(C,E) we use

Theorem (Horozov, van Moerbeke)

There are exactly 6 different genus 2 curves related to Prym(C,E) *via 2:1 isogenies.*

$$
\begin{array}{ccccc}
\mathrm{Jac}(\widetilde{\Gamma}_1) & & \mathrm{Jac}(\widetilde{\Gamma}_2) & & \mathrm{Jac}(\widetilde{\Gamma}_3) \\
\| & \searrow & \downarrow & \swarrow & \| \\
\| & & \mathrm{Prym}(C,\sigma) & & \| \\
\Downarrow & \swarrow & \downarrow & \searrow & \Downarrow \\
\mathrm{Jac}(\Gamma_1) & & \mathrm{Jac}(\Gamma_2) & & \mathrm{Jac}(\Gamma_3) \\
\| & \searrow & \downarrow & \swarrow & \| \\
\| & & \mathrm{Prym}^*(C,\sigma) & & \| \\
\Downarrow & \swarrow & \downarrow & \searrow & \Downarrow \\
\mathrm{Jac}(\widetilde{\Gamma}_1) & & \mathrm{Jac}(\widetilde{\Gamma}_2) & & \mathrm{Jac}(\widetilde{\Gamma}_3)
\end{array}
$$

Given the curves C, E, explicit equations of all the six genus 2 curves $\Gamma_i, \widetilde{\Gamma}_i$ have been calculated in

V. Enolski, Yu. Fedorov.
arXiv:1411.6143v1 [nlin.SI] 22 Nov 2014

Example. The Clebsch integrable case on $e^*(3) = (K, p)$.

The Kirchhoff equations possess four independent quadratic integrals

$$
\begin{aligned}
Q_1 &= (K, K) - (c_2 + c_3)p_1^2 - (c_3 + c_1)p_2^2 - (c_1 + c_2)p_3^2\,,\\
Q_2 &= (p, p)\,,\\
Q_3 &= (K, p)\,,\\
Q_0 &= (K, DK) - \det D(p, D^{-1}p)\,, \qquad D = \operatorname{diag}(c_1, c_2, c_3).
\end{aligned}
$$

Set

$$
\begin{gathered}
Q_\alpha = h_\alpha, \quad Q_0 = h_0, \qquad h_\alpha, h_0 = \text{const},\\
K_\alpha = x_\alpha/x_0\,, \quad p_\alpha = x_{\alpha+3}/x_0, \qquad \alpha = 1, 2, 3
\end{gathered}
$$

Then the integrals define the quadratic forms

$$
\begin{aligned}
\overline{Q}_1(x) &= x_1^2 + x_2^2 + x_3^2 - (c_2 + c_3)x_4^2 - (c_3 + c_1)x_5^2 - (c_1 + c_2)x_6^2 - h_1 x_0^2\,,\\
\overline{Q}_2(x) &= x_4^2 + x_5^2 + x_6^2 - h_2 x_0^2\,,\\
\overline{Q}_3(x) &= x_1 x_4 + x_2 x_5 + x_3 x_6 - h_3 x_0^2\,,\\
\overline{Q}_0(x) &= -c_1 x_1^2 - c_2 x_2^2 - c_3 x_3^2 + c_2 c_3 x_4^2 + c_1 c_3 x_5^2 + c_1 c_2 x_6^2 + h_0 x_0^2
\end{aligned}
$$

and the linear system $\Lambda_3 = \left\{ \sum_{k=0}^{3} t_k \overline{Q}_k(x) = 0 \right\}$.

The discriminant surface $\widetilde{W_1} \subset \Lambda_3(t_0 : t_1 : t_2 : t_3)$ is

$$\begin{vmatrix} t_1\mathbf{I} - t_0 D & t_3\mathbf{I} & \mathbf{0} \\ t_3\mathbf{I} & t_0 D'' + t_2\mathbf{I} - t_1 D' & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & f(t) \end{vmatrix} = 0\,,$$

$$D' = \mathrm{diag}\,(c_2 + c_3, c_3 + c_1, c_1 + c_2)\,, \quad D'' = \mathrm{diag}\,(c_2 c_3, c_3 c_1, c_1 c_2)\,,$$
$$f(t) = -t_0 h_0 - t_1 h_1 - t_2 h_2 - t_3 h_3\,.$$

An unexpected result: $\widetilde{W_1}$ is a union of 3 cones and a plane:
$\widetilde{W_1} = C_1 \cup C_2 \cup C_3 \cup H_d$,

$$C_\alpha = \left\{(t_1 - t_0 c_\alpha)(t_2 + c_\beta c_\gamma t_0 - (c_\beta + c_\gamma)t_1) = t_3^2\right\}\,, \quad (\alpha, \beta, \gamma) = (1, 2, 3),$$
$$H_d = \{t_0 h_0 + t_1 h_1 + t_2 h_2 + t_3 h_3 = 0\}\,.$$

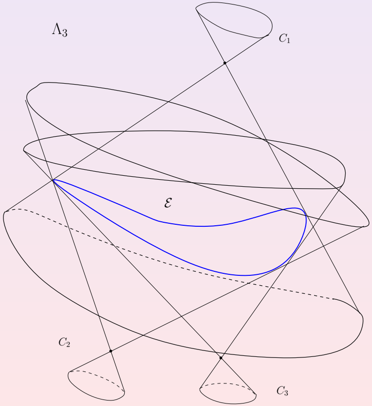
Moreover, $\mathcal{E} = C_1 \cap C_2 \cap C_3$ is not a finite number of points in Λ_3, as one might expect, but a spatial curve

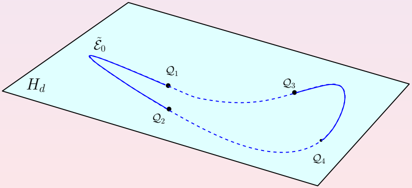
$$\mathcal{E} = \left\{t_2 = t_1^2 t_0\right\} \cap \left\{t_3^2 = (t_1 - c_1 t_0)(t_2 - (c_2 + c_3)t_1 + c_2 c_3 t_0)\right\}$$

The one-to-one projection of $\mathcal{E}$ onto the plane $(t_1 : t_3 : t_0)$ is the elliptic curve

$$\mathcal{E}_0 = \left\{ t_3^2 = (t_1 - c_1)(t_1 - c_2)(t_1 - c_3) \right\} \, .$$

Along $\mathcal{E}$ the discriminant surface $\widetilde{W_1}$ has triple self-intersections, any point of $\mathcal{E} \subset \Lambda_3$ is a quadric in $\mathbb{P}^6$ of rank ≤ 4.
4 points of intersection of H_d with $\mathcal{E}$ correspond to quadrics of rank 3.





Parametrization of $\mathcal{E}$:

$$\mathcal{E} = \left\{ t_0 = 1, \;\; t_1 = t, \;\; t_2 = t^2, \;\; t_3 = \sqrt{\Psi(t)} \right\} ,$$
$$\Psi(t) = (t - c_1)(t - c_2)(t - c_3) , \qquad t \in \mathbb{C} .$$

Then we have family of quadrics in $\mathbb{P}^6$ of rank ≤ 4:

$$\mathcal{E} \; : \; \sum_{\alpha=1}^{3} \left(\sqrt{t - c_\alpha}\, x_\alpha + \frac{\sqrt{\Psi(t)}}{\sqrt{t - c_\alpha}} x_{\alpha+3} \right)^2 = \left(h_2 t^2 + h_1 t + h_0 + 2 h_3 \sqrt{\Psi(t)} \right) x_0^2 .$$

This family was first found by F. Kötter (1891), who used it to obtain explicit solutions of the Clebsch system:

Let $\{ Q_1 = (s_1, \sqrt{\Psi(s_1)}), \dots, Q_4 = (s_1, \sqrt{\Psi(s_1)}) = \mathcal{E} \cap H_d$:

$$\sum_{\alpha=1}^{3} \left(\sqrt{s_j - c_\alpha}\, x_\alpha + \frac{\sqrt{\Psi(s_j)}}{\sqrt{s_j - c_\alpha}} x_{\alpha+3} \right)^2 = 0, \quad j = 1, 2, 3, 4$$

$$\sum_{\alpha=1}^{3}\left(\sqrt{s_j-c_\alpha}\,x_\alpha+\frac{\sqrt{\Psi(s_j)}}{\sqrt{s_j-c_\alpha}}x_{\alpha+3}\right)^2=0,\quad j=1,2,3,4$$

Under a linear change $(x_\alpha, x_{\alpha+3}) \to (\xi_\alpha, \eta_\alpha)$ the above rank 3 quadrics gives rise to the following 3 *independent* relations

$$\sum_{\alpha=1}^{3}(\xi_\alpha+\eta_\alpha)^2=0\,,\quad \sum_{\alpha=1}^{3}\left(d_\alpha\xi_\alpha+\frac{\eta_\alpha}{d_\alpha}\right)^2=0\,,\quad \sum_{\alpha=1}^{3}\xi_\alpha\eta_\alpha=0\,,$$

defining a two-dimensional manifold in $\mathbb{P}^5=(\xi_1:\xi_2:\xi_3:\eta_1:\eta_2:\eta_3)$.

This manifold is isomorphic to the set $\mathcal{T}$ of common tangent lines ℓ of two confocal quadrics in $\mathbb{C}^3=(X_1,X_2,X_3)$ given by

$$\mathcal{Q}_j=\left\{\frac{X_1^2}{d_1^2-\nu_j}+\frac{X_2^2}{d_2^2-\nu_j}+\frac{X_3^2}{d_3^2-\nu_j}=1\right\},\quad j=1,2\,,\quad \nu_1=0\,,\ \nu_2=d_1^2d_2^2d_3^2.$$

Following H. Knörrer, $\mathcal{T}$ is an 8-fold unramified covering of $\mathrm{Jac}(\Gamma)$,

$$\Gamma:\ w^2=z(z-d_1^2)(z-d_2^2)(z-d_3^2)(z-d_1^2d_2^2d_3^2)$$

Ueber die Bewegung eines festen Körpers in einer Flüssigkeit.

(Von Herrn *Fritz Kötter*.)

. . . .

α denselben Werth beilegen und dann die Wurzeln $\sqrt{s_\beta - c_\alpha}$ entsprechend der eben angegebenen Bedingung auswählen. Ferner soll gesetzt werden:

$$(24.)\quad \begin{cases} \xi_\alpha = x_\alpha\left(\dfrac{\sqrt{(s_1-c_1)(s_1-c_2)(s_1-c_3)}}{\sqrt{s_1-c_\alpha}\sqrt{\psi'(s_1)}} + i\,\dfrac{\sqrt{(s_2-c_1)(s_2-c_2)(s_2-c_3)}}{\sqrt{s_2-c_\alpha}\sqrt{\psi'(s_2)}}\right) \\ \qquad\qquad + y_\alpha\left(\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} + i\,\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}\right), \\ \eta_\alpha = x_\alpha\left(\dfrac{\sqrt{(s_1-c_1)(s_1-c_2)(s_1-c_3)}}{\sqrt{s_1-c_\alpha}\sqrt{\psi'(s_1)}} - i\,\dfrac{\sqrt{(s_2-c_1)(s_2-c_2)(s_2-c_3)}}{\sqrt{s_2-c_\alpha}\sqrt{\psi'(s_2)}}\right) \\ \qquad\qquad + y_\alpha\left(\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} - i\,\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}\right), \end{cases}$$

$$(25.)\quad d_\alpha = \frac{\dfrac{\sqrt{s_3-c_\alpha}}{\sqrt{\psi'(s_3)}} + i\,\dfrac{\sqrt{s_4-c_\alpha}}{\sqrt{\psi'(s_4)}}}{\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} + i\,\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}}, \quad -d_\alpha^{-1} = \frac{\dfrac{\sqrt{s_3-c_\alpha}}{\sqrt{\psi'(s_3)}} - i\,\dfrac{\sqrt{s_4-c_\alpha}}{\sqrt{\psi'(s_4)}}}{\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} - i\,\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}}.$$

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$$(30.)\quad Z_\beta = \sqrt{z_\beta(z_\beta-d_1^2)(z_\beta-d_2^2)(z_\beta-d_3^2)\left(\frac{z_\beta}{d_1^2\,d_2^2\,d_3^2}-1\right)} \qquad (\beta=1,\,2),$$

Given

$$\sum_{\alpha=1}^{3}(\xi_\alpha+\eta_\alpha)^2=0\,,\quad \sum_{\alpha=1}^{3}\left(d_\alpha\xi_\alpha+\frac{\eta_\alpha}{d_\alpha}\right)^2=0\,,\quad \sum_{\alpha=1}^{3}\xi_\alpha\eta_\alpha=0\,,$$

and points $(z_1,w_1),(z_2,w_2)$ on

$$\Gamma:\; w^2=z(z-d_1^2)(z-d_2^2)(z-d_3^2)(z-d_1^2d_2^2d_3^2)$$

we have (known) parametrization of $\mathcal{T}$ in terms of theta-functions of Γ with half-integer theta-characteristics

$$\eta_\alpha=\chi(z_1,z_2,w_1,w_2)\sqrt{\frac{(z_1-d_\alpha^2)(z_2-d_\alpha^2)}{(d_\beta^2-d_\alpha^2)(d_\gamma^2-d_\alpha^2)}}=\chi(u)\,C_\alpha\frac{\theta[\Delta+\delta_\alpha](u_1,u_2)}{\theta[\Delta](u_1,u_2)}$$

and a similar expression for $\xi(u_1,u_2)$. Here the coordinates (u_1,u_2) on $\mathrm{Jac}(\Gamma)$ are the Abel image of the divisor $(z_1,w_1),(z_2,w_2)$.

On the other hand

Theorem (L. Heine (1984))

Complex invariant manifolds of the Clebsch system are open subsets of $\mathrm{Prym}(\mathcal{C},\mathcal{E})$ *related to the genus 3 spectral curve*

$$\mathcal{C}\;:\;\mu^2=h_2t^2+h_1t+h_0+2h_3\sqrt{\Psi(t)}$$

HK discretization of the Clebsch case: Experiments with the 4 quadrics

$$\begin{cases} \tilde{M} - M = \epsilon(\tilde{M} \times AM + M \times A\tilde{M} + \tilde{P} \times AP + P \times A\tilde{P}), \\ \tilde{P} - P = \epsilon(\tilde{P} \times AM + P \times A\tilde{M}). \end{cases}$$

Fixing certain values of 4 constants of motion, one arrives at intersection of 4 quadrics in $\mathbb{C}^6(M, P)$ given by (*following A. Pfadler*)

$$Q_2 = -\frac{1392728800943552687\, {M_1}^2}{6350417324533397568} + \frac{1081317358603017 1\, {M_2}^2}{5292014437111646 4}$$

$$-\frac{1392728800943552687\, M_3^2}{7990941800038586064} - \frac{15507960856265463687508220537\, {P_1}^2}{21851230352205513592008}$$

$$+\frac{4587510103810042772196 91\, {P_2}^2}{17480984281764410873 6} - \frac{5645646684352629681188 27\, {P_3}^2}{17480984281764410873 6} + M_2 P_2 = 0,$$

$$Q_3 = \frac{13249117652086408 7\, {M_1}^2}{48849364034876736} - \frac{1324911765208647\, M_2^2}{5658384667373292} + \frac{10813173586017 1\, M_3^2}{52920144371116 4}$$

$$+\frac{387699562943529715935475770 5\, P_1^2}{50286771837029244855 78} - \frac{4587701171035466426445263899\, P_2^2}{1609176698784935835384 96}$$

$$+\frac{5646092416703010165934626089\, P_3^2}{1609176698784935835384 96} + M_3 P_3 = 0,$$

$$
\begin{aligned}
Q_0 &= 1 - \frac{265468\,{P_1}^2}{1823} + \frac{9805\,P_2^2}{1823} - \frac{12071\,P_3^2}{1823} = 0,\\
Q_1 &= \frac{1081317358603017 1\,M_1^2}{52920144371116464} + \frac{38006418090887\,M_2^2}{1225983344597531} + \frac{3800641809088\,M_3^2}{13318236333397643}\\
&- \frac{620321500710913317110079\,P_1^2}{1012440339652188796429} + \frac{4581423701591780363 4157371\,P_2^2}{2024880679304377592859 4}\\
&- \frac{5642167592294337606543313409\,P_3^2}{20248806793043775928594} + M_1\,P_1 = 0.
\end{aligned}
$$

Consider the linear system of quadrics

$$\Lambda_3 \ : \ t_1\,Q_1 + t_2\,Q_2 + t_3\,Q_3 + t_0\,Q_0 = 0.$$

The discriminant surface $\tilde{W}_1 \subset \Lambda_3(t_0 : t_1 : t_2 : t_3)$ is the union of 3 cones C_1, C_2, C_3 and of the plane $\{t_0 = 0\}$.

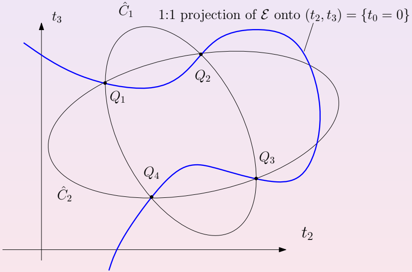
The intersection $C_1 \cap C_2 \cap C_3$ is a spatial elliptic curve $\mathcal{E}$. Thus, according to [AvM], the intersection of the 4 quadrics $Q_0, \ldots, Q_4$ is an Abelian variety.

Setting $t_1 = 1$, we obtain its affine part $E \subset \mathbb{C}^3(t_0, t_2, t_3)$. It is projected one-to-one (no branching) to the plane elliptic curve $\bar{\mathcal{E}}$ on the plane $\{t_0 = 0\}$ with the coordinates (t_2, t_3):

$$\begin{aligned} R(t_2, t_3) = 1 &- \frac{5640867029107772795 9}{7558440019602232212}\, t_2 + \frac{6711460476626436816 41}{8314284021562455433 2}\, t_3 \\ &- \frac{18873094535164528856161777467427675 4071}{47399423197482691225229801089369308 6}\, t_2^2 \\ &+ \frac{24541330110810145197799919675288793 49}{26000780689787543184437630877328 2}\, t_2\, t_3 \\ &- \frac{17712692390214793633592940142597862442316 39}{11375861567395845894055152261448634064}\, t_2^2\, t_3 \\ &+ \frac{20991244131743014452197679585814932137819 71}{11375861567395845894055152261448634064}\, t_2\, t_3^2 \\ &- \frac{12712815541250763955297950132998499083 17}{23699711598741345612614900544684654 3}\, t_3^2 \\ &+ \frac{26961739830389557454138593936714387569 1}{6240187365549010364265031410558768}\, t_2^3 \\ &- \frac{44795827665027233405596748712316153731 1}{6240187365549010364265031410558768}\, t_3^3 = 0 \end{aligned}$$

Applying MAPLE's command `Wierstrassform`, we get $\mathcal{E}$ in a canonical form

$$Y^2 = (X - E_1)(X - R_2)(X - E_3), \quad E_1 + E_2 + E_3 = 0,$$
$$E_1 = 250053786123809372629491722988556174669229146,$$
$$E_2 = -123721775932295447173160163074152674664430737,$$
$$E_3 = -126332010191513925456331559914403500004798409.$$



The 4 intersection points of $\mathcal{E}$ with $\{t_0 = 0\}$ are (up to 10^{-8})

$(t_2 = -1.49051153,\ t_3 = -1.28056092)$, $(t_2 = -1.28977159,\ t_3 = -1.11826304)$, $(t_2 = -0.35655629,\ t_3 = -0.2487487)$, $(t_2 = 0.57166862,\ t_3 = 0.605708301)$

and $t_0 = 0, t_1 = 1$.

Substituting these values into $t_1 Q_1 + t_2 Q_2 + t_3 Q_3 + t_0 Q_0 = 0$, as in the Clebsch case, we get 4 integrals as rank 3 quadrics in $\mathbb{P}^6$:

$$
\begin{aligned}
G_1 &= (0.05192301994\, M_1 + 96.29330714\, P_1)^2 \\
&\quad + (0.0401571127\, i\, M_2 + 18.5583836\, i\, P_2)^2 \\
&\quad + (0.031234934\, M_3 - 20.398727\, P_3)^2 = 0, \\
G_2 &= (0.00556740514\, M_1 - 89.8099126\, P_1)^2 \\
&\quad + (0.03745025367\, i\, M_2 + 17.21965624\, i\, P_2)^2 \\
&\quad + (0.02913249\, M_3 + 19.19255819\, P_3)^2 = 0, \\
G_3 &= (0.5582455209\, M_1 + 0.8952094727\, P_1)^2 \\
&\quad + (0.107287088\, i\, M_2 - 1.661683628\, i\, P_2)^2 \\
&\quad + (0.11904002\, M_3 - 1.04479376\, P_3)^2 = 0, \\
G_4 &= (0.7703328105\, M_1 + 0.6448613906\, P_1)^2 \\
&\quad + (0.1480458476\, i\, M_2 - 1.930706863\, i\, P_2)^2 \\
&\quad + (0.1642646049\, M_3 + 1.843689236\, P_3)^2 = 0
\end{aligned}
$$

The quadrics can be written (not uniquely !) in the "canonical" form

$$\sum_{\alpha=1}^{3} (\xi_\alpha + \eta_\alpha)^2 = 0\,, \quad \sum_{\alpha=1}^{3} \left(d_\alpha \xi_\alpha + \frac{\eta_\alpha}{d_\alpha} \right)^2 = 0\,, \quad \sum_{\alpha=1}^{3} \xi_\alpha \eta_\alpha = 0\,,$$

with $d_1 = 212.8614773, \quad d_2 = 6.2152341, \quad d_3 = 65.13483385$
under the linear change

$$\begin{aligned} M_1 &= 0.841818528\,\xi_1 - 2.703246\,\eta_1, & P_1 &= 0.300039184\,\xi_1 + 2.2106989\eta_1, \\ M_2 &= 0.841818528\,\xi_2 - 2.703246\,\eta_2, & P_2 &= 0.300039184\,\xi_2 + 2.2106989\eta_2, \\ M_3 &= 3.947919782\,\xi_3 + 32.2679122\,\eta_3, & P_3 &= 0.00032459\,\xi_3 - 3.22667443\eta_3. \end{aligned}$$

Then, up to a factor $\chi(u_1, u_2)$ the new coordinates ξ_α, η_α admit parameterization

$$\eta_\alpha = \chi(u)\, C_\alpha \frac{\theta[\Delta + \delta_\alpha](u_1, u_2)}{\theta[\Delta](u_1, u_2)}, \quad \xi_\alpha = \chi(u)\, D_\alpha \frac{\theta[\Delta + \kappa_\alpha](u_1, u_2)}{\theta[\Delta](u_1, u_2)},$$

in terms of theta-functions of
$\Gamma: \; w^2 = z(z - d_1^2)(z - d_2^2)(z - d_3^2)(z - d_1^2 d_2^2 d_3^2)$.
The factor $\chi(u_1, u_2)$ is calculated uniquely by using the last integral, e. g., $Q_0(P)$. Thus, we get an explicit parameterization of the intersection of the 4 quadrics in the HK discretization of the Clebsch case.

Parametrization of $t_2, t_3, t_0\,(t_1 = 1)$ in terms of X, Y on the canonical elliptic curve $Y^2 = (X - E_1)(X - R_2)(X - E_3)$:

$$t_2 = \frac{F_2(X,Y)}{(X - E_1)\rho_2(x)}, \quad t_3 = \frac{F_3(X,Y)}{(X - E_1)\rho_2(x)},$$

$$\rho_2(X) = X^2 + 1756597211462590761736770442155111322135863845930 23\,X$$

$$+738319931128365249871897876115729773581792986271 44307$$

$$302957955897103671043509081728738749160856174475 4793095$$

$$6427425124036653541024404901274263253716269616070656),$$

$$t_0 = \frac{P_2(X)Y + P_4(X)}{\rho_2(X)\,R_2(X,Y)},$$

where F_2, F_3, P_2, R_2 are certain polynomials of the corresponding degree.

Proposition

1) The intersection of A. Pfadler's quadrics $Q_0, \ldots, Q_3$ *is the 2-dim. Prym variety* $\mathrm{Prym}(C, \mathcal{E})$*, where the genus 3 curve* C *is given by* $Z^2 = t_0(X, Y)$*, namely*

$$W^2 = \frac{P_2(X)Y + P_4(X)}{\rho_2(X)\, R_2(X, Y)}, \quad Y^2 = (X - E_1)(X - R_2)(X - E_3).$$

2) $\mathrm{Prym}(C, \mathcal{E})$ *is a 8 -fold covering of* $\mathrm{Jac}(\Gamma)$

$$\Gamma : \; w^2 = z(z - d_1^2)(z - d_2^2)(z - d_3^2)(z - d_1^2 d_2^2 d_3^2),$$
$$d_1 = 212.8614773, \quad d_2 = 6.2152341, \quad d_3 = 65.13483385$$