From the master equation to mean field game asymptotics

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Joint work with Francois Delarue and Kavita Ramanan

Overview

A **mean field game** (MFG) will refer to a game with a continuum of players.

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This talk: Refined MFG asymptotics in the form of a central limit theorem and large deviation principle, as well as non-asymptotic concentration bounds.

Key idea: Use the master equation to quantitatively relate n-player equilibrium to n-particle system of McKean-Vlasov type, building on idea of Cardaliaguet-Delarue-Lasry-Lions '15.

Interacting diffusions

Suppose particles $i = 1, \ldots, n$ interact through their empirical measure according to

$$dX_t^i = b(X_t^i, \bar{\nu}_t^n)dt + dW_t^i, \qquad \bar{\nu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^k},$$

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Under "nice" assumptions on b, we have $\bar{\nu}_t^n \to \nu_t$, where ν_t solves the **McKean-Vlasov** equation,

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$$dX_t = b(X_t, \nu_t)dt + dW_t, \qquad \nu_t = \mathrm{Law}(X_t),$$

or

$$\frac{d}{dt}\langle \nu_t, \varphi \rangle = \langle \nu_t, b(\cdot, \nu_t)\nabla\varphi(\cdot) + \frac{1}{2}\Delta\varphi(\cdot)\rangle.$$

Empirical measure limit theory

There is a rich literature on asymptotics of $\bar{\nu}^n_t$:

1. LLN: $\bar{\nu}^n \to \nu$, where ν solves a McKean-Vlasov equation.
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The idea: Use the more tractable McKean-Vlasov system to analyze the large-n-particle system.

A class of mean field games

Agents $i = 1, \ldots, n$ have state process dynamics

$$dX_t^i = \alpha_t^i dt + dW_t^i,$$

with $W^1, \ldots, W^n$ independent Brownian, $(X_0^1, \ldots, X_0^n)$ i.i.d.

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Agent i chooses α^i to minimize

$$J_i^n(\alpha^1, \ldots, \alpha^n) = \mathbb{E}\left[\int_0^T \left(f(X_t^i, \bar{\mu}_t^n) + \frac{1}{2}|\alpha_t^i|^2\right) dt + g(X_T^i, \bar{\mu}_T^n)\right],$$

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Say $(\alpha^1, \ldots, \alpha^n)$ form an ϵ-Nash equilibrium if

$$J_i^n(\alpha^1, \ldots, \alpha^n) \le \epsilon + \inf_\beta J_i^n(\ldots, \alpha^{i-1}, \beta, \alpha^{i+1}, \ldots), \forall i = 1, \ldots, n$$

The n-player HJB system

The value function $v_i^n(t,\boldsymbol{x})$, for $\boldsymbol{x} = (x_1, \dots, x_n)$, for agent i in the n-player game solves

$$\partial_t v_i^n(t,\boldsymbol{x}) + \frac{1}{2}\sum_{k=1}^{n} \Delta_{x_k} v_i^n(t,\boldsymbol{x}) + \frac{1}{2}|D_{x_i} v_i^n(t,\boldsymbol{x})|^2$$

$$+ \sum_{k \neq i} D_{x_k} v_k^n(t,\boldsymbol{x}) \cdot D_{x_k} v_i^n(t,\boldsymbol{x}) = f\left(x_i, \frac{1}{n}\sum_{k=1}^{n} \delta_{x_k}\right).$$

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A Nash equilibrium is given by

$$\alpha_t^i = -D_{x_i} v_i^n(t, X_t^1, \dots, X_t^n).$$

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But v_i^n is generally hard to find, especially for large n.

Mean field limit $n \to \infty$?

The problem

Given a Nash equilibrium $(\alpha^{n,1}, \ldots, \alpha^{n,n})$ for each n, can we describe the asymptotics of $(\bar{\mu}^n_t)_{t \in [0,T]}$?

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Previous results, limited to LLN

Lasry/ Lions '06, Feleqi '13, Fischer '14, L. '15, Cardaliaguet-Delarue-Lasry-Lions '15, Cardaliaguet '16...

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A related, better-understood problem

Find a mean field game solution directly, and use it to construct an ϵ_n-Nash equilibrium for the n-player game, where $\epsilon_n \to 0$.
See Huang/Malhamé/Caines '06 & many others.

Proposed mean field game limit

A deterministic measure flow $(\mu_t)_{t\in[0,T]} \in C([0,T];\mathcal{P}(\mathbb{R}^d))$ is a mean field equilibrium (MFE) if:

$$\begin{cases} \alpha^* & \in \arg\min_\alpha \mathbb{E}\left[\int_0^T \left(f(X_t^\alpha,\mu_t) + \frac{1}{2}|\alpha_t|^2\right) dt + g(X_T^\alpha,\mu_T)\right], \\ dX_t^\alpha & = \alpha_t dt + dW_t, \\ & \end{cases}$$

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Law of large numbers

Under strong assumptions, there exists a unique MFE μ, and $\bar{\mu}^n \to \mu$ in probability in $C([0,T];\mathcal{P}(\mathbb{R}^d))$.

Constructing the MFG value function

1. Fix $t \in [0, T)$ and $m \in \mathcal{P}(\mathbb{R}^d)$.
2. Solve the MFG starting from (t, m), i.e., find (α^*, μ) s.t.

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3. Define the value function, for $x \in \mathbb{R}^d$, by

$$U(t, x, m) = \mathbb{E}\left[\int_t^T \left(f(X_s^{\alpha^*}, \mu_s) + \frac{1}{2}|\alpha_s^*|^2\right) ds + g(X_T^{\alpha^*}, \mu_T) \middle| X_t^{\alpha^*} = x\right]$$

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Note: This definition requires uniqueness!

Toward the master equation

The strategy is analogous to classical stochastic optimal control:

1. Show the value function satisfies a dynamic programming principle (DPP).
2. Use the DPP to identify a PDE for the value function.
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The second step requires a notion of derivative on the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures as well as an analog of Itô's formula for certain measure-valued processes.

Derivatives on $\mathcal{P}(\mathbb{R}^d)$

Definition

$u : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if $\exists \frac{\delta u}{\delta m} : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ continuous s.t.

$$\lim_{h \downarrow 0} \frac{u(m + t(\widetilde{m} - m)) - u(m)}{t} = \int_{\mathbb{R}^d} \frac{\delta u}{\delta m}(m, y)\, d(\widetilde{m} - m)(y).$$

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Define also

$$D_m u(m, y) = D_y \left(\frac{\delta u}{\delta m}(m, y) \right).$$

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Key lemma: For $x_1, \ldots, x_n \in \mathbb{R}^d$,

$$D_{x_i} u \left(\frac{1}{n} \sum_{k=1}^{n} \delta_{x_k} \right) = \frac{1}{n} D_m u \left(\frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}, x_i \right).$$

Key tool: The master equation

Using the DPP along with an Itô formula for functions of measures, one derives the master equation:

$$\begin{aligned}
\partial_t U(t,x,m) - \int_{\mathbb{R}^d} D_x U(t,y,m) \cdot D_m U(t,x,m,y)\, m(dy) \\
+ f(x,m) - \frac{1}{2}\left|D_x U(t,x,m)\right|^2 + \frac{1}{2}\Delta_x U(t,x,m) \\
+ \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{div}_y D_m U(t,x,m,y)\, m(dy) = 0,
\end{aligned}$$

Refer to Cardaliaguet-Delarue-Lasry-Lions '15, Chassagneux-Crisan-Delarue '14, Carmona-Delarue '14, Bensoussan-Frehse-Yam '15

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Assume henceforth that there is a smooth classical solution with bounded derivatives!

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See also explicitly solvable models: Carmona-Fouque-Sun '13, L.-Zariphopoulou '17

A first n-particle approximation

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Old idea: Consider the system of n independent processes,

$$dX_t^i = \underbrace{-D_x U(t, X_t^i, \mu_t)}_{\alpha_t^i} dt + dW_t^i.$$

These controls α_t^i can be proven to form an ϵ_n-equilibrium for the n-player game, where $\epsilon_n \to 0$.

A better n-particle approximation

Key idea of Cardaliaguet et al.: Consider the McKean-Vlasov system

$$dY_t^i = \underbrace{-D_x U(t, Y_t^i, \bar{\nu}_t^n)}_{\alpha_t^i} dt + dW_t^i, \qquad \bar{\nu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{Y_t^k}.$$

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Classical theory says that $\bar{\nu}^n \to \nu$, where ν solves the McKean-Vlasov equation,

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We had the same equation for the MFE μ, so uniqueness implies

$$\mu \equiv \nu.$$

So to prove $\bar{\mu}^n \to \mu$, it suffices to show $\bar{\mu}^n$ and $\bar{\nu}^n$ are **close**.

A better *n*-particle approximation

Key result of Cardaliaguet et al. '15

Recalling that $\bar{\mu}^n_t$ denotes the n-player Nash equilibrium empirical measure, $\bar{\mu}^n$ and $\bar{\nu}^n$ are very close.

Note: This requires smoothness assumptions on the master equation U, but not on the n-player HJB system!

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Proof idea: Show that

$$u^n_i(t, x_1, \dots, x_n) := U\left(t, x_i, \frac{1}{n}\sum_{k=1}^n \delta_{x_k}\right)$$

nearly solves the *n*-player HJB system.

The n-player HJB system revisited

We defined

$$u_i^n(t, x_1, \ldots, x_n) := U\left(t, x_i, \frac{1}{n}\sum_{k=1}^{n} \delta_{x_k}\right).$$

Use the master equation U to find

$$\partial_t u_i^n(t, \boldsymbol{x}) + \frac{1}{2}\sum_{k=1}^{n} \Delta_{x_k} u_i^n(t, \boldsymbol{x}) + \frac{1}{2}|D_{x_i} u_i^n(t, \boldsymbol{x})|^2$$
$$+ \sum_{k \neq i} D_{x_k} u_k^n(t, \boldsymbol{x}) \cdot D_{x_k} u_i^n(t, \boldsymbol{x}) = f\left(x_i, \frac{1}{n}\sum_{k=1}^{n} \delta_{x_k}\right) + r_i^n(t, \boldsymbol{x}),$$

where r_i^n is continuous, with $\|r_i^n\|_\infty \leq C/n$.

Nash system vs. McKean-Vlasov system

The n-player Nash equilibrium state processes solve

$$dX_t^i = -D_{x_i} v_i^n(t, X_t^1, \ldots, X_t^n) dt + dW_t^i.$$

Compare this to the McKean-Vlasov system,

$$dY_t^i = -D_x U(t, Y_t^i, \bar{\nu}_t^n) dt + dW_t^i, \quad \text{where} \quad \bar{\nu}_t^n = \frac{1}{n} \sum_{k=1} \delta_{Y_t^k}.$$

Use Lipshitz property of $D_x U$ and Gronwall to bound

$$\frac{1}{n} \sum_{i=1}^n |X_t^i - Y_t^i|^2 \leq \frac{C}{n} \sum_{i=1}^n \int_0^t |(D_{x_i} v_i^n - D_{x_i} u_i^n)(s, X_s^1, \ldots, X_s^n)|^2 ds.$$

Nash system vs. McKean-Vlasov system

We have estimated

$$\frac{1}{n}\sum_{i=1}^{n}|X_t^i - Y_t^i|^2 \leq \frac{C}{n}\sum_{i=1}^{n}\int_0^t |\mathcal{Z}_s^{i,i} - \overline{\mathcal{Z}}_s^{i,i}|^2 ds,$$

where

$$\mathcal{Y}_t^i = v_i^n(t, \boldsymbol{X}_t), \qquad \mathcal{Z}_t^{i,j} = D_{x_j} v_i^n(t, \boldsymbol{X}_t),$$
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The rest of the argument relies on BSDE-type estimates.

Key observation: Recalling $u_i^n(t,\boldsymbol{x}) = U(t, x_i, \frac{1}{n}\sum_{k=1}^n \delta_{x_k})$, the bounds on master equation derivatives yield

$$|\overline{\mathcal{Z}}_t^{i,i}| \le C, \qquad |\overline{\mathcal{Z}}_t^{i,j}| \le C/n, \text{ for } i \ne j.$$

Toward refined mean field game asymptotics

Main idea: Estimate the “distance” between the Nash EQ empirical measure $\bar{\mu}^n$ and the McKean-Vlasov empirical measure $\bar{\nu}^n$, and then **transfer known results on McKean-Vlasov limits**.

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Note: In linear-quadratic systems, we can instead describe the asymptotics of the mean $\int_{\mathbb{R}^d} x \, d\bar{\mu}^n_t(x)$ in a self-contained manner.

Fluctuations

Theorem

The sequences $\sqrt{n}(\bar{\mu}^n_t - \mu_t)$ and $\sqrt{n}(\bar{\nu}^n_t - \mu_t)$ both "converge" to the unique solution of the SPDE:

$$\partial_t S_t(x) = \mathcal{A}^*_{t,\mu_t} S_t(x) - \mathrm{div}_x(\sqrt{\mu_t(x)}\dot{B}(t,x)),$$

where B is a space-time Brownian motion and

$$\mathcal{A}_{t,m}\varphi(x) := \mathcal{L}_{t,m}\varphi(x) - \int_{\mathbb{R}^d} \frac{\delta}{\delta m}\left(D_x U(t,y,m)\right)(x) \cdot \nabla\varphi(y)\, m(dy),$$

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Provides a second-order approximation $\bar{\mu}_t^n \approx \mu_t + \frac{1}{\sqrt{n}} S_t$.

Proof idea

Show $S_t^n = \sqrt{n}(\bar{\mu}_t^n - \bar{\nu}_t^n) \to 0$, then use Kurtz-Xiong '04 to identify limit of $\sqrt{n}(\bar{\nu}_t^n - \mu_t)$. For nice φ,

$$
\begin{aligned}
|\langle S_t^n, \varphi\rangle| &\le \frac{1}{\sqrt{n}} \sum_{i=1}^n |\varphi(X_t^i) - \varphi(Y_t^i)| \le \dots \\
&\le \frac{C}{\sqrt{n}} \sum_{i=1}^n \int_0^t \big(|X_s^i - Y_s^i| + W_2(\bar{\mu}_s^n, \bar{\nu}_s^n) \\
&\qquad + |D_{x_i} v^{n,i}(s, \boldsymbol{X}_s) - D_x U(s, X_s^i, \bar{\mu}_s^n)| \big)\, ds.
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$$\le \frac{C}{\sqrt{n}} \sum_{i=1}^n \int_0^t \big(|X_s^i - Y_s^i| + W_2(\bar{\mu}_s^n, \bar{\nu}_s^n) + |D_{x_i} v^{n,i}(s, \boldsymbol{X}_s) - D_x U(s, X_s^i, \bar{\mu}_s^n)| \big)\, ds.$$

Key point: Master equation estimates yield

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[\sup_{t\in[0,T]} |X_t^i - Y_t^i| \right] \le \frac{C}{n},$$

not $C/\sqrt{n}$! Similarly for other terms.

Proof idea

Show $S^n_t = \sqrt{n}(\bar{\mu}^n_t - \bar{\nu}^n_t) \to 0$, then use Kurtz-Xiong '04 to identify limit of $\sqrt{n}(\bar{\nu}^n_t - \mu_t)$. For nice φ,

$$|\langle S^n_t, \varphi\rangle| \le \frac{1}{\sqrt{n}} \sum_{i=1}^n |\varphi(X^i_t) - \varphi(Y^i_t)| \le \dots$$

$$\le \frac{C}{\sqrt{n}} \sum_{i=1}^n \int_0^t \big(|X^i_s - Y^i_s| + W_2(\bar{\mu}^n_s, \bar{\nu}^n_s) + |D_{x_i} v^{n,i}(s, \boldsymbol{X}_s) - D_x U(s, X^i_s, \bar{\mu}^n_s)|\big)\, ds.$$

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not $C/\sqrt{n}$! Similarly for other terms. Yields $\mathbb{E}|\langle S^n_t, \varphi\rangle| \le C/\sqrt{n}$.

Large deviations

Theorem

The sequences $\bar{\mu}^n$ and $\bar{\nu}^n$ both satisfy a large deviation principle on $C([0,T];\mathcal{P}(\mathbb{R}^d))$, with the same (good) rate function.

$$I(m.) = \begin{cases} \frac{1}{2}\int_0^T \|\partial_t m_t - \mathcal{L}^*_{t,m_t} m_t\|_S^2 dt & \textit{if } m \textit{ abs. cont.} \\ \infty & \textit{otherwise,} \end{cases}$$

where $\|\cdot\|_S$ acts on Schwartz distributions by

$$\|\gamma\|_S^2 = \sup_{\varphi \in C_c^\infty} \langle \gamma, \varphi\rangle^2 / \langle \gamma, |\nabla \varphi|^2\rangle.$$

Heuristically:

$$\mathbb{P}\left(\bar{\mu}^n \in A\right) \approx \exp\left(-n \inf_{m.\in A} I(m.)\right).$$

Large deviations

Proof idea: Show exponential equivalence

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sup_{t\in[0,T]} W_2(\bar{\mu}_t^n, \bar{\nu}_t^n) > \epsilon \right) = -\infty, \ \forall \epsilon > 0,$$

where W_2 is Wasserstein distance, then identify LDP $\bar{\nu}^n$ using Dawson-Gärtner '87 or Budhiraja-Dupuis-Fischer '12.

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Key challenge: Bounding $W_2(\bar{\mu}_t^n, \bar{\nu}_t^n)$ requires **exponential** estimates for terms like

$$\frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n \int_0^T |(D_{x_j} v_i^n - D_{x_j} u_i^n)(t, X_t^1, \ldots, X_t^n)|^2 dt.$$

Non-asymptotic estimates

Theorem (Dimension-free concentration)

$\exists C,\ \delta > 0$ *such that for* $\forall\ a > 0,\ \forall\ n \geq C/a$ *and all* 1*-Lipshitz functions* $\Phi : (C([0,T];\mathbb{R}^d))^n \to \mathbb{R}$ *we have*

$$\mathbb{P}\Big(|\Phi(X^1,\dots,X^n) - \mathbb{E}\,\Phi(X^1,\dots,X^n)| > a\Big) \leq 2ne^{-\delta n a^2} + 2e^{-\delta a^2}.$$

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Corollary

$\exists C,\ \delta > 0$ *such that for* $\forall\ a > 0,\ \forall\ n \geq C/a$ *we have*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} W_2(\bar{\mu}^n_t, \mu_t) > a\Big) \leq 2ne^{-\delta n^2 a^2} + 2e^{-\delta n a^2}.$$

Proof idea.

The map $(x_1,\dots,x_n) \mapsto W_2(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}, \mu_t)$ is $n^{-1/2}$-Lipschitz. □

Non-asymptotic estimates

Quantitatively compare n-player and k-player games:

Corollary

$\exists C,\ \delta > 0$ *such that for* $\forall\ a > 0,\ \forall\ n, k \geq C/a$ *we have*

$$\mathbb{P}\Big(\sup_{t \in [0,T]} W_2(\bar{\mu}_t^n, \bar{\mu}_t^k) > a \Big) \leq 2n e^{-\delta n^2 a^2} + 2e^{-\delta n a^2} + 2k e^{-\delta k^2 a^2} + 2e^{-\delta k a^2}.$$

Non-asymptotic estimates

Proof of concentration theorem.

Use McKean-Vlasov results after showing

$$\mathbb{P}\left(\sqrt{\frac{1}{n}\sum_{i=1}^{n}\|X^i - Y^i\|_\infty^2} > a\right) \leq 2n\exp(-\delta a^2 n^2).$$

Justify dimension-free concentration for McKean-Vlasov systems by showing $P_n := \mathrm{Law}(Y^1, \ldots, Y^n)$ satisfies a transport-entropy inequality with constant independent of n, i.e., $\exists C > 0$ s.t.

$$W_1(P_n, Q) \leq \sqrt{CH(Q|P_n)}, \quad \forall Q \ll P_n.$$

Use results of Djellout-Guillin-Wu '04. □

The moral of the story

Sufficiently smooth solution of master equation
$\Longrightarrow$ refined asymptotics for mean field game equilibria,
by comparing the n-player equilibrium to an n-particle system and then applying existing results on McKean-Vlasov systems.

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Major challenges

- Requires a lot of regularity for the master equation, permitting Lipshitz-BSDE-type estimates.
- Are there counterexamples without smoothness? E.g., can we always expect $\bar{\mu}^n$ and $\bar{\nu}^n$ to be exponentially equivalent?