Linear time list recovery via expander codes

Brett Hemenway and Mary Wootters

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Outline

Our Results

One slide version

- Inner code + expander graph $\Rightarrow$ Expander code

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- [SS96, Spi96, Zem01, BZ02, BZ05, BZ06]

Inner code has decent distance $\Rightarrow$ So does the expander code and it's decodable in linear time!

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- [HOW13]

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- This work:

Inner code has decent list-recoverability	$\Rightarrow$	So does the expander code and it's list-recoverable in linear time!

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Inner code has decent list-recoverability $\Rightarrow$ So does the expander code and it's list-recoverable in linear time!

- List-recoverable: decodable from *uncertainty* (instead of errors).

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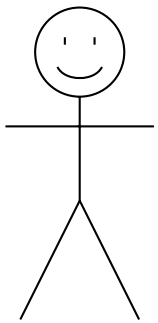
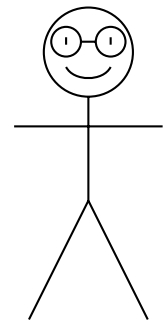
- List-recoverable: decodable from *uncertainty* (instead of errors).
- Known linear-time list-recoverable codes have rate $< 1/2$.
- Expander codes can have rate $1 - \varepsilon$!

- We can plug our codes into [Meir'14] and get the optimal* rate/error trade-off, for any rate.

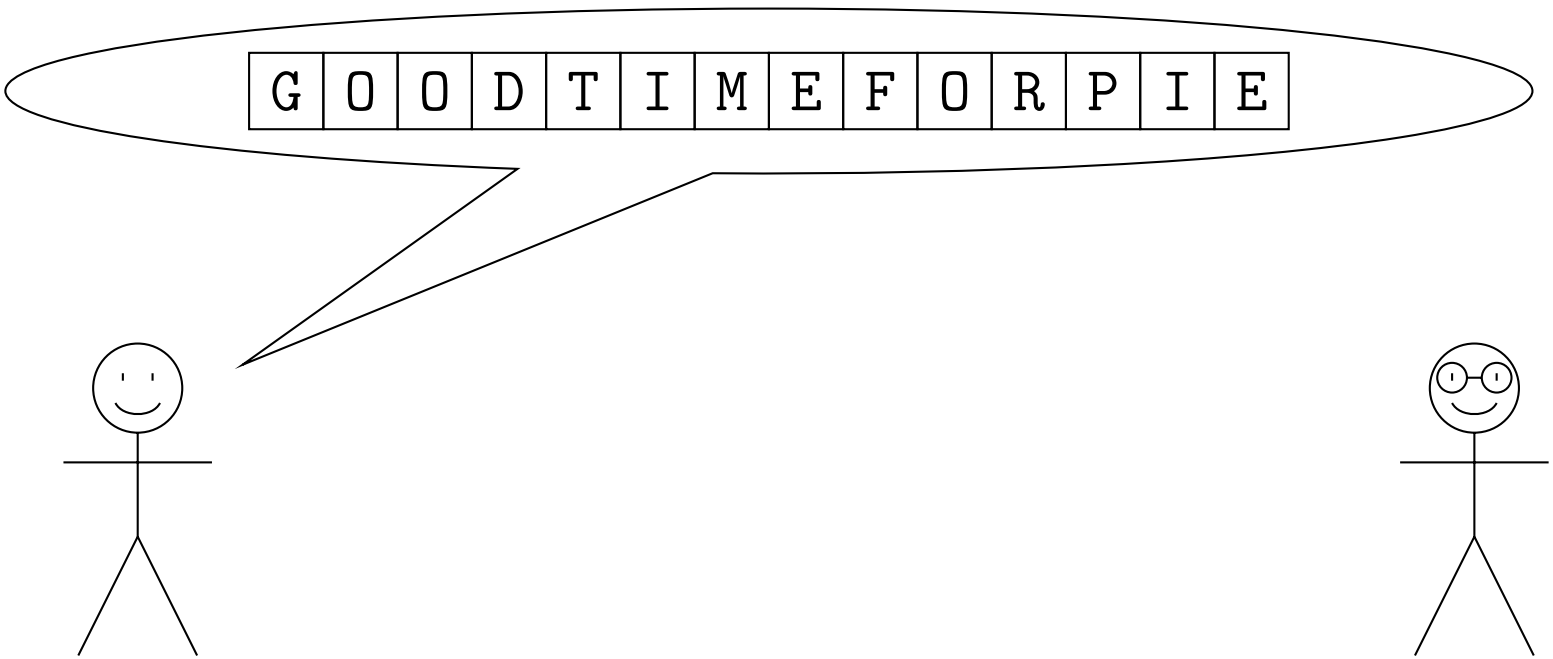
Outline

List Decoding





List Decoding



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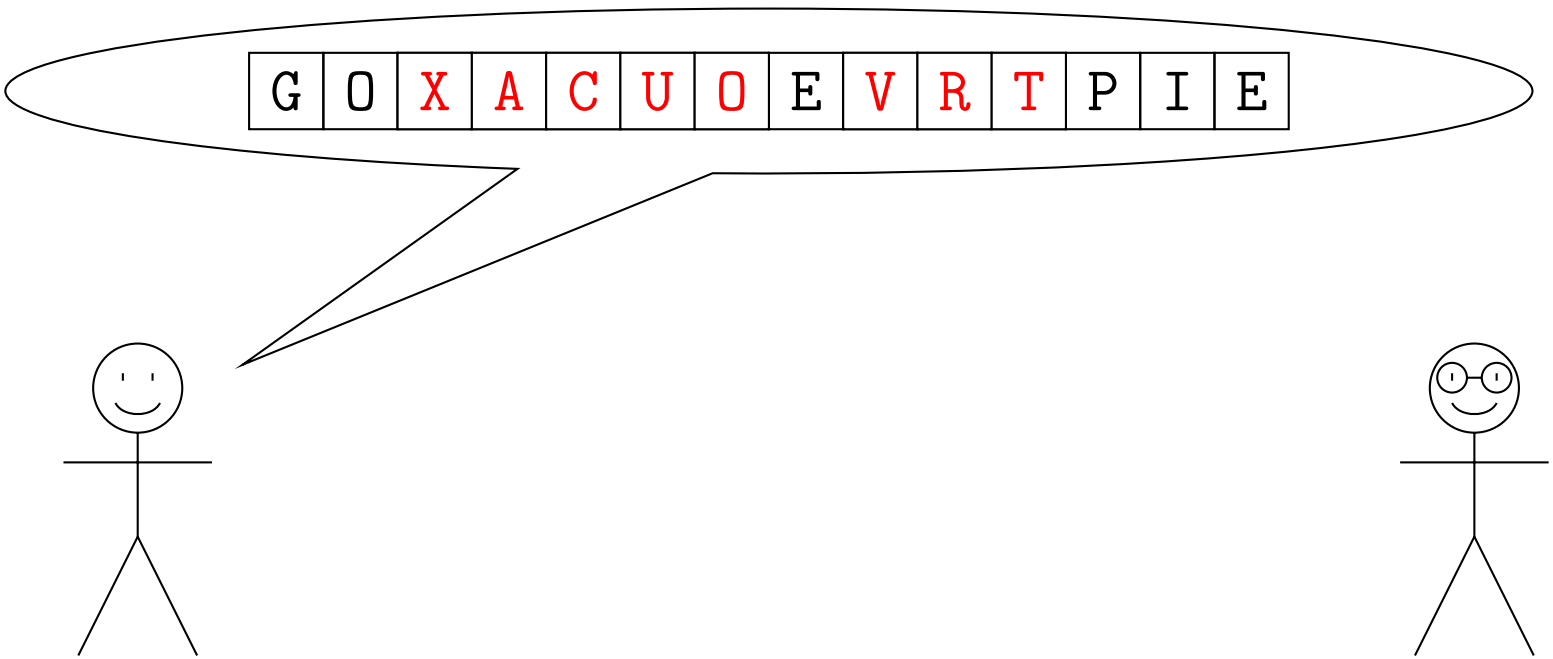

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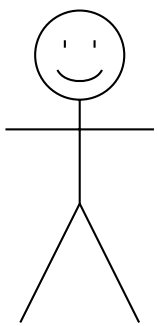

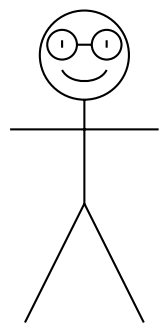

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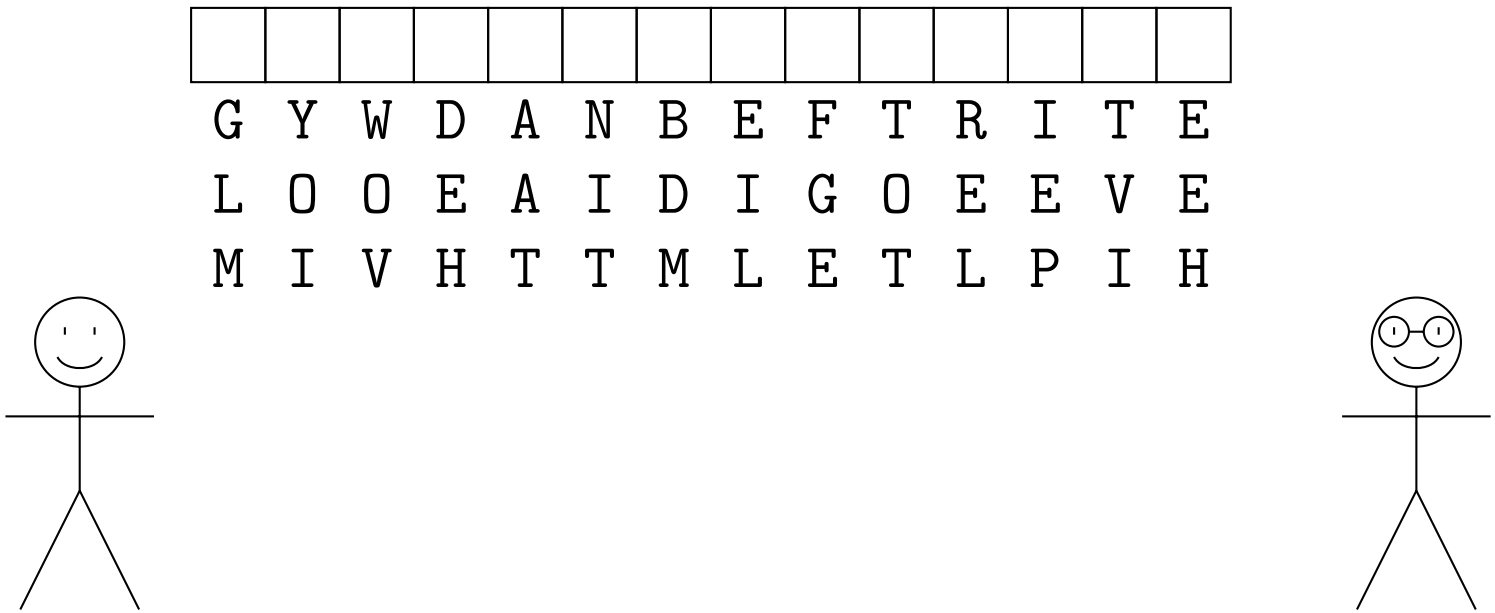

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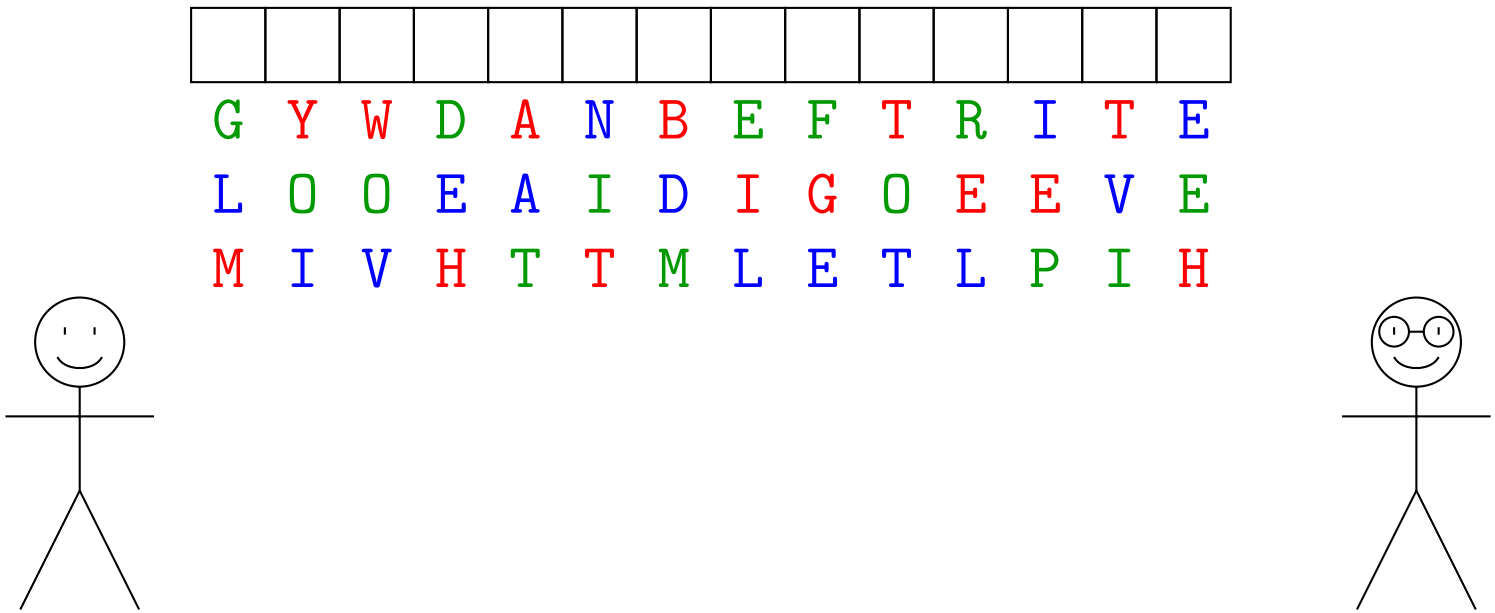
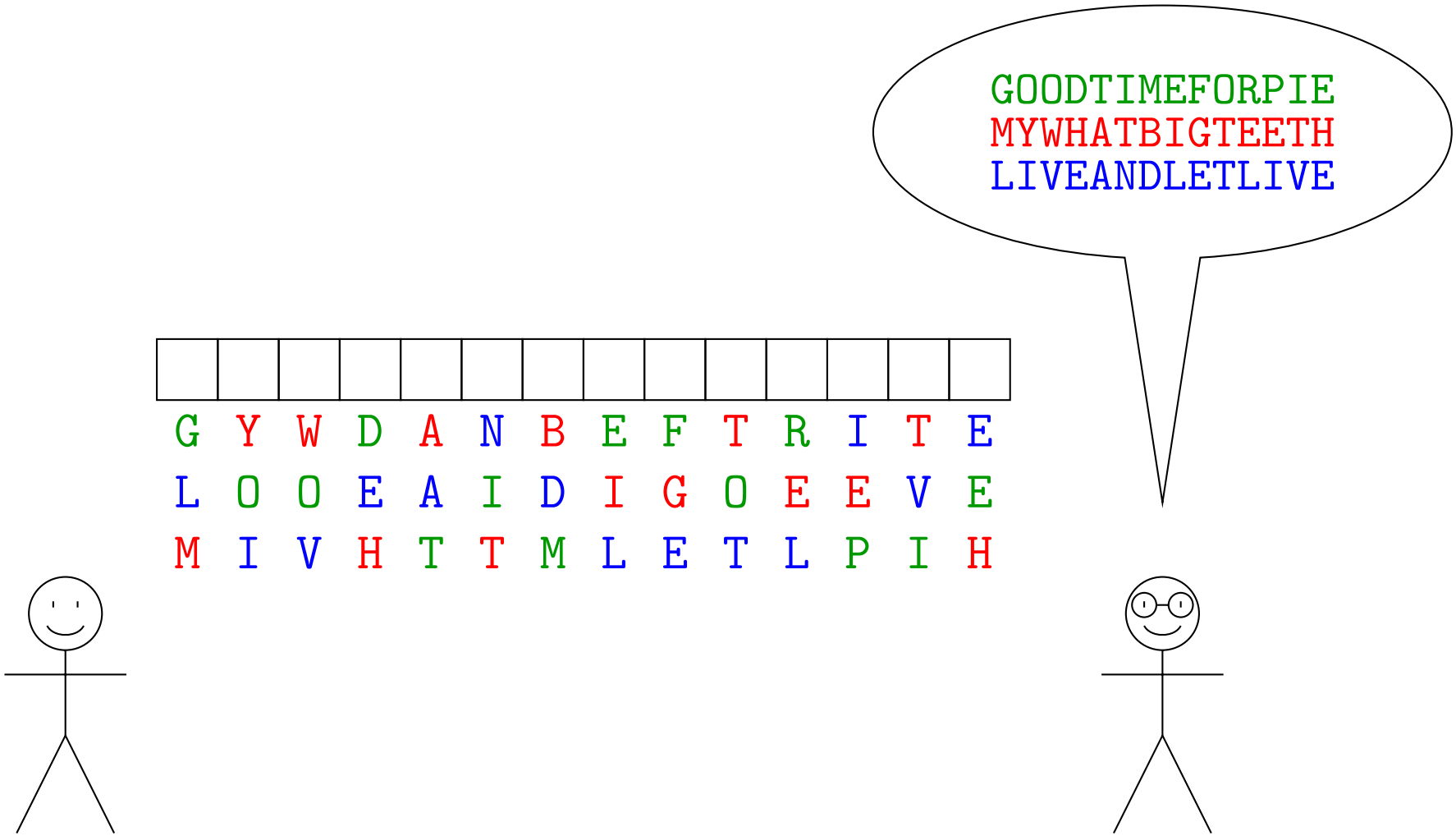
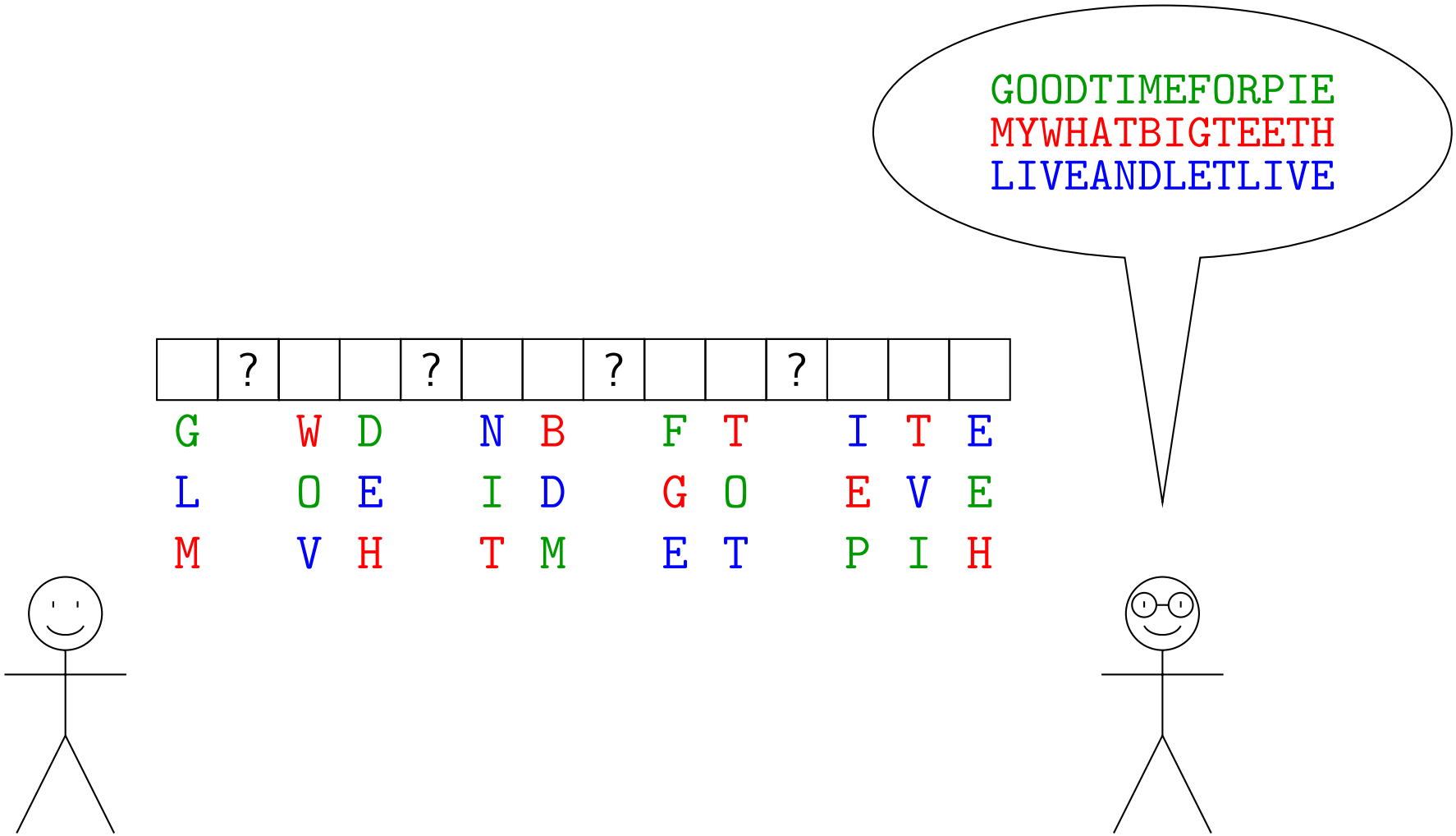


List Recovery



List Recovery From Erasures



List Recovery

Definition

Definition

$\mathcal{C} \in \Sigma^N$ is (α, ℓ, L)-list-recoverable (from erasures) if:

- for any set of lists $S_1, \ldots, S_N$
  so that at least αN of them have size $\leq \ell$,
- there are at most L codewords $c \in \mathcal{C}$ so that $c_i \in S_i$ for all i.

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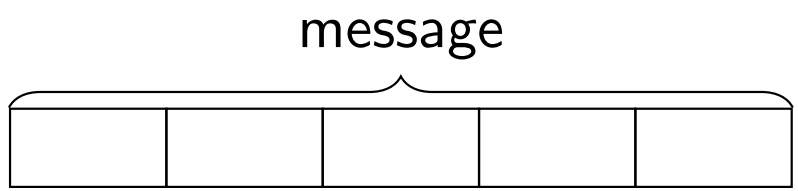
- $\alpha \longleftarrow$ fraction of codeword not erased
- $\ell \longleftarrow$ number of symbols in each slot
- $L \longleftarrow$ number of codewords that match

Note: $L \geq \ell$.

List decoding concatenated codes

An application of list recovery



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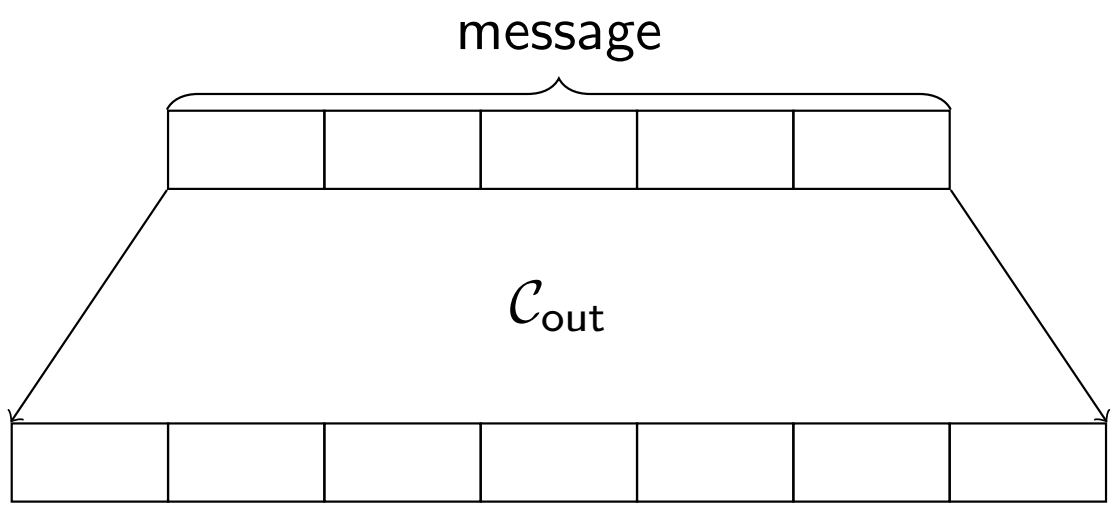

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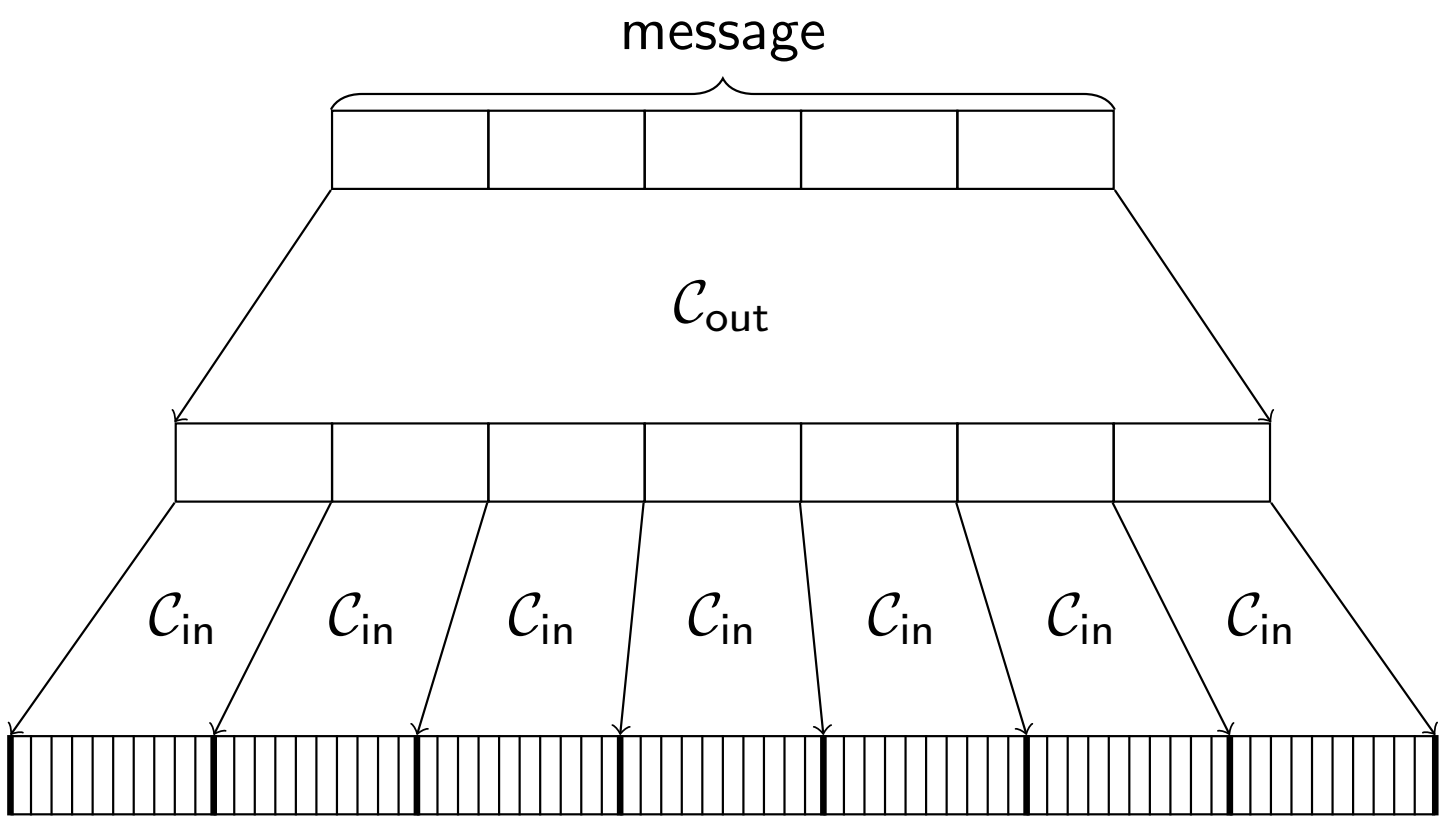

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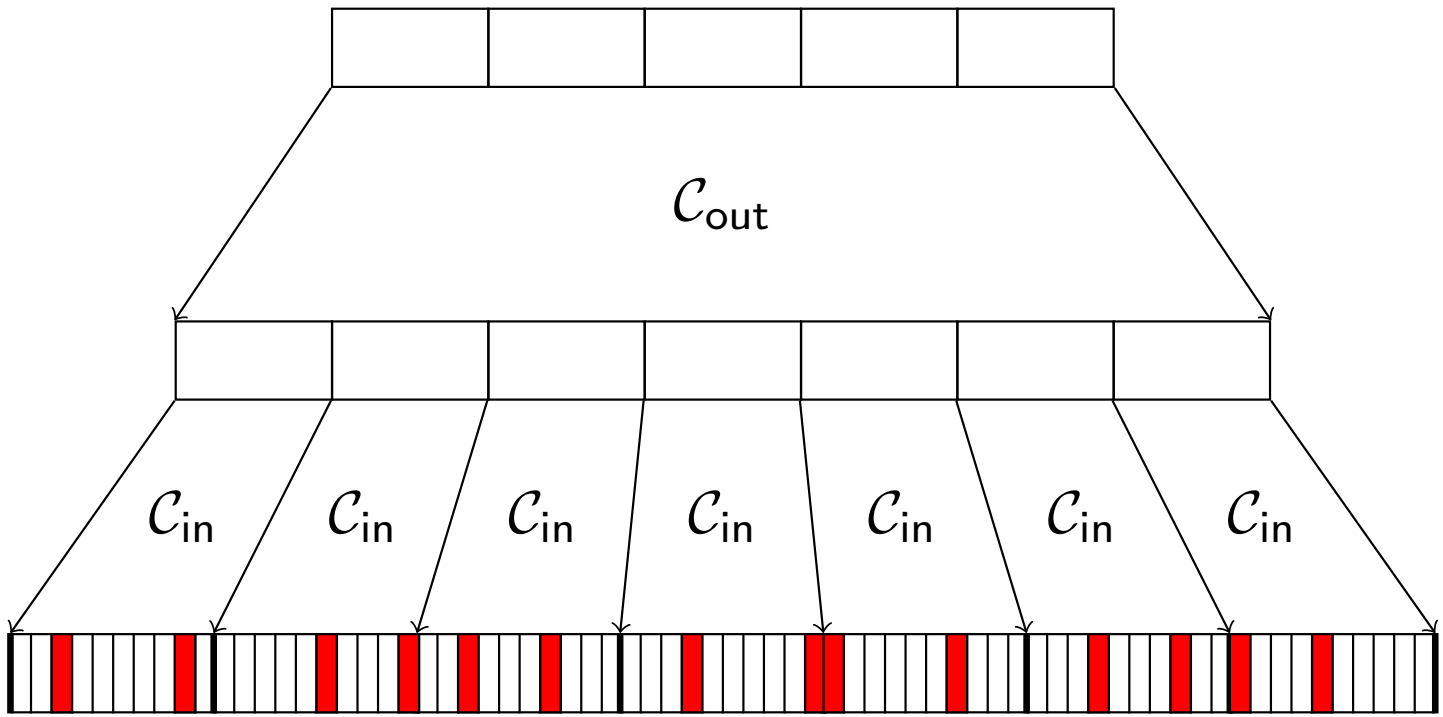

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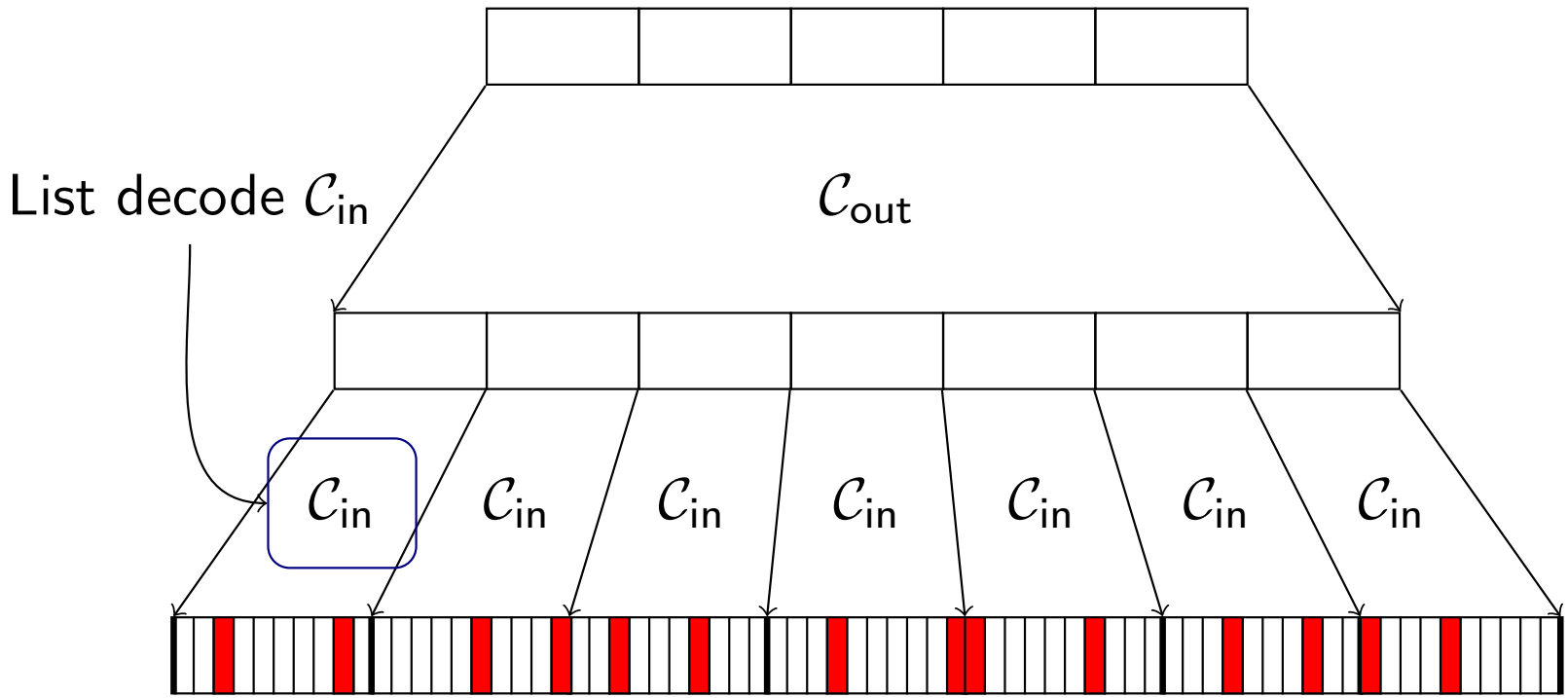

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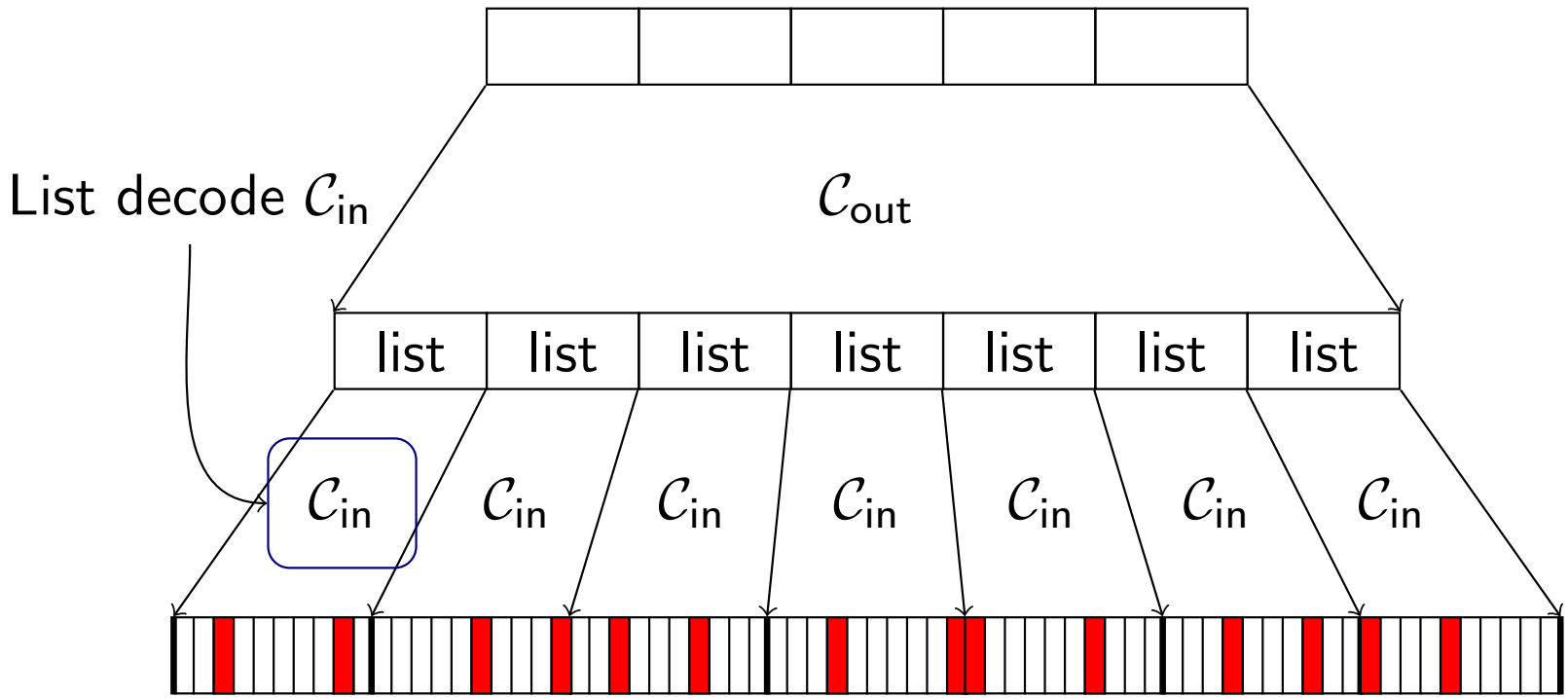

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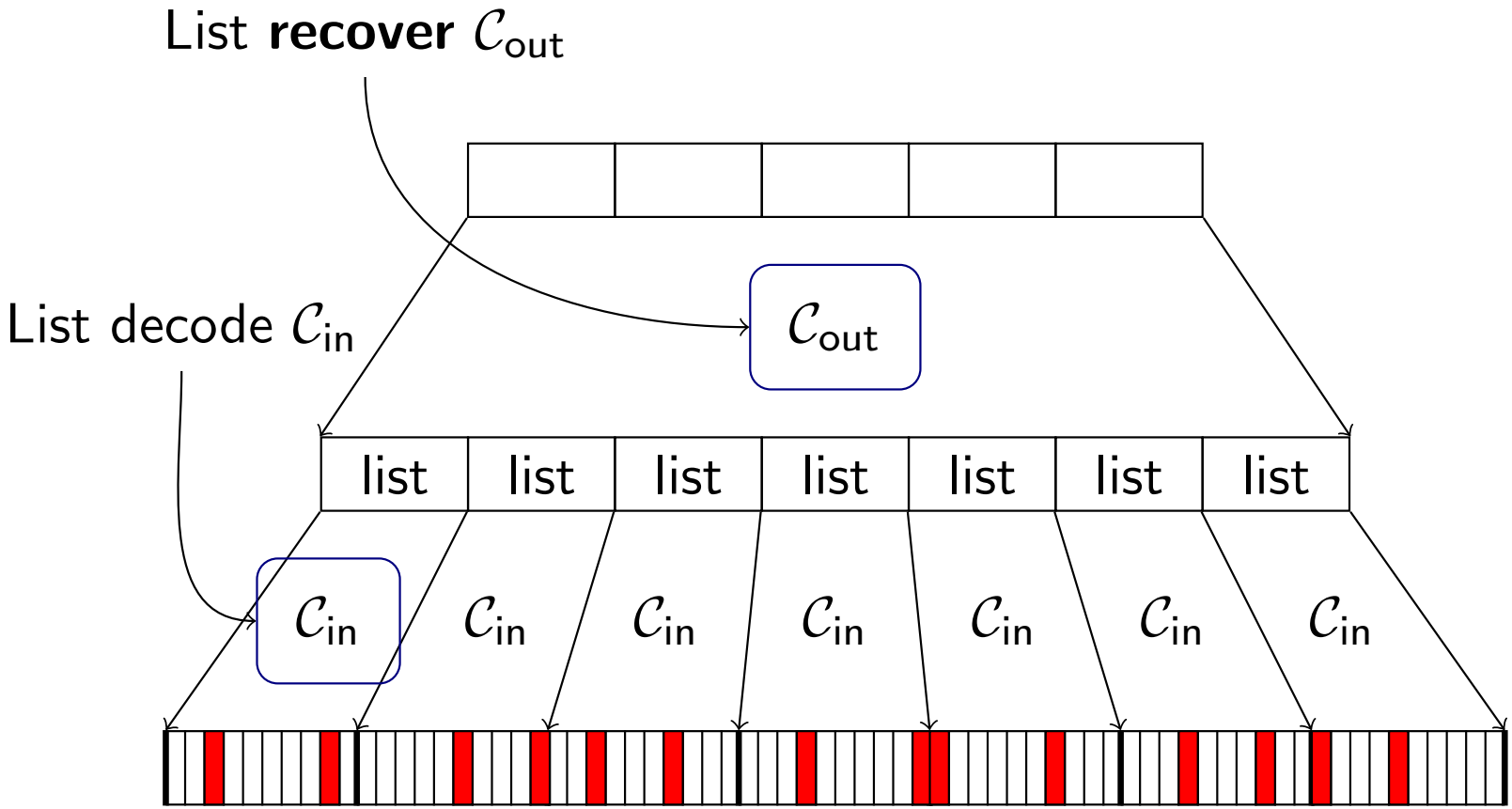

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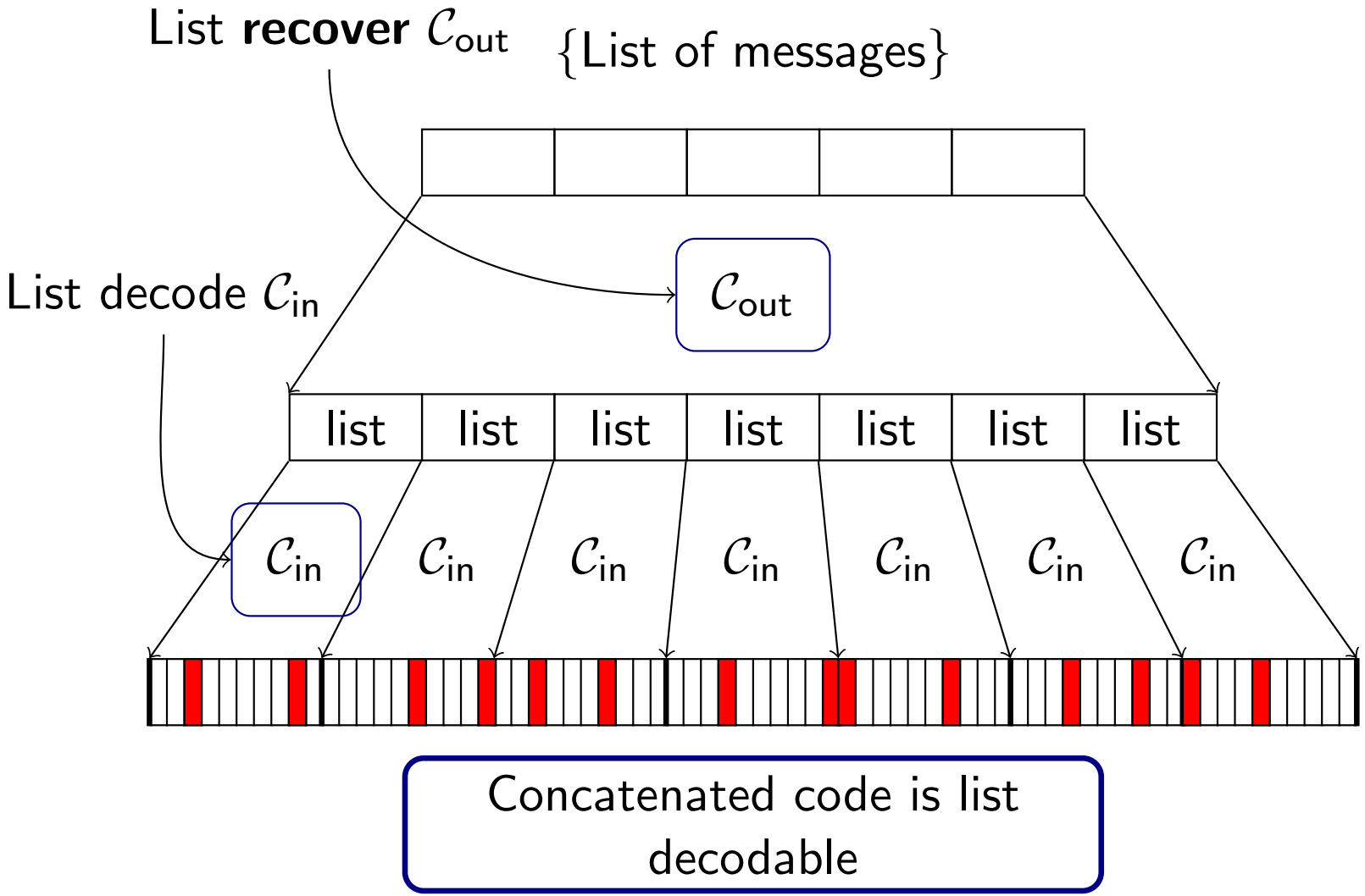


List Recovery

Applications

- Related to list decoding [GI02, GI03, GI04]
- Compressed sensing [NPR12, GNP$^+$13]
- Group testing [INR10]
- Erasure model is weaker than error model (erasure model was studied before in [GI04])

Outline

Tanner Codes [Tanner'81]

Given:

- A d-regular graph G with n vertices and $N = \frac{nd}{2}$ edges
- An *inner code* $\mathcal{C}_0$ with block length d over Σ

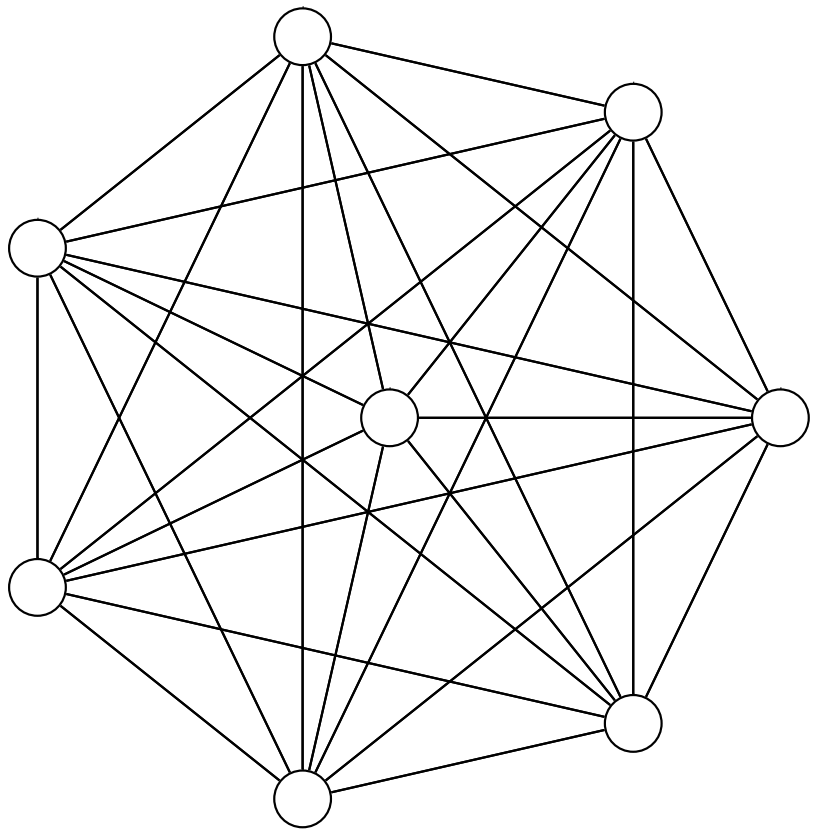
We get a *Tanner code* $\mathcal{C}$.

- $\mathcal{C}$ has block length N and alphabet Σ.
- Codewords are labelings of edges of G.
- A labeling is in $\mathcal{C}$ if the labels on each vertex form a codeword of $\mathcal{C}_0$.
- (We fix an arbitrary ordering of edges at each vertex)

Example [Tanner'81]

G is K_8, and $\mathcal{C}_0$ is the $[7, 4, 3]$-Hamming code.

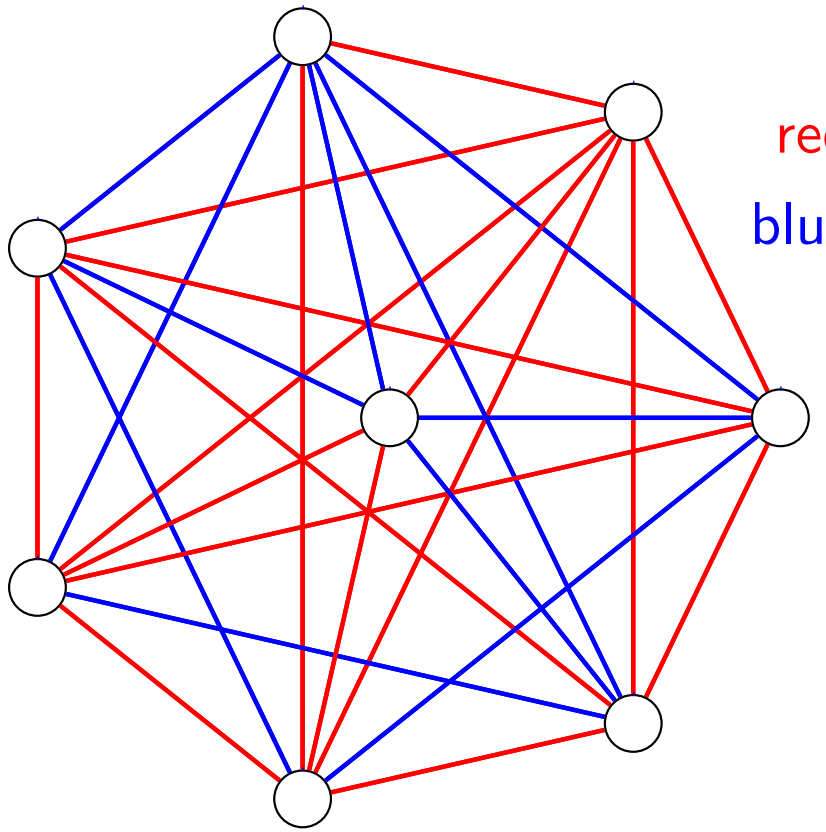
$$N = \binom{8}{2} = 28 \text{ and } \Sigma = \{0, 1\}$$



Example [Tanner'81]

G is K_8, and $\mathcal{C}_0$ is the $[7, 4, 3]$-Hamming code.

A codeword of $\mathcal{C}$ is a labeling of edges of G.



red $\mapsto 0$

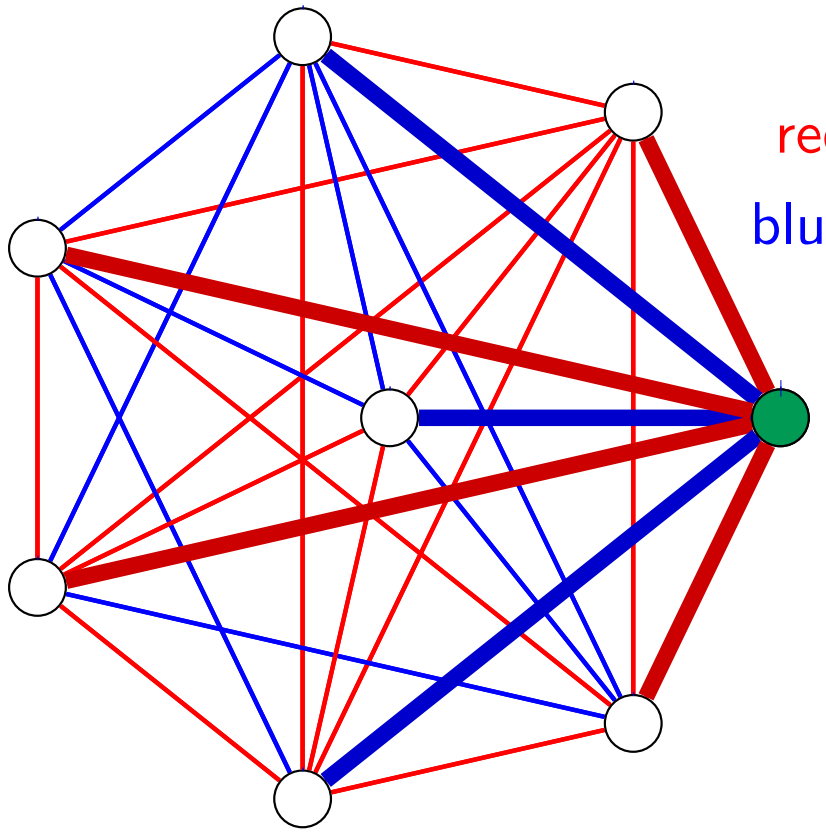
blue $\mapsto 1$

$$(0,0,0,0,0,0,0,1,1,0,1,1,1,0,1,0,0,1,0,1,0,0,0,1,0,0,1,1) \in \mathcal{C} \subset \{0,1\}^{28}$$

Example [Tanner'81]

G is K_8, and $\mathcal{C}_0$ is the $[7, 4, 3]$-Hamming code.

These edges form a codeword in the Hamming code



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Encoding Tanner Codes

Encoding is Easy!

1. Generate parity-check matrix
   Requires:
   - Edge-vertex incidence matrix of graph
   - Parity-check matrix of inner code
2. Calculate a basis for the kernel of the parity-check matrix
3. This basis defines a generator matrix for the linear Tanner Code
4. Encoding is just multiplication by this generator matrix

Linearity

If the inner code $\mathcal{C}_0$ is linear, so is the Tanner code $\mathcal{C}$

- $\mathcal{C}_0 = \mathrm{Ker}(H_0)$ for some *parity check* matrix H_0.

$$x \in \mathcal{C}_0 \iff \boxed{H_0} \quad \boxed{x} \quad = 0$$

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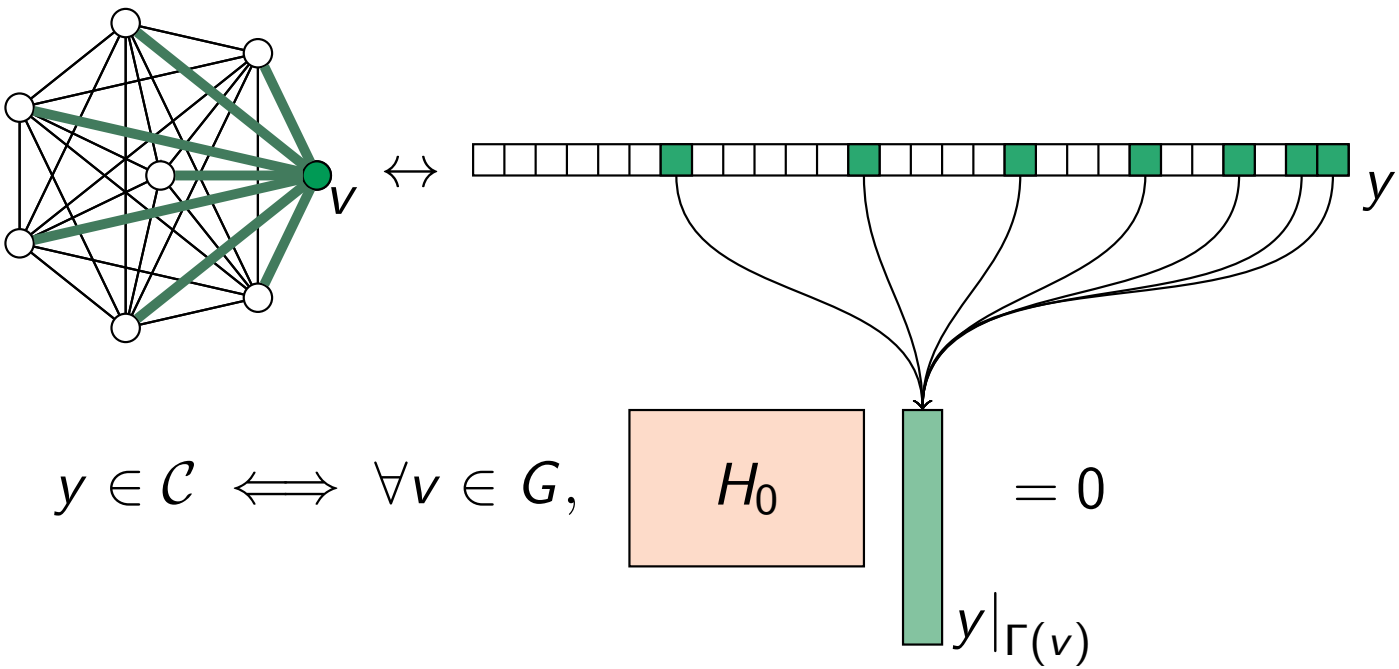
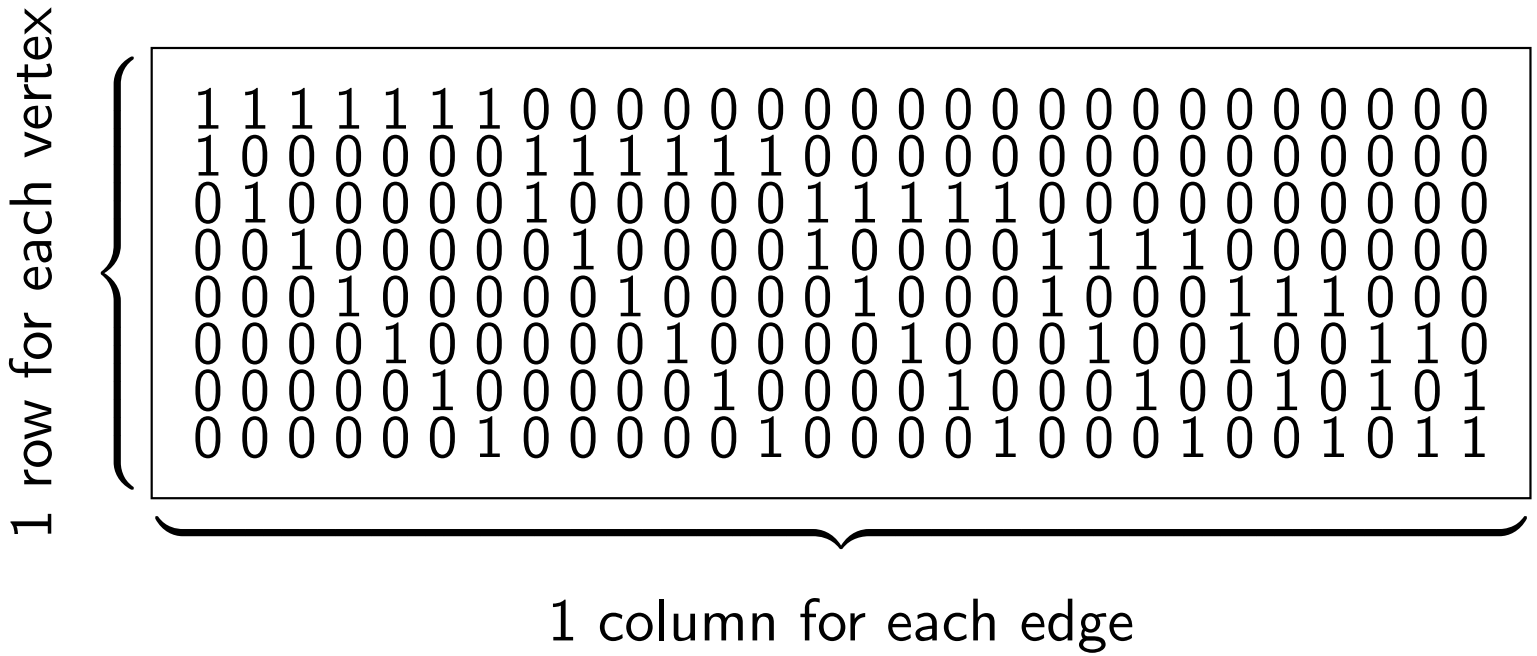
- So codewords of the Tanner code $\mathcal{C}$ also are defined by linear constraints:



Example: vertex edge incidence matrix of K_8



- Columns have weight 2
  (Each edge hits two vertices)
- Rows have weight 7
  (Each vertex has degree seven)

Example: parity-check matrix of a Tanner code

K_8 and the $[7, 4, 3]$-Hamming code

Parity-check
of Hamming code

$$\begin{matrix} 1&0&1&0&1&0&1 \\ 0&1&1&0&0&1&1 \\ 0&0&0&1&1&1&1 \end{matrix}$$

$$\begin{matrix} 1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&1&1&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1&0 \\ 0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1 \\ 0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1 \end{matrix}$$

Edge-vertex incidence matrix of K_8

Example: parity-check matrix of a Tanner code

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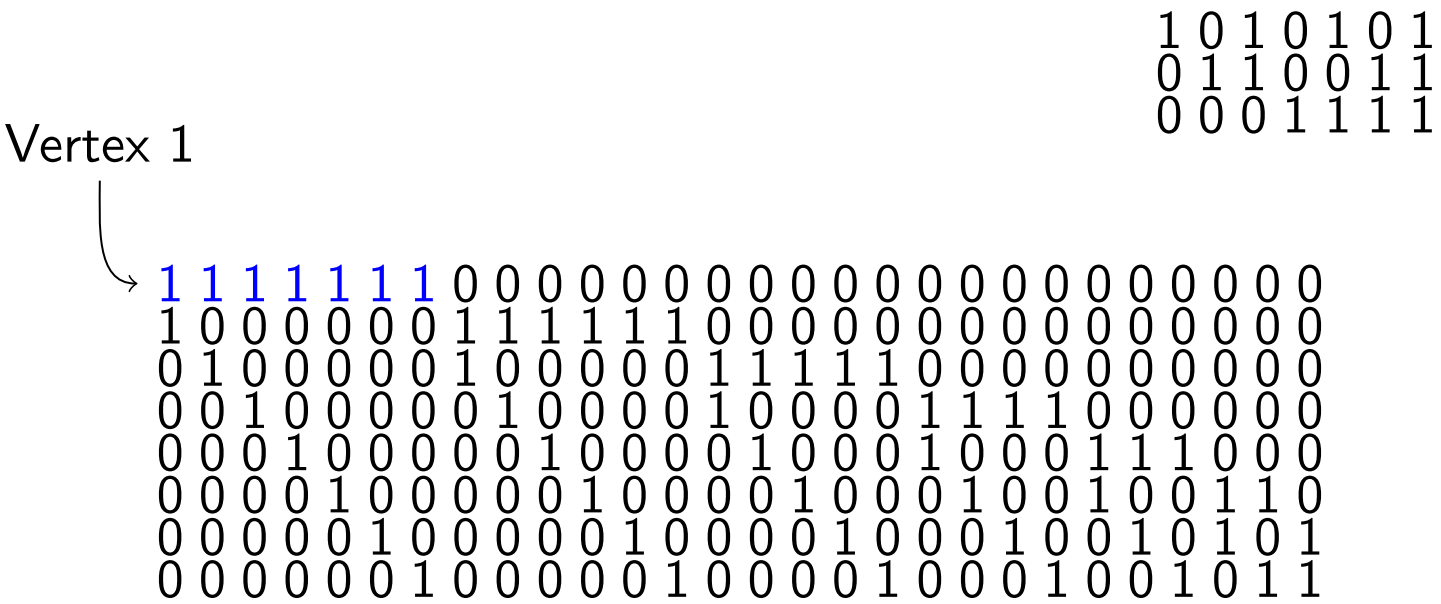

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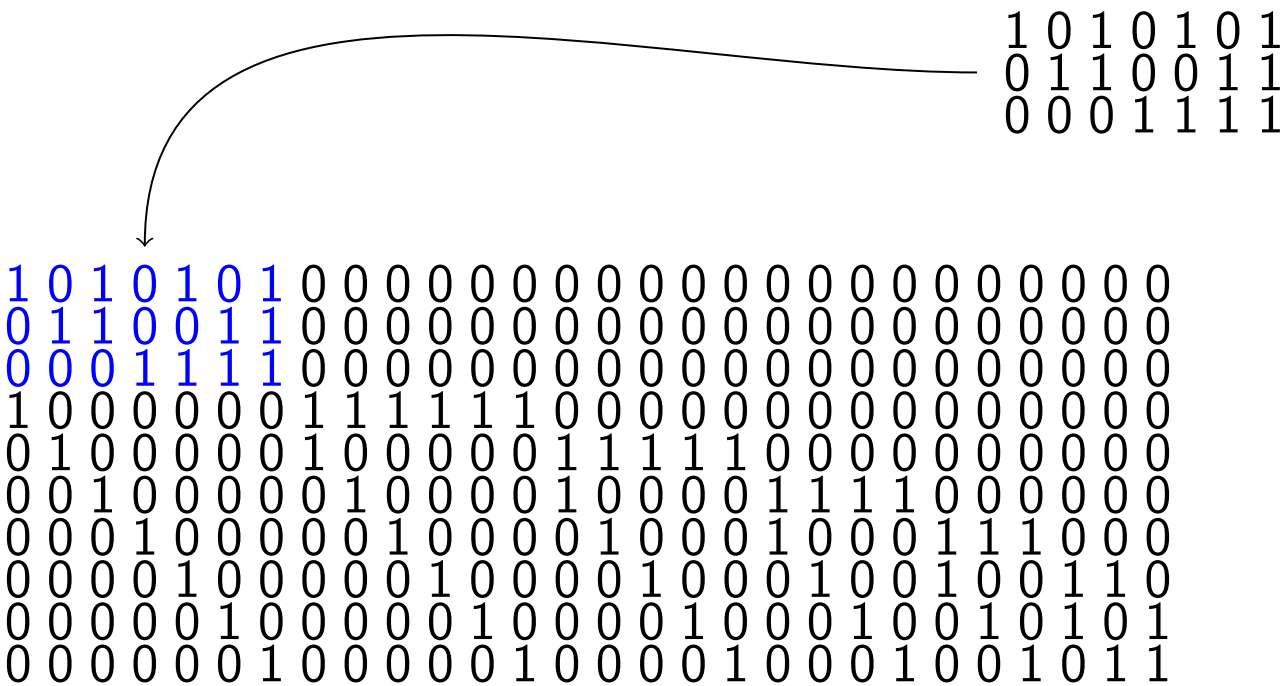

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$$\begin{array}{ccccccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

$$\begin{array}{cccccccccccccccccccccccccccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
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\end{array}$$

Example: parity-check matrix of a Tanner code

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0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

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1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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$$
\begin{pmatrix}
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0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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\end{pmatrix}
$$

If the inner code has good rate, so does the outer code

Say that $\mathcal{C}_0$ is linear

- If $\mathcal{C}_0$ has rate r_0, it satisfies $(1 - r_0)d$ linear constraints.
- Each of the n vertices of G must satisfy these constraints.

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- $\mathcal{C}$ is defined by at most $n \cdot (1 - r_0)d$ constraints.
- Length of $\mathcal{C} = N = \#$ edges $= nd/2$
- The rate of $\mathcal{C}$ is

$$R = \frac{k}{N} \geq \frac{N - n \cdot (1 - r_0)d}{N} = 2r_0 - 1.$$

Better rate bounds?

- The lower bound $R > 2r_0 - 1$ is *independent* of the ordering of edges around a vertex
- Tanner already noticed that order matters.
  Let G be the complete bipartite graph with 7 vertices per side
  Let $\mathcal{C}_0$ be the $[7, 4, 3]$ hamming code
  Then different "natural" orderings achieve a Tanner code with
  - $[49, 16, 9]$ $(\frac{16}{49} \approx .327)$
  - $[49, 12, 16]$ $(\frac{12}{49} \approx .245)$
  - $[49, 7, 17]$ $(\frac{7}{49} \approx .142)$ Meets lower bound of $2 \cdot \frac{4}{7} - 1$

Expander codes

When the underlying graph is an *expander graph*, the Tanner code is a *expander code.*

- Expander codes admit very fast decoding algorithms [Sipser and Spielman 1996]
- Further improvements in [Sipser'96, Zemor'01, Barg and Zemor'02,'05,'06]

Outline

Linear-time list-recoverable codes

Theorem (Main Theorem)

- *Suppose that $\mathcal{C}_0$ is (α_0, ℓ, L_0)-list recoverable from erasures.*
- *Suppose G is an expander graph of degree d.*
  - *d needs to be big enough compared to ℓ, L_0.*

- *Then the expander code $\mathcal{C}$ is (α, ℓ, L)-list recoverable from erasures.*
  - *It can be list-recovered in linear time.*
  - *Above, α, L are constants (independent of n).*

...

Comparison with Previous Work

Code	Rate	Decoding time
Random code	$1 - \varepsilon$	lots
Folded RS (+friends) [GR08, Gur11, DL12]	$1 - \varepsilon$	$\mathrm{poly}(n)$
[GI03, GI04]	$1/\mathrm{poly}(\ell)$	$O(n)$
This work (Random inner code)	$1 - \varepsilon$	$O(n)$

- All list sizes L are constants
  - but some are *huge* constants

Improving the rate/error tradeoff

Theorem ("Dream theorem" of Meir'14)

Constructing codes of rate $\to 1$ *with property* $\mathcal{P}$ *and decent distance*

$$\Rightarrow$$

Constructing codes with $\mathcal{P}$ *that approach the Singleton bound.*

Improving the rate/error tradeoff

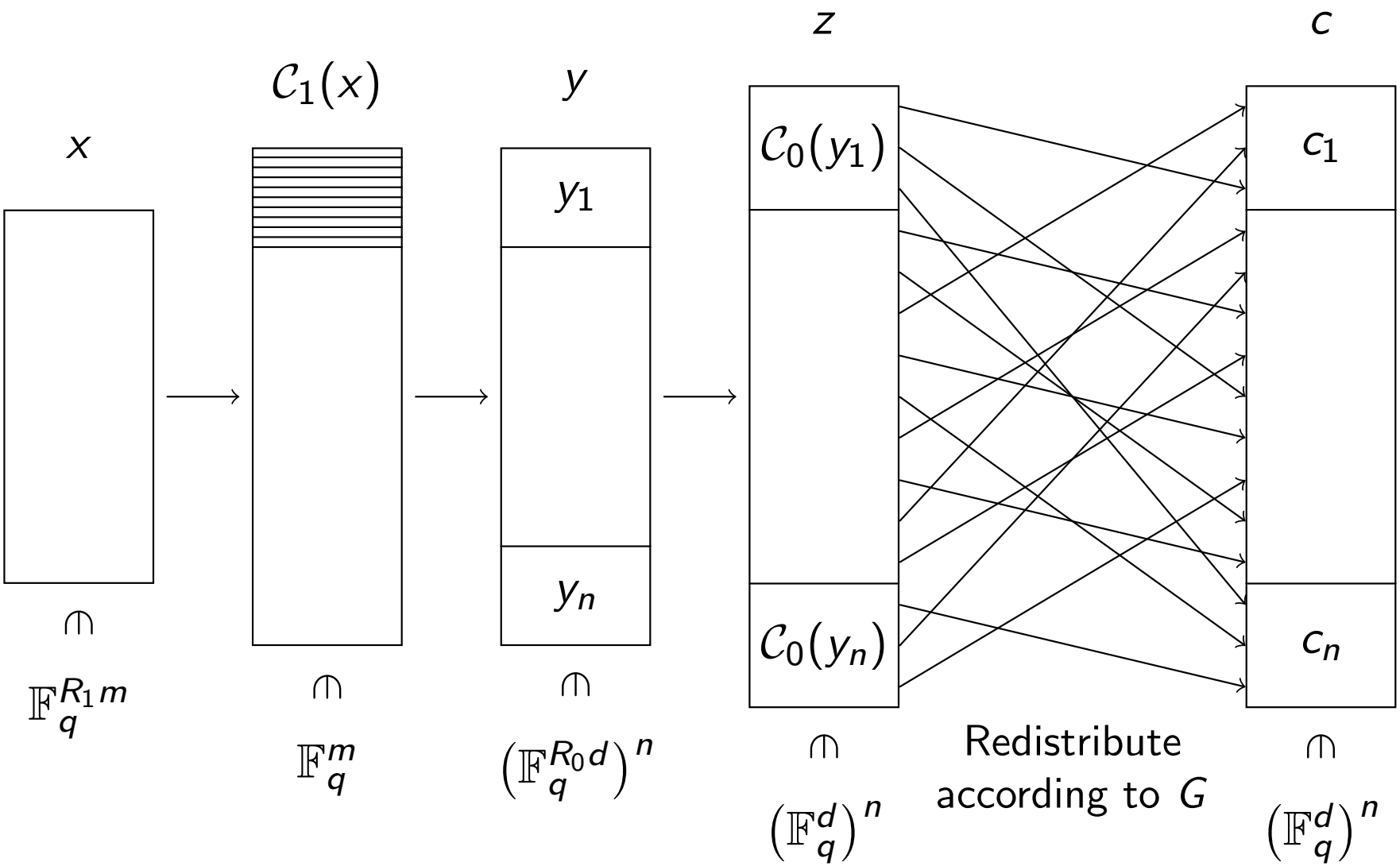
Theorem

For any $R > 0$, $\ell > 0$, *and* $\varepsilon > 0$,
and for any large enough L, d, q *(depending only on* ℓ, ε*),*
there is a family of codes with:

- *rate at least* R
- $(R + \varepsilon, \ell, L)$*-list-recoverable in linear time.*

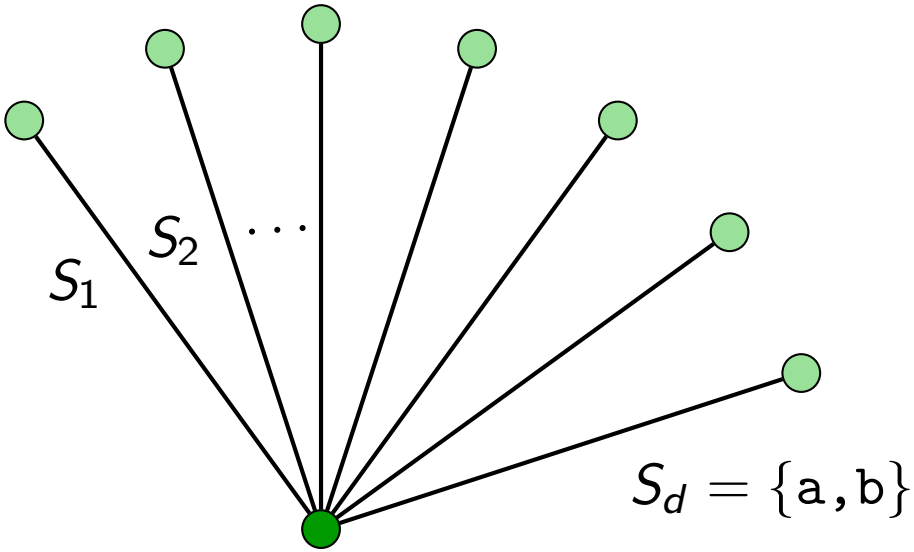
Improving the rate/error tradeoff

The Or Meir Construction [Mei14]



The algorithm

Suppose that $\mathcal{C}_0$ is list-recoverable.



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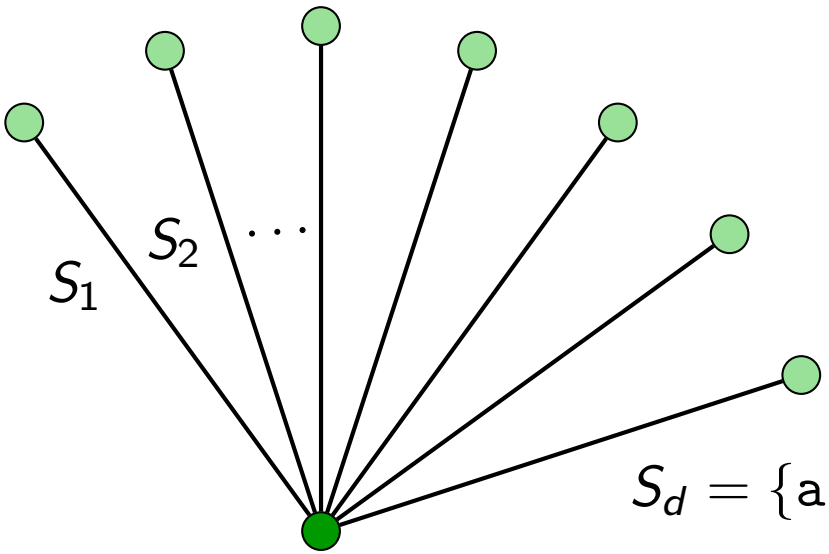

c_1	a d c d c f b f c d c f a
$\vdots$	d b c r e t a b c b c d b
c_L	a b x d e f a b z d e d a

1. List recover locally at this vertex: get $\{c_1, \ldots, c_L\}$

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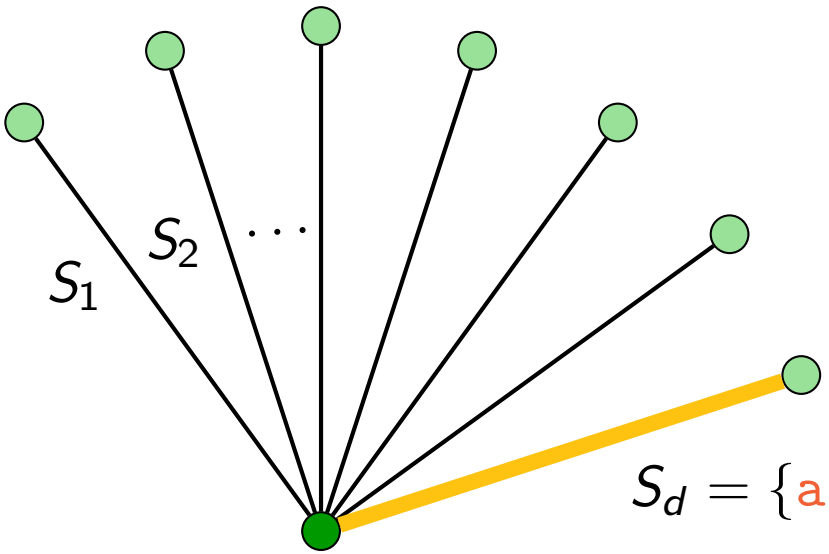

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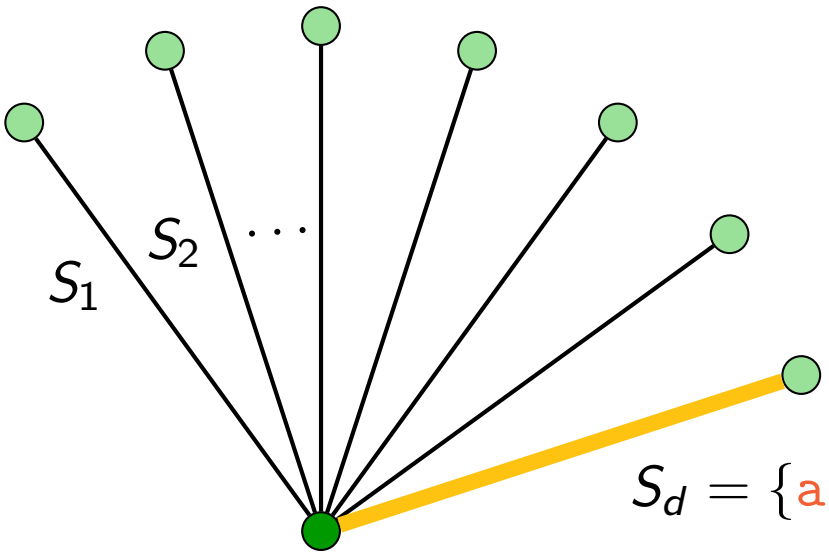
2. What if we know this symbol is "a"?

The algorithm

Suppose that $\mathcal{C}_0$ is list-recoverable.



1. List recover locally at this vertex: get $\{c_1, \ldots, c_L\}$

c_1	a	d	c	d	c	f	b	f	c	d	c	f	a
$\vdots$	d	b	c	r	e	t	a	b	c	b	c	d	b
c_L	a	b	x	d	e	f	a	b	z	d	e	d	a

$S_d = \{\mathtt{a}, \mathtt{b}\}$

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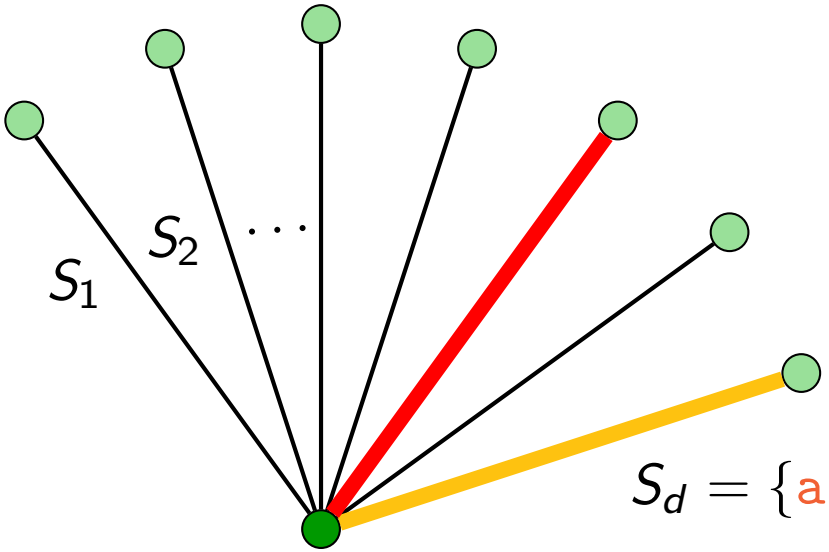

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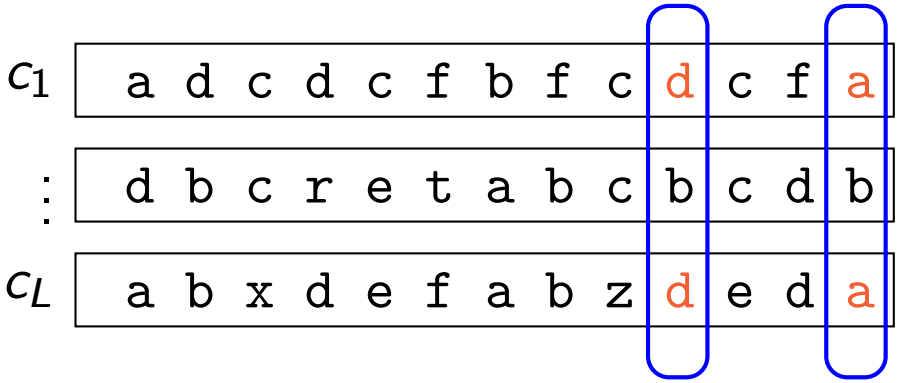

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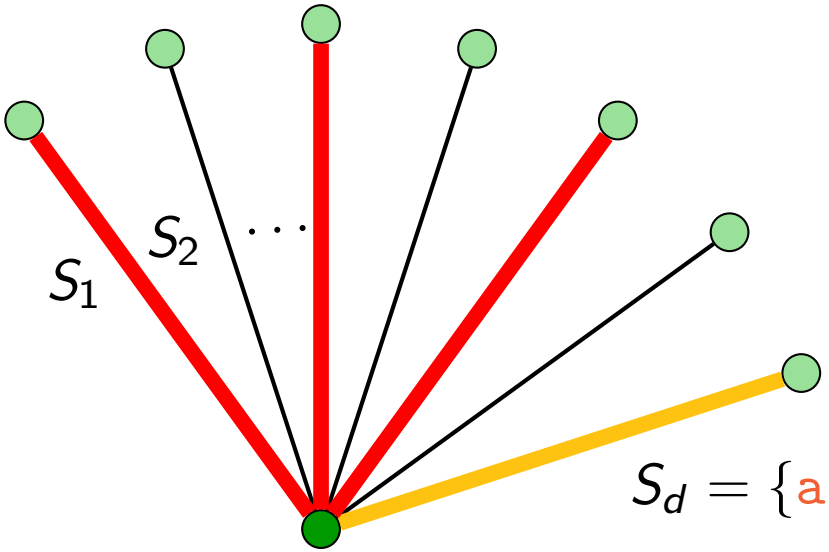

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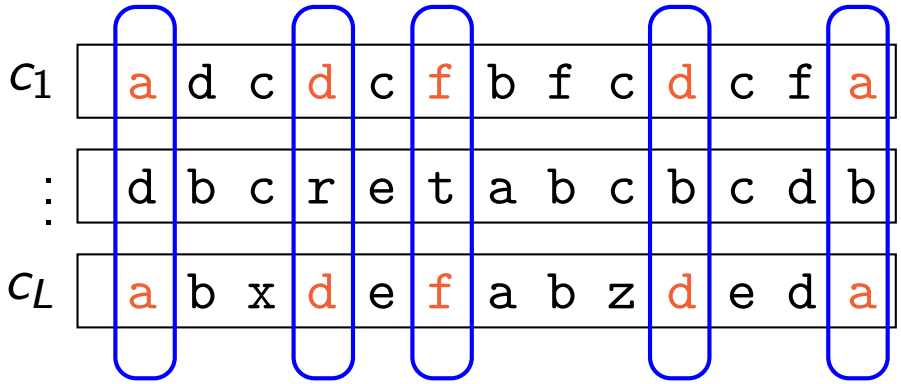
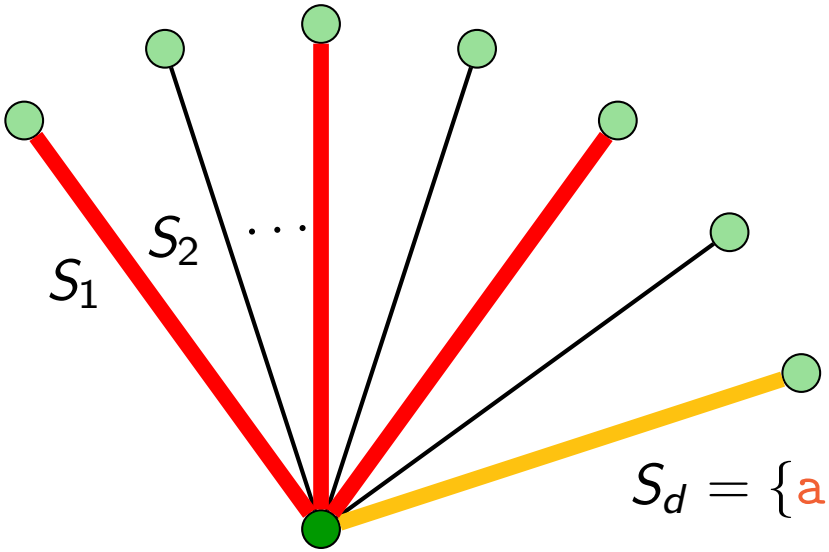


3. Then we know a bunch of other symbols too.

The algorithm

Suppose that $\mathcal{C}_0$ is list-recoverable.



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2. What if we know this symbol is "a"?

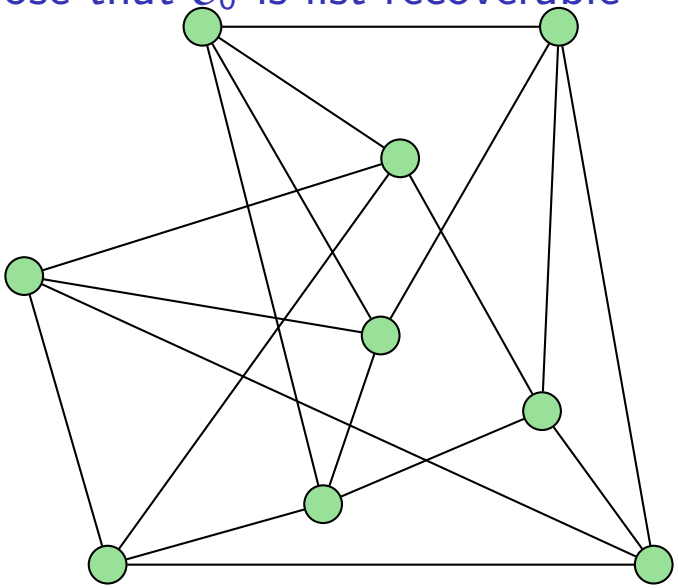
c_1	a	d	c	d	c	f	b	f	c	d	c	f	a
$\vdots$	d	b	c	r	e	t	a	b	c	b	c	d	b
c_L	a	b	x	d	e	f	a	b	z	d	e	d	a

3. Then we know a bunch of other symbols too.

(L^ℓ possible columns. Each choice fixes about d/L^ℓ others)

The algorithm

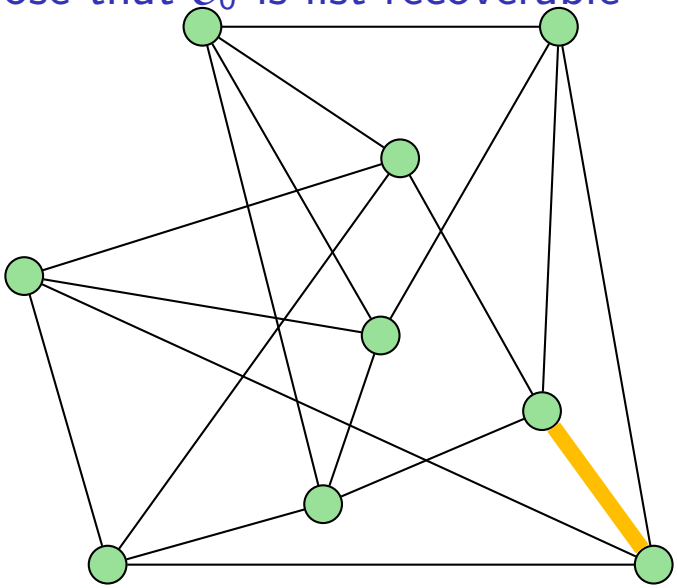
Suppose that $\mathcal{C}_0$ is list-recoverable



- Make one decision.

The algorithm

Suppose that $\mathcal{C}_0$ is list-recoverable



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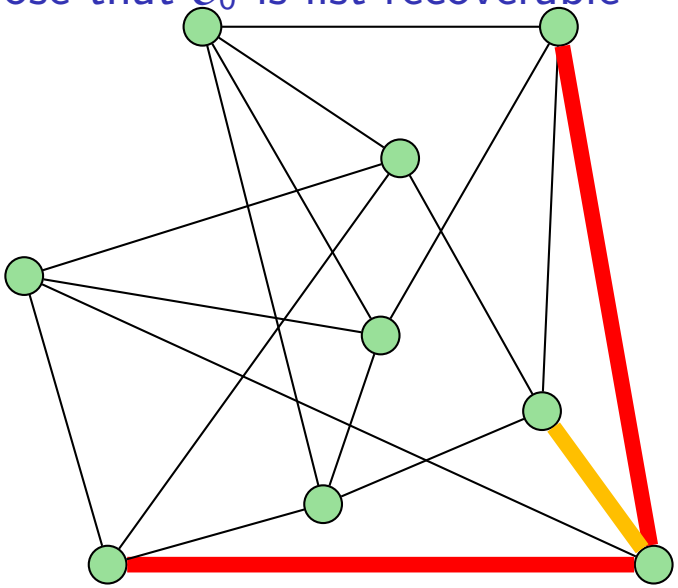

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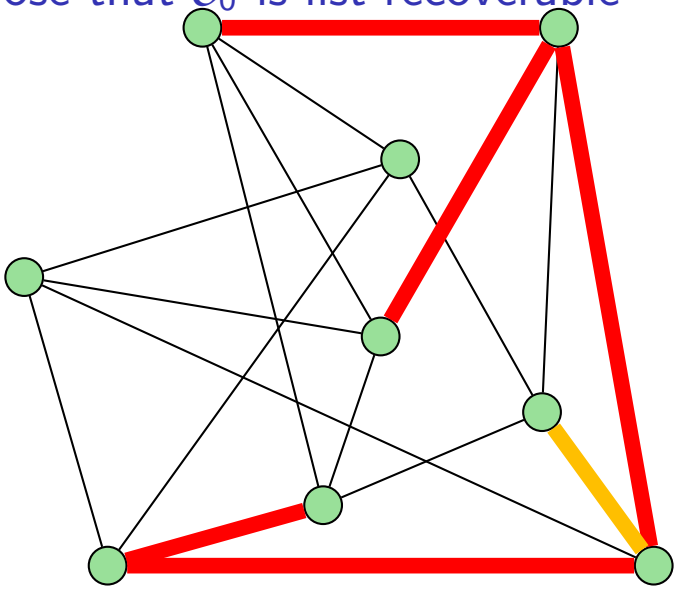

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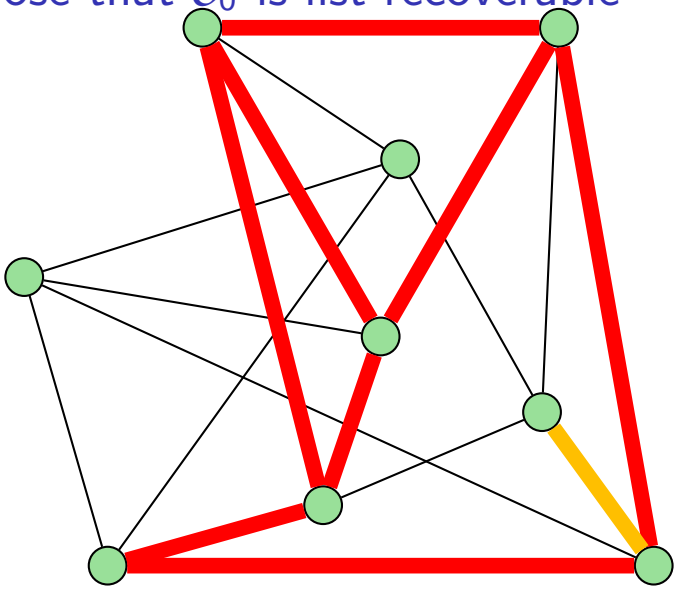
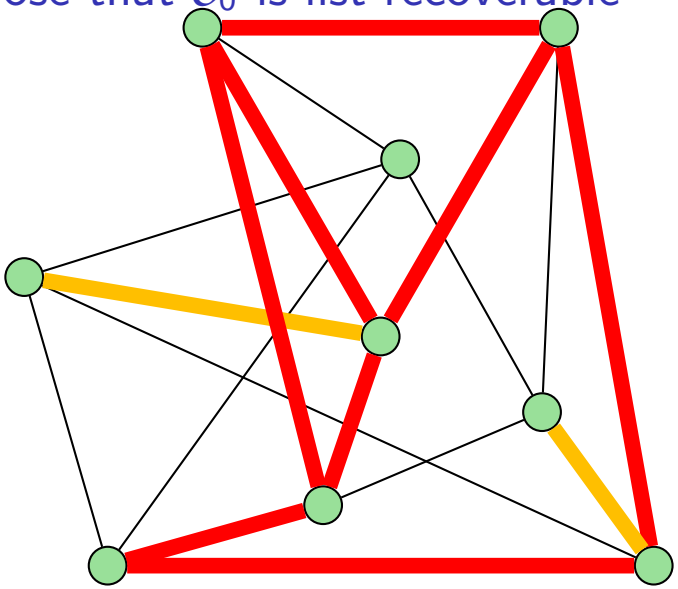


- Make one decision.
- Determine a bunch of other edges.

The algorithm

Suppose that $\mathcal{C}_0$ is list-recoverable



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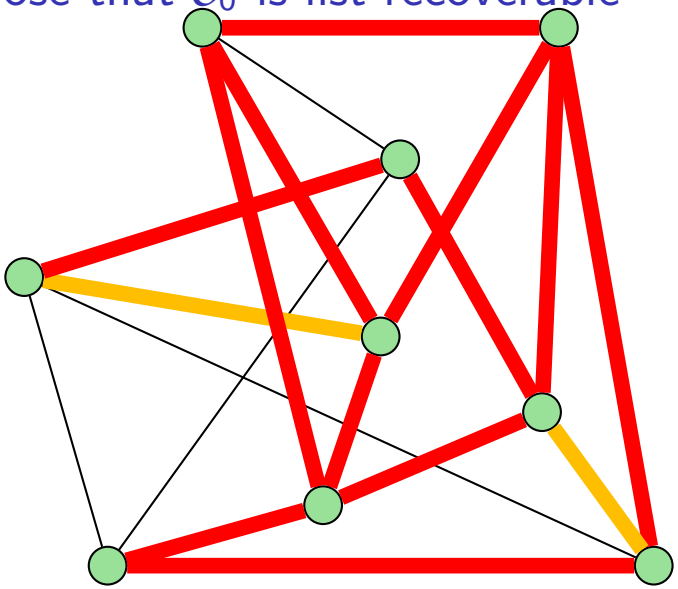

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- Determine some more edges.

The algorithm

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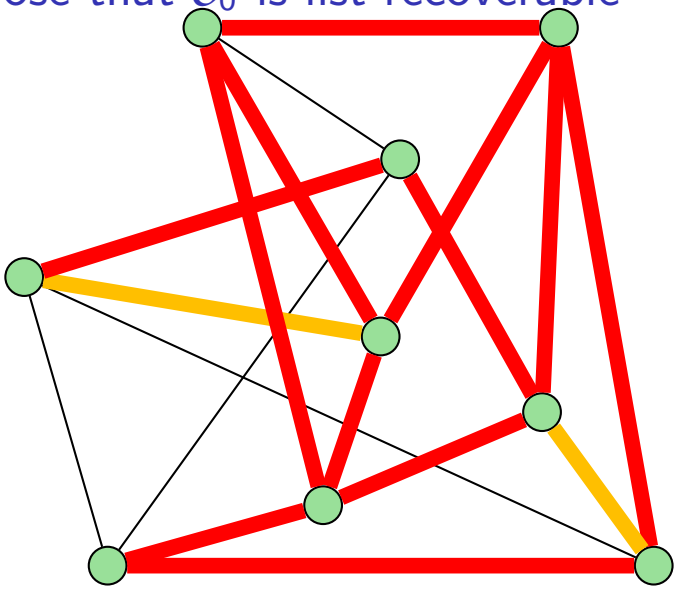

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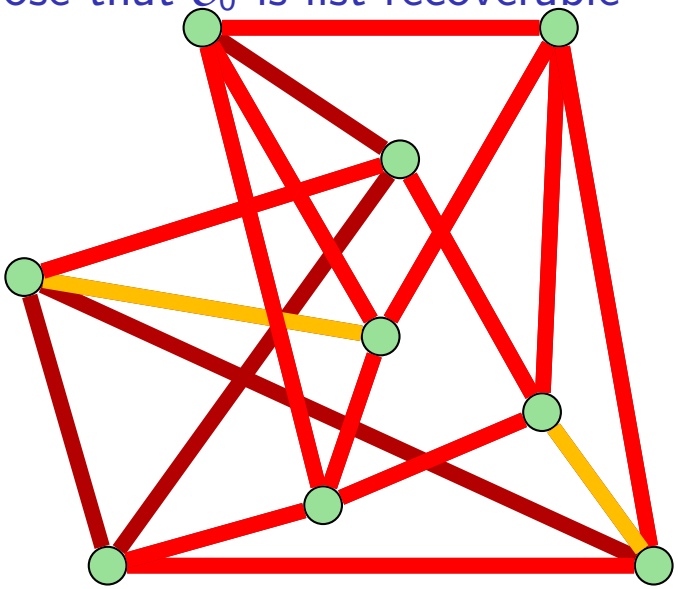

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- Determine a bunch of other edges.
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- Determine some more edges.
- ...
- Correct the rest using regular decoding alg.

The algorithm

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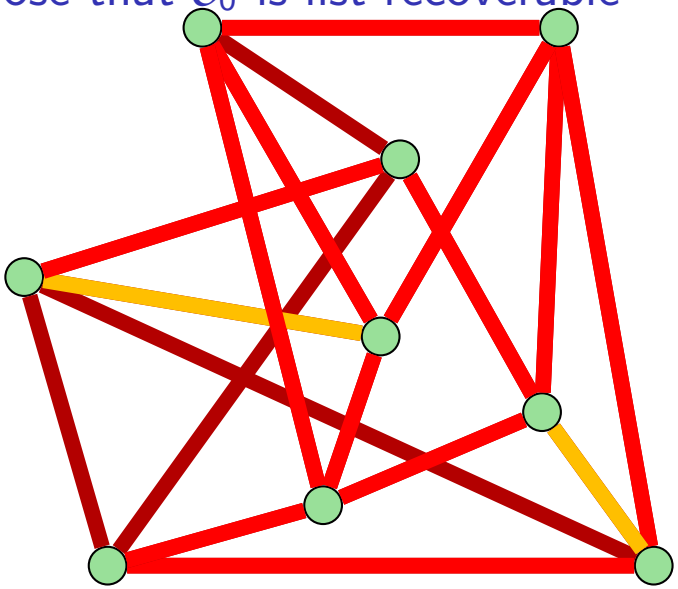

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- ...
- Correct the rest using regular decoding alg.

$$\text{Number of possibilities} = \ell^{\text{number of } \textbf{decision edges}} = \ell^{O(1)}.$$

Why does propagation work?

Equivalence classes of edges

- $L > \ell$ so choosing single symbol won't determine CW
- L CWs, ℓ choices per index
- L^ℓ possible "columns"
  (For each index i, L^ℓ maps from CW to symbol)
  (L^ℓ maps from $[L] \to \ell$)
- Edges are in same equivalance class (wrt vertex) if knowing one determines all the others
- Each vertex has d edges
  At most L^ℓ equivalence classes
  Average equivalence class is of size d/L^ℓ

Why does propagation work?

Large Equivalance Classes

- Average size of an equivalence class is d/L^{ℓ}
- What if we get unlucky and pick an edge in a small equivalence class?
- By the expansion property of the graph there is a large (constant fraction) subset of edges that all have large equivalence class
- Probability a uniformly chosen edge covers an $\varepsilon(\varepsilon - \lambda)$ fraction of the graph is at least $1 - 3\varepsilon\ell^{L}$

Results

- $(R + \eta, \ell, L)$-list recoverable codes
  - Any $R > 0$
  - Any $\ell > 0$
  - Any $\eta > 0$
  - L depends only on ℓ, η
- Linear-time recovery algorithm

Outline

Moral of the story

Inner code has decent _____

$\Rightarrow$

So does the expander code and there's an efficient algorithm!

Moral of the story

Inner code has decent list-recoverability

$\Rightarrow$

So does the expander code and it's list-recoverable in linear time!

Moral of the story

Inner code has decent list-recoverability $\Rightarrow$ So does the expander code and it's list-recoverable in linear time!

Open Questions:

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Open Questions:

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- Handle errors?
- Other applications with erasures?

Moral of the story

Inner code has decent list-recoverability $\Rightarrow$ So does the expander code and it's list-recoverable in linear time!

Open Questions:

- What other properties can expander codes improve?
- Handle errors?
- Other applications with erasures?
- Better inner code? Better constants?

The end

Thanks!

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