Pedestrian macroscopic models:

game-theoretic vs mechanistic viewpoints

Pierre Degond

Department of Mathematics,

Imperial College London

pdegond@imperial.ac.uk (see http://sites.google.com/site/degond/)

'PEDIGREE' ANR Collaboration

Toulouse Math. Institute (IMT)

P. Degond, J. Fehrenbach, J. Hua

S. Motsch (ASU)

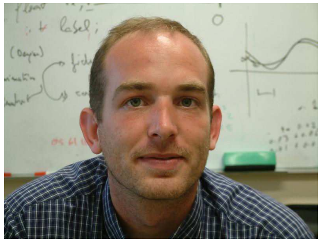
Animal Cognition Lab, Toulouse (CRCA)

M. Moussaid (Berlin), G. Theraulaz, M. Moreau

Theoretical Physics Lab, Orsay (LPT)

C. Appert-Rolland, A. Jelic (ICTP)

INRIA project Lagadic, Rennes



S. Donikian, S. Lemercier, J. Pettré

Summary

1. Issues & context

Issues

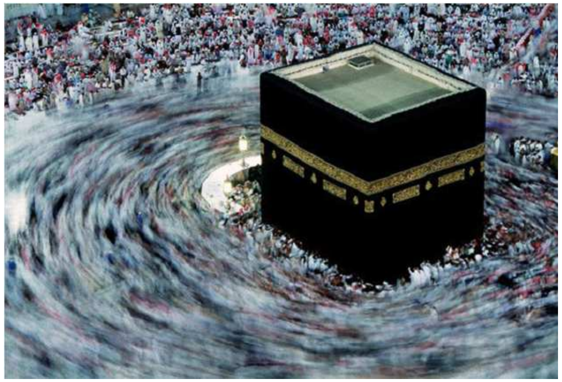
Safety

Avoid crowd disasters

e.g. Duisburg love parade

Cambodia water festival

Demonstration control



Design, comfort, efficiency

Terminals, shopping malls, etc.

Individual-Based Models (IBM)

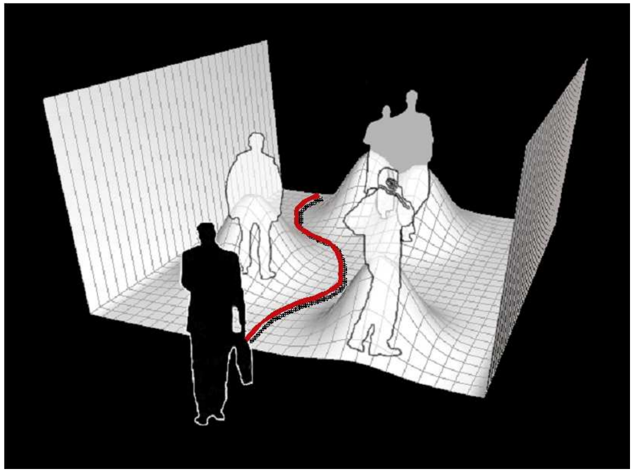
Each individual followed in time

Social force model [Helbing & Molnar, Phys. Rev. E51, 1995]

Analogy with physics:

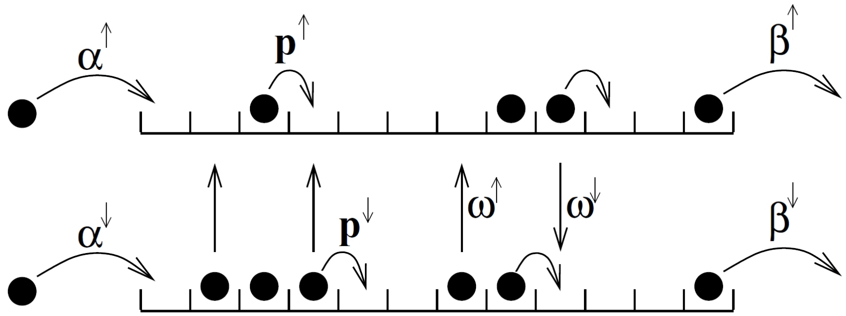
Attractive/repulsive forces

others ...



Cellular automata

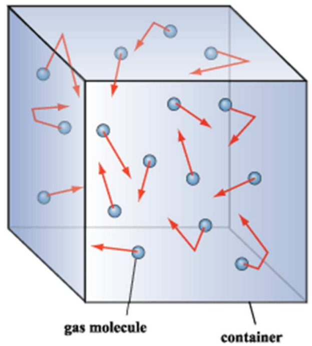
[Burstedde et al, Physica A 295, 2001]



Macroscopic models

Inspired by gas kinetics

[Henderson, Transp. Res. 8, 1974]



Static/dynamic field (∼ chemotaxis)

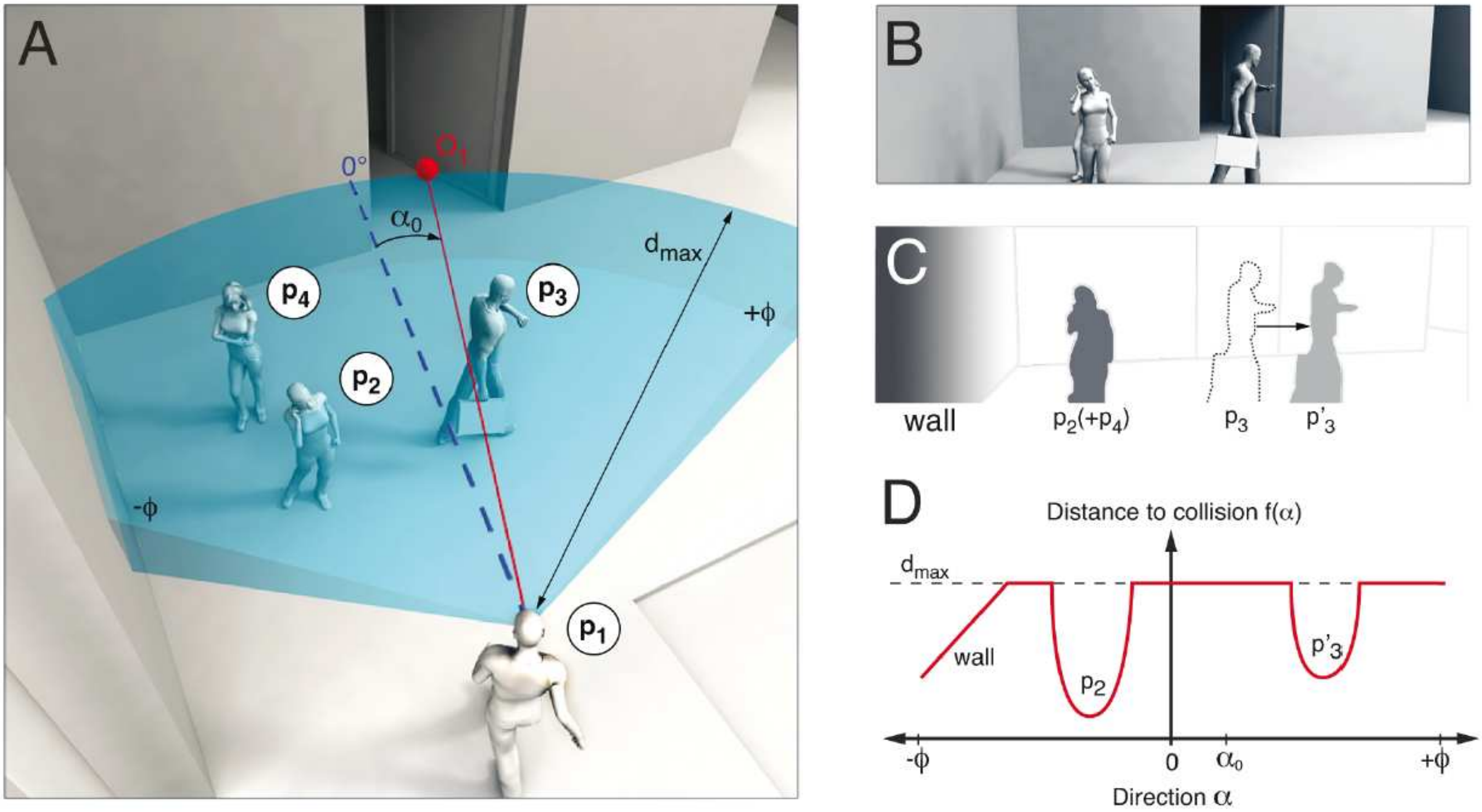
[Hughes et al, Transp. Res. B36, 2002]

Inspired from road traffic

[Colombo et al, MMAS 28, 2005]

2. The Heuristic-Based Model (HBM)

[Moussaïd, Helbing, Theraulaz, PNAS 2011]



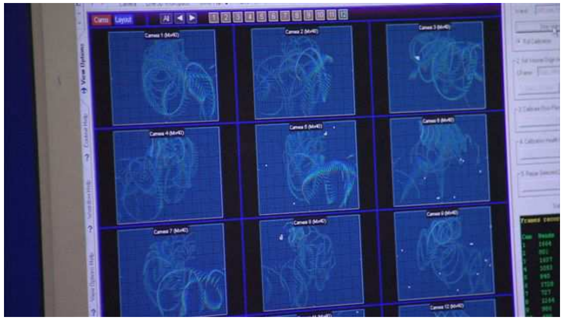
Experiments

Motion capture system

- Sensors reflect infra-red light
- Reflection point camera recorded
- Triangulation $\rightarrow$ coordinates

Circular arena

- Avoids boundary effects

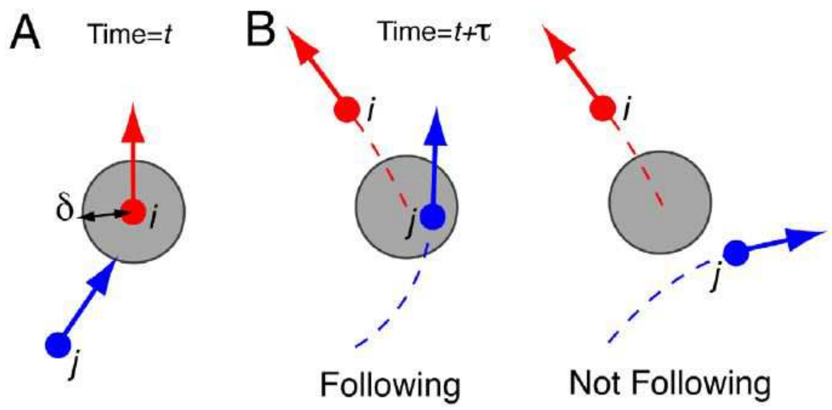


Experiments vs model

Lane formation

Lane definition

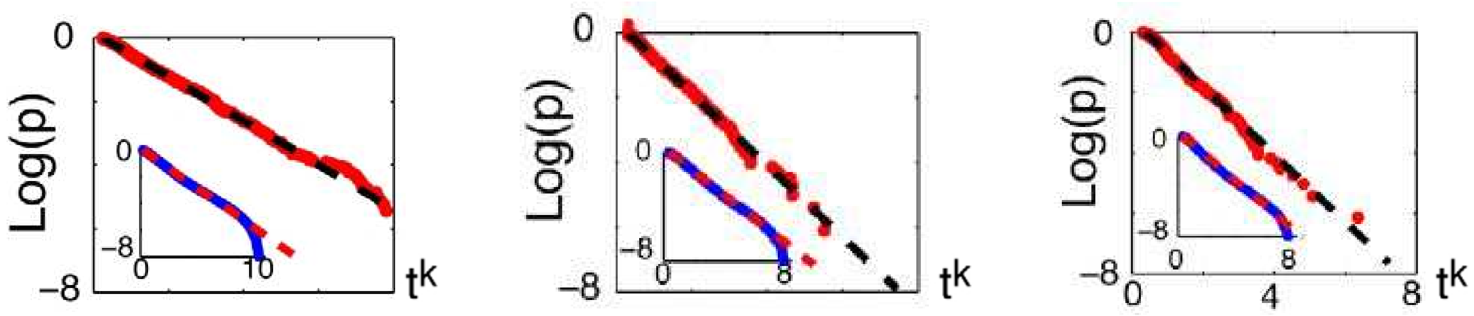
by clustering method



Cluster lifetime statistics

$p(t)dt =$ probability that lifetime $\in [t, t + dt]$

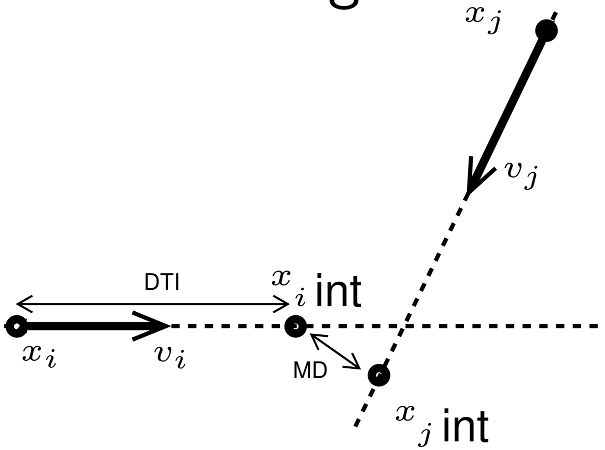
Streched exponential $p(t) = p_0\, e^{at^k}$, $k = 0.4$



In insert: results of model (See [Moussaid et al, PlosCB 2012])

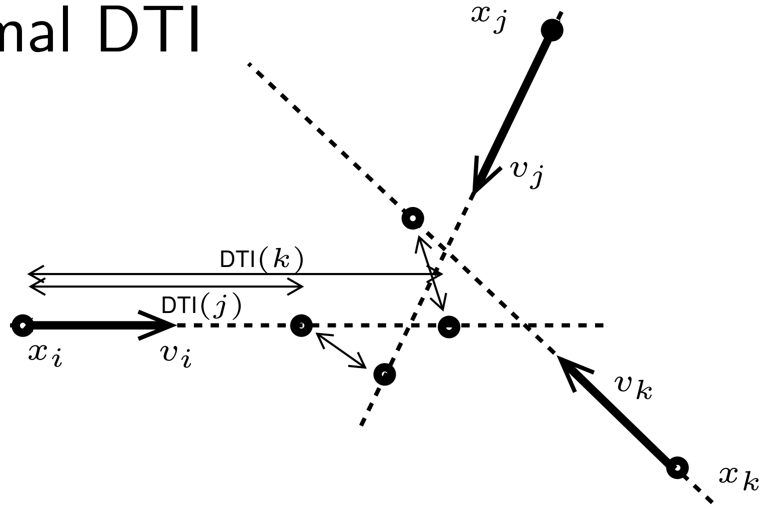
Pedestrians have constant speed

- Evaluation assumes pedestrians move on straight lines
- Distance to Interaction (DTI)
- Minimal Distance (MD)



In case of multiple encounters

- Take the minimal DTI

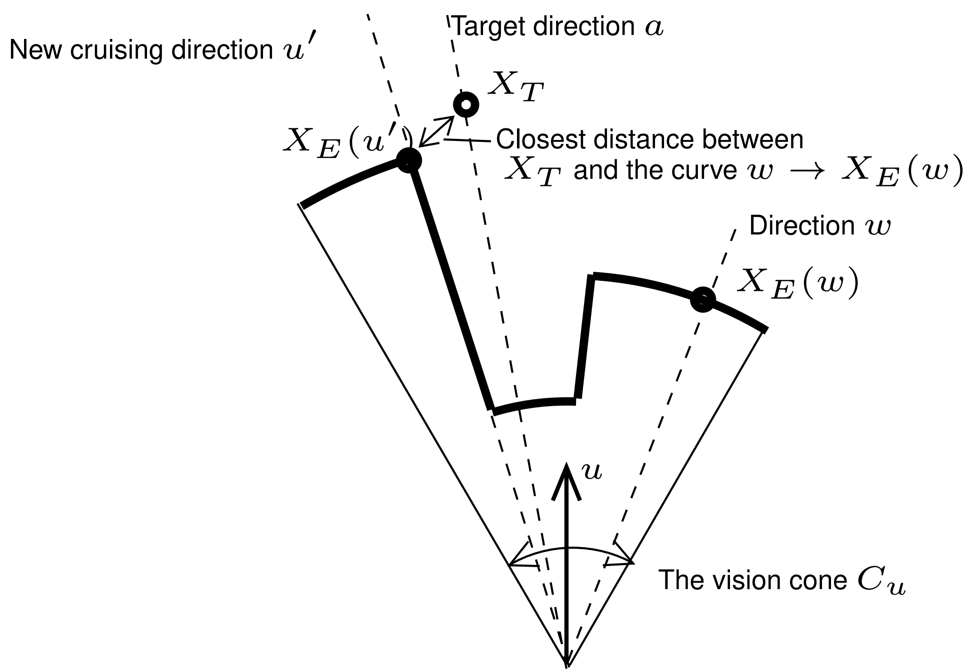


Optimisation: Discrete time step

New cruising direction u' chosen such that

Estimation $X_E(u')$ minimizes distance to target X_T

$$\|X_E(w) - X_T\|^2 \text{ among test directions } w$$



N Particles (pedestrians) $i = 1, \ldots, N$

Position $x_i(t)$, velocity $u_i(t)$, Target direction $a_i(t)$

with $|u_i(t)| = 1$, $|a_i(t)| = 1$, i.e. u_i, $a_i \in \mathbb{S}^1$

$$\dot{x}_i = c u_i,$$

$$du_i = F_i \, dt + P_{u_i^\perp}(\sqrt{2d} \circ dB_i(t))$$

Speed c, noise intensity d, Stratonowich sense $\circ$

Force $F_i \perp u_i$, $P_{u_i^\perp}$ maintains $|u_i| = 1$

Test velocity directions $w \in \mathbb{S}^1 \quad \rightarrow \quad$ Potential $\Phi_i(w,t)$

$$\Phi_i(w,t) = \frac{k}{2} \left| D_i(w) w - L a_i \right|^2$$

Reaction rate k, horizon L

$D_i(w)$ maximal walkable distance in direction w

Force $F_i(t)$ defined by steepest descent of Φ_i

$$F_i(t) = -\nabla_w \Phi_i(u_i(t), t)$$

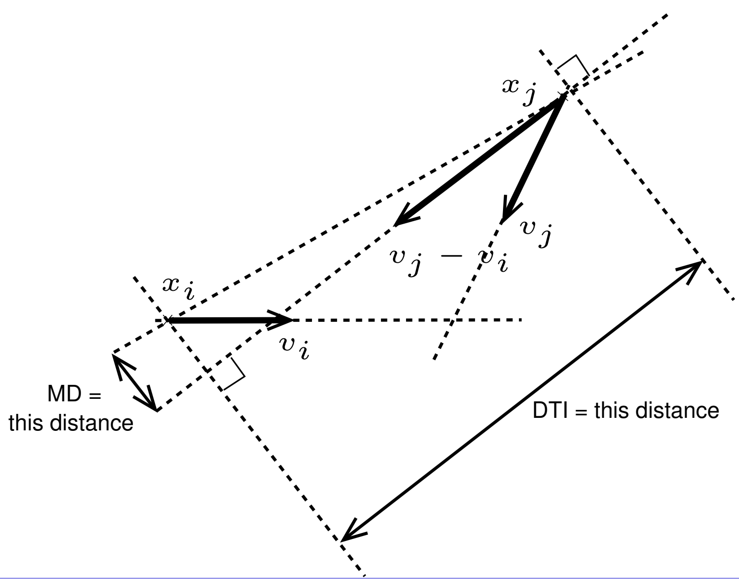
Maximal walkable distance $D_i(w)$

DTI of 'i' against 'j' when 'i' walks in direction w: $D_{ij}(w)$

$$D_i(w) = \text{"}\min_j\text{"}\, D_{ij}(w)$$

For continuum model, replace 'min' by average

e.g. harmonic average in some interaction region



3. The Mean-Field Model

Distribution function $f(x,u,a,t)$ $x \in \mathbb{R}^2$, u, $a \in \mathbb{S}^1$

Probability to find pedestrians at x

with velocity u and target velocity a at time t

$$\partial_t f + \nabla_x \cdot (cuf) + \nabla_u \cdot (Ff) = d\Delta_u f$$

$$F = -\nabla_w \Phi_{(x,a,t)}(u)$$

$$\Phi_{(x,a,t)}(w) = \frac{k}{2} |D_{(x,t)}(w)\, w - La|^2$$

$D_{(x,t)}(w)$ walkable distance of subject at x

in direction w: functional of f

Case: local interactions / no blind zone

Supposes interaction region "very small"

$$D^{-1}_{(x,t)}(w) = \frac{\int_{(v,b)\in \mathbb{T}^2} K(|v-w|)\, f(x,v,b,t)\, dv\, db}{\int_{(v,b)\in \mathbb{T}^2} f(x,v,b,t)\, dv\, db}$$

where K is analytically known (related to the DTI)

If blind zone, $K = K(u, |v-w|)$

Then $D = D_{(x,u,t)}(x)$ and $\Phi = \Phi_{(x,u,a,t)}(w)$

Dependence of Φ on u problematic

Subsequent macroscopic theory cannot be developed

Other closures can be done

4. Macroscopic model

Let $D(u)$ be arbitrary and define

$$Q_D(f) = -\nabla_u \cdot (F_D f) + d\Delta_u f$$

$$F_D(u,a) = -\nabla_u \Phi_D(u,a), \quad \Phi_D(u,a) = \frac{k}{2}|D(u)\,u - La|^2$$

For $f(u,a)$ arbitrary, define

$$D_f^{-1}(u) = \frac{\int_{(v,b)\in\mathbb{T}^2} K(|v-u|)\, f(v,b)\, dv\, db}{\int_{(v,b)\in\mathbb{T}^2} f(x,v,b,t)\, dv\, db}$$

Then mean-field model can be written

$$\partial_t f + \nabla_x \cdot (cuf) = \frac{1}{\varepsilon} Q_{D_f}(f)$$

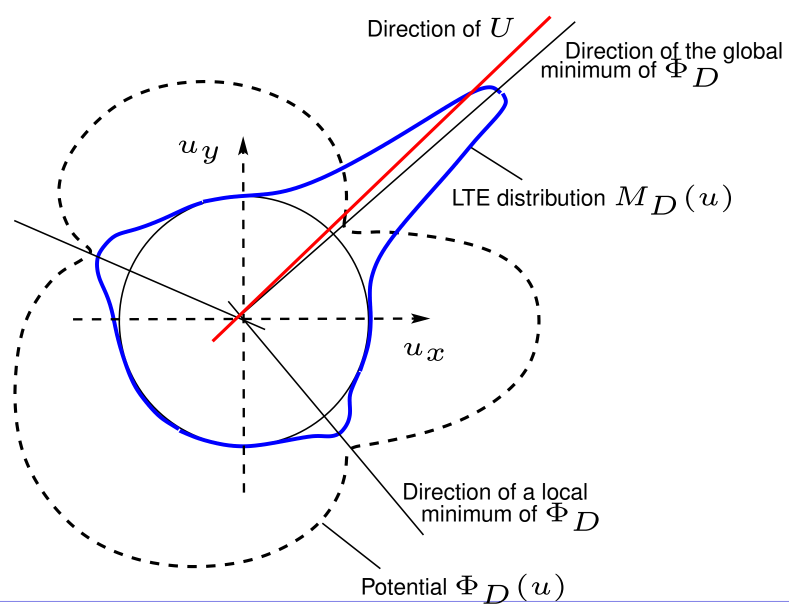
For given $D(u)$, solutions f of $Q_D(f) = 0$ are of the form

$$f(u,a) = \rho(a)\, M_D(u,a)$$

with $\rho(a)$ arbitrary and

$$M_D(u,a) = \frac{1}{Z_D(a)} \exp\Big(- \frac{\Phi_D(u,a)}{d}\Big)$$

where $Z_D(a)$ is s.t.
$\int M_D(u,a)\, du = 1$



Solutions f of $Q_{D_f}(f) = 0$:

are GVM $\quad f = \rho(a)\, M_D(u,a)$

such that $\quad D = D_{\rho M_D}$

Leads to a fixed point equation

$$D^{-1}(u) = \frac{\int_{(v,b)\in\mathbb{T}^2} K(|v-u|)\,\rho(b)\,M_D(v,b)\,dv\,db}{\int_{(v,b)\in\mathbb{S}^1} \rho(b)\,db}$$

Mathematical theory open

Here we assume that for any function $\rho(a)$:

there exists a 'distinguished' solution D_ρ

Hydrodynamic limit

When $\varepsilon \to 0$, formally we have

$$f^\varepsilon \to \rho_{(x,t)}(a)\, M_{D_{\rho_{(x,t)}}}(u,a)$$

where $\rho_{(x,t)}(a)$ satisfies the continuity eq.

$$\partial_t \rho_{(x,t)}(a) + \nabla_x \cdot (c\rho_{(x,t)}(a) U_{\rho_{(x,t)}}(a)) = 0$$

and $U_{\rho_{(x,t)}}(a)$ is the mean equilibrium velocity

$$U_\rho(a) = \int_{u \in \mathbb{S}^1} M_{D_\rho}(u,a)\, u\, du$$

5. Relation to game theory

Spatially homogeneous case:

For probability $f(u,a)$, introduce the 'cost function'

$$\mu_f(u,a) = \Phi_{D_f}(u,a) + d \ln f(u,a)$$

Non-cooperative anonymous game with a continuum of players (aka 'Mean-Field Game [Lasry & Lions])

each pedestrian (player) tries to minimize its cost

by acting on its own decision variable u

f_{NE} is a Nash Equilibrium if

No player can reduce its cost by acting on its control variable u

f_{NE} is a Nash Equilibrium iff $\exists K$ s.t.

$$\mu_{f_{\text{NE}}}(u,a) = K, \quad \forall (u,a) \in \text{Supp}(f_{\text{NE}})$$
$$\mu_{f_{\text{NE}}}(u,a) \geq K, \quad \forall (u,a) \in \mathbb{T}^2$$

The following statements are equivalent:

f is an equilibrium of the kinetic model

and is therefore a GVM distribution

f is a Nash Equilibrium for the Mean-Field Game

defined by cost function μ_f

Hydrodynamic model

Spatially inhomogeneous case

Hydrodynamic model is obtained by

- Taking the continuity equation (i.e. taking the first moment of kinetic eq. wrt u)
- Closing the model by taking the local Nash Equilibrium

See a general framework for

- Kinetic models coupled with Mean-Field Games in

  [D., Liu, Ringhofer, J. Nonlinear Sci. 2012, JSP 2014, Phil. Trans. Roy. Soc A 2014; D. Herty, Liu CMS 2017]

6. Conclusion

Summary

Heuristic-Based model of Moussaid, Helbing Theraulaz

Derivation of

- Time continuous IBM
- Mean-Field Model
- Hydrodynamic Model

Equilibria $\equiv$ Nash equilibria of a Mean-Field Game

Perspective: calibration, validation, elaboration

- PhD thesis of R. Sanchez-Bailo, co-mentored w. J. Carrillo
- Collaboration with Buro Happold