Mean-Field optimization problems and non-anticipative optimal transport

Beatrice Acciaio
London School of Economics

based on ongoing projects with J. Backhoff, R. Carmona and P. Wang

Robust Methods in Probability & Finance
ICERM, Brown University, June 19-23, 2017

Outline

N-player stochastic differential game

→ N players with private state processes given by the solutions to

$$dX_t^{N,i} = b_t(X_t^{N,i}, \alpha_t^{N,i}, \bar{\nu}_t^{N,i})dt + dW_t^i, \quad i = 1, ..., N$$

- $W^1, ..., W^N$ independent Wiener processes
- $\alpha^{N,1}, ..., \alpha^{N,N}$ controls of the N players
- $\bar{\nu}_t^{N,i} = \frac{1}{N-1} \sum_{j \neq i} \delta_{X_t^{N,j}}$ empirical distrib. states of the other players

→ The objective of player i is to choose a control $\alpha^{N,i}$ that minimizes

$$\mathbb{E}\left[\int_0^T f_t(X_t^{N,i}, \alpha_t^{N,i}, \bar{\eta}_t^{N,i})dt + g(X_T^{N,i}, \bar{\nu}_T^{N,i})\right]$$

- $\bar{\eta}_t^{N,i} = \frac{1}{N-1} \sum_{j \neq i} \delta_{(X_t^{N,j}, \alpha_t^{N,j})}$ empirical joint distrib. of states & controls

→ Statistically identical players: same functions b_t, f_t, g

From N-player game to McKean-Vlasov control problem

- rarely expect existence of global minimizers
- resort to approximation by **asymptotic arguments:**

SDE State Dynamics for N players	$\xrightarrow{\text{optimization}}$	Nash equilibrium for N players
$\lim_{N\to\infty} \downarrow$		$\downarrow \lim_{N\to\infty}$
		Mean-Field Game
McKean-Vlasov dynamics	$\xrightarrow{\text{optimization}}$	controlled McK-V dyn

Vast literature: Caines, Carmona, Delarue, Huang, Lachapelle, Lacker, Lasry, Lions, Malhamé, Pham, Sznitman, Wei,...

The red path: approximating cooperative equilibria

Main idea:

- all agents adopt the same feedback control: $\alpha_t^{N,i} = \phi(t, X_t^{N,i})$
- in the limit (# players $\to \infty$) the private states of players evolve independently of each other
- distribution of private state converges toward distribution of the solution to the **McKean-Vlasov control problem:**

$$\inf_\alpha \mathbb{E}\left[\int_0^T f_t\left(X_t, \alpha_t, \mathcal{L}(X_t, \alpha_t)\right) dt + g\left(X_T, \mathcal{L}(X_T)\right)\right]$$

$$\text{subject to} \quad dX_t = b_t\left(X_t, \alpha_t, \mathcal{L}(X_t)\right) dt + dW_t$$

- under suitable conditions, the optimal feedback controls are ϵ-optimal for large systems of players

The blue path: approximating competitive equilibria

Main idea:

- Seek for Nash equilibria for the N-player game
- Model behaviour of a representative agent, and solve the **Mean-Field Game problem**:
  1) for every fixed joint law η, with first marginal ν, solve

  $$\inf_\alpha \mathbb{E}\Big[\int_0^T f_t(X_t, \alpha_t, \eta_t)\,dt + g(X_T, \nu_T)\Big]$$
  $$\text{s.t. } dX_t = b_t(X_t, \alpha_t, \nu_t)\,dt + dW_t$$

  2) fixed point problem: η s.t. for the solution $\mathcal{L}(X, \alpha) = \eta$
- under suitable conditions, the optimal feedback provides an approximate Nash equilibrium for large system of players
- for potential games, MFG can be formulated as MKV

McKean-Vlasov control problem

→ As a result of either approximation path, we shall study the following **McKean-Vlasov control problem:**

$$\inf_{\alpha} \mathbb{E}\left[\int_0^T f_t\left(X_t, \alpha_t, \mathcal{L}(X_t, \alpha_t)\right) dt + g\left(X_T, \mathcal{L}(X_T)\right)\right]$$

subject to

$$dX_t = b_t\left(X_t, \alpha_t, \mathcal{L}(X_t)\right) dt + dW_t, \quad X_0 = 0$$

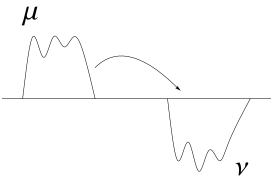
Classical approaches:

- HJB/PDE (Lasry-Lions): forward-backward system of PDEs
- probabilistic: Pontryagin maximum principle, adjoint FBSDEs

Our approach: use Optimal Transport theory

Classical Monge-Kantorovich optimal transport

Given two Polish probability spaces $(\mathcal{X}, \mu), (\mathcal{Y}, \nu)$, move the mass from μ to ν minimizing the cost of transportation $c : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$



$$\mathrm{OT}(\mu, \nu, c) := \inf \left\{ \mathbb{E}^{\pi}[c(x, y)] : \pi \in \Pi(\mu, \nu) \right\},$$

$\Pi(\mu, \nu)$: probability measures on $\mathcal{X} \times \mathcal{Y}$ with marginals μ and ν.

Monge transport: all mass sitting on x is transported into $y = T(x)$.

Kantorovich transport: mass can split.

Causal(≡non-anticipative) optimal transport

Idea: introduce time, and move the mass in a non-anticipative way: what is transported into the 2nd coordinate at time t, depends on the 1st coordinate only up to t (+ sth independent)

Let $\mathcal{F}^{\mathcal{X}} = \left(\mathcal{F}_t^{\mathcal{X}}\right)_t$ on $\mathcal{X}$, $\mathcal{F}^{\mathcal{Y}} = \left(\mathcal{F}_t^{\mathcal{Y}}\right)_t$ on $\mathcal{Y}$ be right-cont. filtrations.

Definition (Causal transport plans $\Pi_c(\mu, \nu)$)

A transport plan $\pi \in \Pi(\mu, \nu)$ is called causal between $(\mathcal{X}, \mathcal{F}^{\mathcal{X}}, \mu)$ and $(\mathcal{Y}, \mathcal{F}^{\mathcal{Y}}, \nu)$ if, for all t and $D \in \mathcal{F}_t^{\mathcal{Y}}$, the map $\mathcal{X} \ni x \mapsto \pi^x(D)$ is measurable w.t.to $\mathcal{F}_t^{\mathcal{X}}$ (π^x regular conditional kernel w.r.to $\mathcal{X}$).

The concept goes back to Yamada-Watanabe (1971); see also Jacod (1980), Kurtz (2014), Lassalle (2015), Backhoff et al. (2016).

$$\mathrm{COT}(\mu, \nu, c) := \inf \left\{ \mathbb{E}^{\pi}[c(X, Y)] : \pi \in \Pi_c(\mu, \nu) \right\}$$

Example: weak-solutions of SDEs

- $\mathcal{X} = \mathcal{Y} = C_0[0, \infty)$
- $\mathcal{F}$ right-continuous canonical filtration

Example (Yamada-Watanabe'71)

Assume weak-existence of the solution to the SDE:

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \quad b, \sigma \text{ Borel measurable.}$$

Then $\mathcal{L}(B, Y)$ is a causal transport plan between the spaces $(C_0[0, \infty), \mathcal{F}, \mathcal{L}(B))$ and $(C_0[0, \infty), \mathcal{F}, \mathcal{L}(Y))$.

- **Transport perspective:** from an observed trajectory of B, the mass can be split at each moment of time into Y only based on the information available up to that time.
- **No splitting of mass:**

  Monge transport $\Longleftrightarrow$ **strong solution** $Y = F(B)$.

McKean-Vlasov control problem and Causal Transport

→ Recall our McKean-Vlasov control problem:

$$\inf_{\alpha} \mathbb{E}\left[\int_0^T f_t\left(X_t, \alpha_t, \mathcal{L}(X_t, \alpha_t)\right) dt + g\left(X_T, \mathcal{L}(X_T)\right)\right]$$

subject to

$$dX_t = b_t\left(X_t, \alpha_t, \mathcal{L}(X_t)\right) dt + dW_t, \quad X_0 = 0$$

→ The joint distribution $\mathcal{L}(W, X)$ is a causal transport plan between $(\mathcal{C}_0[0, T], \mathcal{F}, \mathcal{L}(W))$ and $(\mathcal{C}_0[0, T], \mathcal{F}, \mathcal{L}(X))$

McKean-Vlasov control problem

Definition. A weak solution to the McKean-Vlasov control problem is a tuple $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, W, X, \alpha)$ such that:

(i) $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ supports X, BM W, α is $\mathcal{F}$-progress. meas.

(ii) the state equation $dX_t = b_t\left(X_t, \alpha_t, \mathbb{P}\circ X_t^{-1}\right) dt + dW_t$ holds

(iii) if $(\Omega', (\mathcal{F}'_t)_{t\in[0,T]}, \mathbb{P}', W', X', \alpha')$ is another tuple s.t. (i)-(ii) hold,

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T f_t\left(X_t, \alpha_t, \mathbb{P}\circ(X_t,\alpha_t)^{-1}\right) dt + g\left(X_T, \mathbb{P}\circ X_T^{-1}\right)\right]$$
$$\leq \mathbb{E}^{\mathbb{P}'}\left[\int_0^T f_t\left(X'_t, \alpha'_t, \mathbb{P}'\circ(X'_t,\alpha'_t)^{-1}\right) dt + g\left(X'_T, \mathbb{P}'\circ X'^{-1}_T\right)\right]$$

Assumptions

→ We need some **convexity assumptions**.

→ In the case of linear drift:

$$dX_t = (c_t^1 X_t + c_t^2 \alpha_t + c_t^3 \mathbb{E}[X_t])dt + dW_t,$$

$c_t^i \in \mathbb{R}, c_t^2 > 0$, the assumptions reduce to: for all x, a, η,

- f_t is bounded from below uniformly in t
- $f_t(x, ., \eta)$ is convex
- $f_t(x, a, .)$ is $\prec_{\mathrm{conv}}$-monotone

Example: Inter-bank systemic risk model

[Carmona-Fouque-Sun 2013]

- Inter-bank borrowing/lending, where the log-monetary reserve of each bank, asymptotically, is governed by the MKV eq.

$$dX_t = [k(\mathbb{E}[X_t] - X_t) + \alpha_t]dt + dW_t, \; X_0 = 0$$

$k \geq 0$ rate of m-r in the interaction from b&l between banks

- All banks can control their rate of borrowing/lending to a central bank with the same policy α, to minimize the cost

$$\mathbb{E}\Big[\int_0^T \Big(\frac{1}{2}\alpha_t^2 - q\alpha_t(\mathbb{E}[X_t]-X_t) + \frac{c}{2}(\mathbb{E}[X_t]-X_t)^2\Big)dt + \frac{d}{2}(\mathbb{E}[X_T]-X_T)^2\Big]$$

$q > 0$ incentive to borrowing ($\alpha_t > 0$) or lending ($\alpha_t < 0$),
$c, d > 0$ penalize departure from average

Characterization via non-anticipative optimal transport

- we use transport problems in the path space $\mathcal{C} := \mathcal{C}_0[0,T]$
- γ: Wiener measure on $\mathcal{C}$, $(\omega, \overline{\omega})$: generic element on $\mathcal{C} \times \mathcal{C}$
- here for simplicity control = drift

Theorem

Under the above assumptions, the weak MKV problem is **equivalent** *to the variational problem*

$$\inf_{\nu \in \tilde{\mathcal{P}}} \inf_{\pi \in \Pi_c(\gamma,\nu)} \mathbb{E}^\pi \Big[\int_0^T f_t\Big(\overline{\omega}_t, (\dot{\widehat{\overline{\omega} - \omega}})_t, p_t\Big((\overline{\omega}, \dot{\widehat{\overline{\omega} - \omega}})_{\#}\pi\Big)\Big) dt + g(\overline{\omega}_T, \nu_T) \Big]$$

where $p_t(\eta) = \eta_t$ for $\eta \in \mathcal{P}(\mathcal{C})$, and

$$\tilde{\mathcal{P}} = \{\nu \in \mathcal{P}(\mathcal{C}) : \nu\text{-a.s. pathwise quadr.var. } \exists \text{ and } \langle \omega \rangle_t = t \ \forall\ t\}$$

Characterization via non-anticipative optimal transport

'**Equivalence**' means that the above variational problem and

$$\inf \; \mathbb{E}^{\mathbb{P}}\left[\int_0^T f_t\left(X_t, \alpha_t, \mathbb{P}\circ(X_t,\alpha_t)^{-1}\right)dt + g\left(X_T, \mathbb{P}\circ X_T^{-1}\right)\right]$$

have the same value, where the infimum is taken over tuples $(\Omega, (\mathcal{F}_t), \mathbb{P}, W, X, \alpha)$ s.t. $dX_t = b_t\left(X_t, \alpha_t, \mathbb{P}\circ X_t^{-1}\right)dt + dW_t$, and that moreover the optimizers are related via:

- $\nu^* = \mathcal{L}(X^*)$
- $\pi^* \longleftrightarrow \alpha^*$, with $\pi = \mathcal{L}(W^*, X^*)$

Characterization via non-anticipative optimal transport

→ Weak solutions of MKV control problem given by infimum over tuples $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, W, X, \alpha)$.

Corollary

1. *The infimum can be taken over tuples s.t. α is $\mathcal{F}^X$-measurable (weak closed loop).*
2. *If the infimum is attained, then the optimal α is of weak closed loop form.*

Remark. The outer minimization in VP can be done over $\{\nu \ll \gamma\}$ instead of $\tilde{\mathcal{P}}$, whenever the drift is guaranteed to be square integr. (e.g. drift = control, and $f_t(x, a, \eta) \geq K|a|^2 \;\; \forall\, x, \eta$ and for a large).

Example: separable cost

Separable cost: when running cost = $f_t(x,a) + \tilde{f}_t(\nu_t, x)$,

$$\inf_{\nu \in \tilde{\mathcal{P}}} \left\{ \mathrm{COT}(\gamma, \nu, c(f)) + P_{\tilde{f}}(\nu) \right\}, \quad P_{\tilde{f}}(\nu) \text{ penalty term}$$

↑ standard causal transport (A.-Backhoff-Zalashko 2016)

Example: take $k = q = 0$ in the example above, then

- state dynamics: $dX_t = \alpha_t dt + dW_t$
- cost: $\mathbb{E}\left[\int_0^T \left(\frac{1}{2}\alpha_t^2 + \frac{c}{2}(\mathbb{E}[X_t] - X_t)^2 \right) dt + \frac{d}{2}(\mathbb{E}[X_T] - X_T)^2 \right]$

⇒ COT w.r.t. Cameron-Martin distance (Lassalle 2015):

$$\inf_{\pi \in \Pi_c(\gamma,\nu)} \mathbb{E}^\pi[|\overline{\omega} - \omega|_H^2] = \mathcal{H}(\nu|\gamma),$$

thus

$$\inf_{\nu \ll \gamma} \left\{ \mathcal{H}(\nu|\gamma) + \frac{c}{2}\int_0^T \mathrm{Var}(\nu_t) dt + \frac{d}{2}\mathrm{Var}(\nu_T) \right\}$$

Conclusions

Done so far:

- connection of McKean-Vlasov control problems to non-anticipative transport problems
- characterization of weak McKean-Vlasov solutions via non-anticipative transport

Work in progress:

- The optimization over $\Pi_c(\gamma, \nu)$ is not a standard causal transport problem $\Rightarrow$ new analysis for existence/duality
- Use our characterization theorem in order to find
  - existence and uniqueness of weak MKV solutions
  - explicit formulation of solutions to MKV control problems
- Time-discretization and numerical scheme

Thank you for your attention!