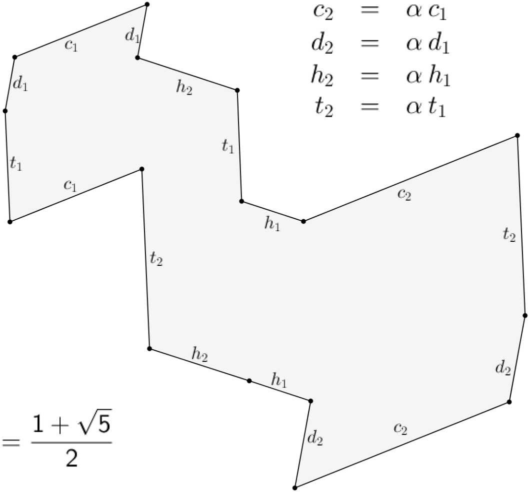
Large $GL(2, \mathbb{R})$ invariant subvarieties of the Hodge bundle and billiards in polygons

Alex Wright

Eskin-Mirzakhani-Mohammadi 2013: All $GL(2,\mathbb{R})$ orbit closures are defined by linear equations.



$$
\begin{aligned}
c_2 &= \alpha\, c_1 \\
d_2 &= \alpha\, d_1 \\
h_2 &= \alpha\, h_1 \\
t_2 &= \alpha\, t_1
\end{aligned}
$$

$$\alpha = \frac{1+\sqrt{5}}{2}$$

If $\mathrm{Jac}(X)$ has an endomorphism with ω as an eigenform, there exists $A : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$, $r \in \mathbb{R}$, so

$$\int_{A\gamma} \omega = r \int_{\gamma} \omega.$$

If two zeros p, q of ω have $n(p-q) = 0$ in $\mathrm{Jac}(X)$, then if $\gamma_{p,q}$ is a path from p to q, there exists $\gamma \in H_1(X, \mathbb{Z})$ so

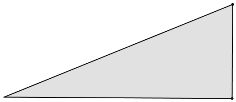
$$\int_{\gamma_{p,q}} \omega = \frac{1}{n} \int_{\gamma} \omega.$$

McMullen 2003: The locus of eigenforms for real multiplication in genus 2 is $GL(2,\mathbb{R})$ invariant. Not true in bigger genus.

Möller 2006: All the linear equations defining a closed orbit come from $\mathrm{Jac}(X)$.

Filip 2015: True for all orbit closures.

Veech 1989: Some unfoldings of triangles have closed $GL(2, \mathbb{R})$ orbit.



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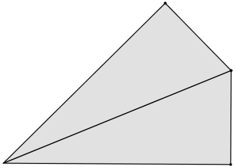

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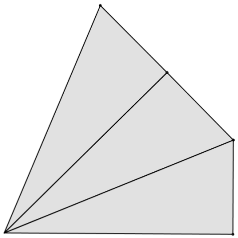

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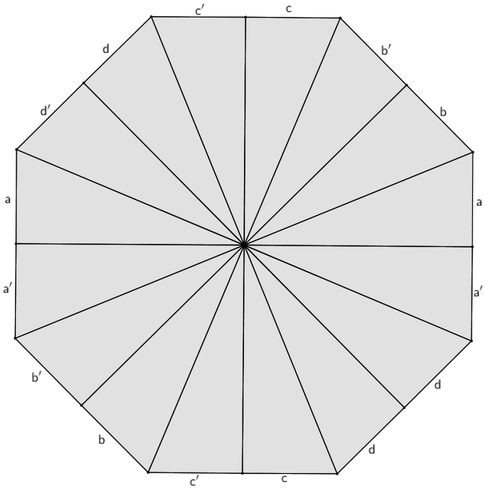


Projection to $\mathcal{M}_g$ is a Teichmüller curve, an isometric embedding $\mathbb{H}/\Gamma \to \mathcal{M}_g$.

$\Gamma \subset SL(2, \mathbb{R})$ is the stabilizer of (X, ω), and must be a lattice (Smillie 1995).

A factor of $\mathrm{Jac}(X)$ has RM, zeros a torsion packet.

The unfolding of every triangle has

- Γ nontrivial,
- $\mathrm{Jac}(X)$ has RM, zeros a torsion packet.

Maybe all unfoldings have closed orbit?

Kenyon-Smillie, Puchta (2001): No!
Classification in acute case.

What are the orbit closures?!?

Triangle (a_1, a_2, a_3) has angles $\frac{a_i}{k}\pi$, where $k = \sum a_i$.

Theorem (Mirzakhani-W)

Infinitely many triangles unfold to translation surfaces with dense orbit, including at least 74 percent of the 1436 non-isosceles triangles with k odd and less than 50*.*

$(1,2,8)$	$(1,3,7)$	$(2,4,5)$			
$(1,4,8)$	$(2,3,8)$	$(3,4,6)$			
$(4,5,6)$					
$(1,2,14)$	$(1,4,12)$	$(1,5,11)$	$(2,3,12)$	$(2,4,11)$	$(2,6,9)$
$(2,7,8)$	$(3,4,10)$	$(3,6,8)$	$(4,6,7)$		
$(1,2,16)$	$(1,3,15)$	$(1,4,14)$	$(1,5,13)$	$(1,6,12)$	$(1,7,11)$
$(2,3,14)$	$(2,4,13)$	$(2,5,12)$	$(2,6,11)$	$(2,7,10)$	$(2,8,9)$
$(3,4,12)$	$(3,5,11)$	$(3,6,10)$	$(3,7,9)$	$(4,6,9)$	$(4,7,8)$
$(5,6,8)$					
$(1,4,16)$	$(1,5,15)$	$(3,5,13)$	$(4,7,10)$	$(6,7,8)$	
$(1,2,20)$	$(1,3,19)$	$(1,4,18)$	$(1,5,17)$	$(1,6,16)$	$(1,7,15)$
$(1,8,14)$	$(1,9,13)$	$(2,3,18)$	$(2,4,17)$	$(2,5,16)$	$(2,6,15)$
$(2,7,14)$	$(2,8,13)$	$(2,9,12)$	$(2,10,11)$	$(3,4,16)$	$(3,5,15)$
$(3,6,14)$	$(3,7,13)$	$(3,8,12)$	$(3,9,11)$	$(4,5,14)$	$(4,6,13)$
$(4,7,12)$	$(4,8,11)$	$(4,9,10)$	$(5,6,12)$	$(5,7,11)$	$(5,8,10)$
$(6,7,10)$	$(6,8,9)$				
$(1,3,21)$	$(1,4,20)$	$(1,6,18)$	$(1,7,17)$	$(1,8,16)$	$(1,10,14)$
$(2,3,20)$	$(2,4,19)$	$(2,5,18)$	$(2,7,16)$	$(2,8,15)$	$(2,10,13)$
$(3,4,18)$	$(3,5,17)$	$(3,6,16)$	$(3,7,15)$	$(3,8,14)$	$(3,9,13)$
$(3,10,12)$	$(4,5,16)$	$(4,6,15)$	$(4,8,13)$	$(4,9,12)$	$(4,10,11)$
$(5,6,14)$	$(6,7,12)$	$(6,8,11)$	$(6,9,10)$	$(7,8,10)$	

"Universal" translation surfaces:
After an affine change of coordinates, looks arbitrarily close to any other translation surface.

Can apply volumes of strata, Siegel-Veech constants, etc to the counting problems in billiards that motivated them.

For any tuple $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ of angles of a rational polygon, there is a variety $\mathcal{M}(\boldsymbol{\theta})$ that is the orbit closure of the unfolding of the generic polygon with these angles.

Open problem: Compute $\mathcal{M}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$.

Case when $\theta_i \in \frac{\pi}{2}\mathbb{Z}$, resp. $\frac{\pi}{3}\mathbb{Z}$ done (Eskin-Athreya-Zorich, resp. Mirzakhani-W).

Eskin-McMullen-Mukamel-W: $\mathcal{M}(\boldsymbol{\theta})$ smaller than expected when $\boldsymbol{\theta}$ is

$$\left(\frac{\pi}{5},\frac{\pi}{5},\frac{\pi}{5},\frac{7\pi}{5}\right) \quad \left(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6},\frac{3\pi}{2}\right) \quad \left(\frac{\pi}{8},\frac{\pi}{8},\frac{\pi}{4},\frac{3\pi}{2}\right)$$

$$\left(\frac{\pi}{8},\frac{\pi}{4},\frac{\pi}{4},\frac{11\pi}{8}\right) \quad \left(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{3},\frac{4\pi}{3}\right) \quad \left(\frac{\pi}{10},\frac{\pi}{5},\frac{\pi}{5},\frac{3\pi}{2}\right)$$

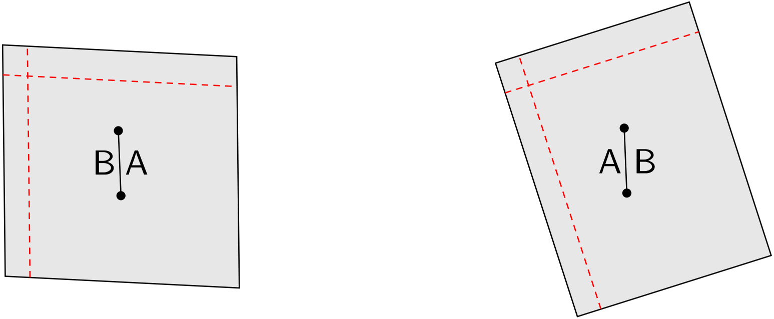
Open problem: Are there infinitely many $\mathcal{M}(\theta_1, \ldots, \theta_n)$ that fail to be as big as possible when $n = 4$? Are there any at all when $n > 4$?

As big as possible means a connected component of a stratum, or locus with an involution.

A measure of the size of an orbit closure is rank, an integer between 1 and the genus.

Closed orbits have rank 1. Strata have rank equal to the genus. The six new examples have rank 2.

Rank is half the dimension, after subtracting the dimension of the space of deformations that do not change absolute periods.



Alternatively, rank is the dimension of the eigenspace in $H^{1,0}(X)$ containing ω.

Full rank means that the defining equations must only involve relative periods, without restricting absolute periods: only torsion relations, no endomorphisms.

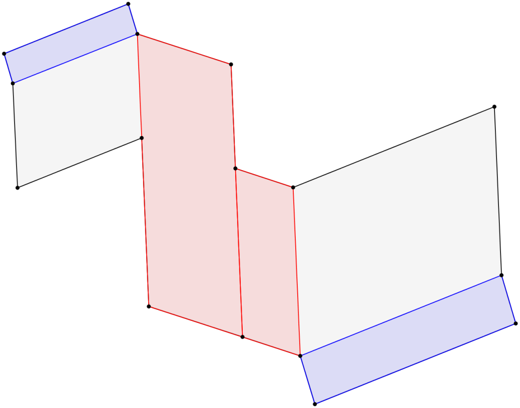
An example is the hyperelliptic locus in a stratum.

Theorem (Mirzakhani-W)

Any full rank orbit closure is a connected component of a stratum or the hyperelliptic locus.

Filip: equivalent to saying all other orbit closures contain only (X, ω) where $\mathrm{Jac}(X)$ has extra endomorphisms.

W: Can deform cylinders and stay in orbit closure.



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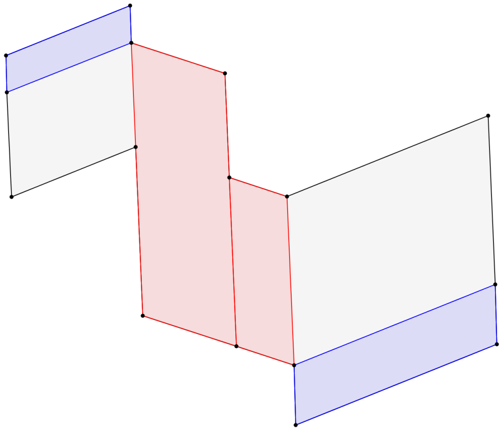

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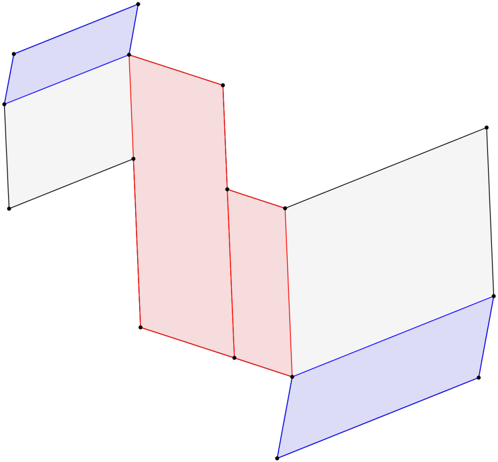


Typically there are generically parallel cylinders. The same deformations must be done simultaneously to generically parallel cylinders.

Rigidity: ratios of circumferences of generically parallel cylinders are constant.

Theorem (Mirzakhani-W)

Any orbit closure closed under all cylinder deformations is a connected component of a stratum.

If $\mathcal{M}$ is full rank but not a stratum, produce a flat geometry certificate that $\mathcal{M}$ is not a stratum.

Ex: a pair of homologous cylinders whose ratio of areas is constant.

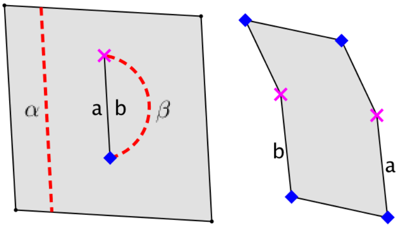
Hope:

1. Degenerate.
2. Use certificate to see boundary $\partial\mathcal{M}$ not a stratum.
3. Use induction to see $\partial\mathcal{M}$ hyperelliptic.
4. Show $\mathcal{M}$ hyperelliptic.

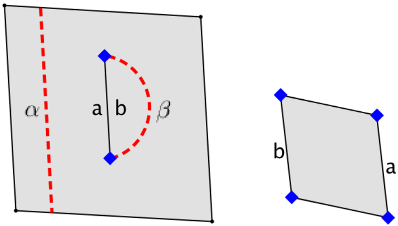
Use "what you see is what you get" partial compactification, and results of Mirzakhani-W on boundary.

Quotient Hodge bundle over $\overline{\mathcal{M}}_{g,n}$ by $(X, \omega) \sim (X', \omega')$ if equal after removing components where ω is identically zero.

Difficulty: Maybe the degeneration of something full rank is not full rank?

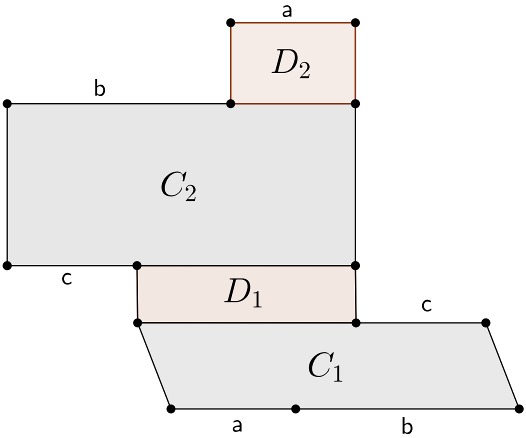


$$\int_\alpha \omega = 2\int_\beta \omega$$



$$\int_\alpha \omega = 2\int_\beta \omega$$

Difficulty: Maybe the degeneration of something not hyperelliptic is hyperelliptic?



But it works.

Analysis aided by remembering marked points and using results of Apisa.

At first glance, full rank result seems useless for triangles.

Trivial rank bound: $\mathcal{M}(\theta_1, \ldots, \theta_n)$ has rank at least $n - 2$.

For triangles, rank at least 1.

So, need non-trivial lower bounds on rank.

McMullen 2003: Locus of eigenforms for real multiplication in genus 3 is not $GL(2,\mathbb{R})$ invariant.

Proof uses the variational formula for the derivative of the period map, to show that the $GL(2,\mathbb{R})$ orbit is not tangent to the real multiplication locus.

$$\int_X \omega_i \omega_j \frac{\overline{\omega_1}}{\omega_1} \neq 0$$

The same technique, together with Filip's work on VHS, can be used to beat the trivial rank bound in some cases.

For most triangles (asymptotically 100 percent?), this shows rank is at least 2. (Apply to classification of obtuse triangles giving closed orbits?)

To get to full rank for some triangles, must use finer structure of orbit closure.

Roughly, list all endomorphism types for all possible orbit closures containing the unfolding, and show $GL(2, \mathbb{R})$ orbit is not tangent to each of them. The final result gives a simple algorithm.

Theorem (Mirzakhani-W)

Any full rank orbit closure is a connected component of a stratum or the hyperelliptic locus.

Theorem

Any orbit closure closed under all cylinder deformations is a connected component of a stratum.

Both false for $GL(2,\mathbb{R})$ invariant loci of strata $\mathcal{Q}$ of quadratic differentials!

Eskin-McMullen-Mukamel-W:
$\mathcal{M} \subsetneq \mathcal{Q}$, $\text{rank}(\mathcal{M}) = \text{rank}(\mathcal{Q})$.

Matheus-Yoccoz: $\mathcal{M} \subsetneq \mathcal{Q}$ closed under all cylinder deformations.