Mean-Field optimization problems and non-anticipative optimal transport

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Outline

1. McKean-Vlasov control problem and motivation
2. Our toolkit: causal transport
3. Characterization of MKV solutions via causal transport
4. Conclusions and ongoing research
N-player stochastic differential game

→ $N$ players with private state processes given by the solutions to

$$dX^N_{t,i} = b_t(X^N_{t,i}, \alpha^N_{t,i}, \bar{\nu}^N_{t,i})dt + dW^i_t, \quad i = 1, \ldots, N$$

- $W^1, \ldots, W^N$ independent Wiener processes
- $\alpha^N_{1,i}, \ldots, \alpha^N_{N,N}$ controls of the $N$ players
- $\bar{\nu}^N_{t,i} = \frac{1}{N-1} \sum_{j \neq i} \delta X^N_{t,j}$ empirical distrib. states of the other players

→ The objective of player $i$ is to choose a control $\alpha^N_{N,i}$ that minimizes

$$\mathbb{E} \left[ \int_0^T f_t(X^N_{t,i}, \alpha^N_{t,i}, \bar{\eta}^N_{t,i})dt + g(X^N_{T,i}, \bar{\nu}^N_{T,i}) \right]$$

- $\bar{\eta}^N_{t,i} = \frac{1}{N-1} \sum_{j \neq i} \delta (X^N_{t,j}, \alpha^N_{t,j})$ empirical joint distrib. of states & controls

→ Statistically identical players: same functions $b_t, f_t, g$
rarely expect existence of global minimizers
resort to approximation by **asymptotic arguments:**

\[
\lim_{N \to \infty} \text{SDE State Dynamics for N players} \quad \xrightarrow{\text{optimization}} \quad \text{Nash equilibrium for N players}
\]

\[
\lim_{N \to \infty} \text{McKean-Vlasov dynamics} \quad \xrightarrow{\text{optimization}} \quad \text{controlled McK-V dyn}
\]

Vast literature: Caines, Carmona, Delarue, Huang, Lachapelle, Lacker, Lasry, Lions, Malhamé, Pham, Sznitman, Wei,...
Main idea:

- all agents adopt the same feedback control: \( \alpha_{t}^{N,i} = \phi(t, X_{t}^{N,i}) \)
- in the limit (\# players \( \to \infty \)) the private states of players evolve independently of each other
- distribution of private state converges toward distribution of the solution to the **McKean-Vlasov control problem**:

\[
\inf_{\alpha} \mathbb{E} \left[ \int_{0}^{T} f_{t}(X_{t}, \alpha_{t}, \mathcal{L}(X_{t}, \alpha_{t})) \, dt + g(X_{T}, \mathcal{L}(X_{T})) \right] \\
\text{subject to} \quad dX_{t} = b_{t}(X_{t}, \alpha_{t}, \mathcal{L}(X_{t})) \, dt + dW_{t}
\]

- under suitable conditions, the optimal feedback controls are \( \epsilon \)-optimal for large systems of players
Main idea:

- Seek for Nash equilibria for the $N$-player game
- Model behaviour of a representative agent, and solve the Mean-Field Game problem:
  1) for every fixed joint law $\eta$, with first marginal $\nu$, solve

$$\inf_{\alpha} \mathbb{E} \left[ \int_0^T f_t (X_t, \alpha_t, \eta_t) \, dt + g (X_T, \nu_T) \right]$$

s.t. $dX_t = b_t (X_t, \alpha_t, \nu_t) \, dt + dW_t$

2) fixed point problem: $\eta$ s.t. for the solution $\mathcal{L}(X, \alpha) = \eta$

- under suitable conditions, the optimal feedback provides an approximate Nash equilibrium for large system of players
- for potential games, MFG can be formulated as MKV
As a result of either approximation path, we shall study the following McKean-Vlasov control problem:

\[
\inf_{\alpha} \mathbb{E} \left[ \int_0^T f_t(X_t, \alpha_t, \mathcal{L}(X_t, \alpha_t)) \, dt + g(X_T, \mathcal{L}(X_T)) \right]
\]

subject to

\[
dX_t = b_t(X_t, \alpha_t, \mathcal{L}(X_t)) \, dt + dW_t, \quad X_0 = 0
\]

Classical approaches:

- HJB/PDE (Lasry-Lions): forward-backward system of PDEs
- probabilistic: Pontryagin maximum principle, adjoint FBSDEs

Our approach: use Optimal Transport theory
Given two Polish probability spaces \((X, \mu), (Y, \nu)\), move the mass from \(\mu\) to \(\nu\) minimizing the cost of transportation \(c : X \times Y \to [0, \infty]\). 

\[
\text{OT}(\mu, \nu, c) := \inf \{\mathbb{E}^\pi[c(x, y)] : \pi \in \Pi(\mu, \nu)\},
\]

\(\Pi(\mu, \nu)\): probability measures on \(X \times Y\) with marginals \(\mu\) and \(\nu\).

**Monge transport:** all mass sitting on \(x\) is transported into \(y = T(x)\).

**Kantorovitch transport:** mass can split.
**Causal (≡ non-anticipative) optimal transport**

**Idea:** introduce time, and move the mass in a non-anticipative way: what is transported into the 2nd coordinate at time \( t \), depends on the 1st coordinate only up to \( t \) (+ sth independent)

Let \( \mathcal{F}^X = \left( \mathcal{F}_t^X \right)_t \) on \( X \), \( \mathcal{F}^Y = \left( \mathcal{F}_t^Y \right)_t \) on \( Y \) be right-cont. filtrations.

**Definition (Causal transport plans \( \Pi_c(\mu, \nu) \))**

A transport plan \( \pi \in \Pi(\mu, \nu) \) is called causal between \( (X, \mathcal{F}^X, \mu) \) and \( (Y, \mathcal{F}^Y, \nu) \) if, for all \( t \) and \( D \in \mathcal{F}_t^Y \), the map \( X \ni x \mapsto \pi^x(D) \) is measurable w.r.t. \( \mathcal{F}_t^X \) (\( \pi^x \) regular conditional kernel w.r.t. \( X \)).

The concept goes back to Yamada-Watanabe (1971); see also Jacod (1980), Kurtz (2014), Lassalle (2015), Backhoff et al. (2016).

\[
\text{COT}(\mu, \nu, c) := \inf \left\{ \mathbb{E}^\pi[c(X, Y)] : \pi \in \Pi_c(\mu, \nu) \right\}
\]
Example: weak-solutions of SDEs

- $\mathcal{X} = \mathcal{Y} = C_0[0, \infty)$
- $\mathcal{F}$ right-continuous canonical filtration

Example (Yamada-Watanabe’71)

Assume weak-existence of the solution to the SDE:

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \quad b, \sigma \text{ Borel measurable.}$$

Then $\mathcal{L}(B, Y)$ is a causal transport plan between the spaces $(C_0[0, \infty), \mathcal{F}, \mathcal{L}(B))$ and $(C_0[0, \infty), \mathcal{F}, \mathcal{L}(Y))$.

- **Transport perspective:** from an observed trajectory of $B$, the mass can be split at each moment of time into $Y$ only based on the information available up to that time.

- **No splitting of mass:**
  
  Monge transport $\iff$ strong solution $Y = F(B)$.
Recall our McKean-Vlasov control problem:

$$\inf_{\alpha} \mathbb{E} \left[ \int_{0}^{T} f_t(X_t, \alpha_t, \mathcal{L}(X_t, \alpha_t)) \, dt + g(X_T, \mathcal{L}(X_T)) \right]$$

subject to

$$dX_t = b_t(X_t, \alpha_t, \mathcal{L}(X_t)) \, dt + dW_t, \quad X_0 = 0$$

The joint distribution $\mathcal{L}(W, X)$ is a causal transport plan between $(C_0[0, T], \mathcal{F}, \mathcal{L}(W))$ and $(C_0[0, T], \mathcal{F}, \mathcal{L}(X))$
McKean-Vlasov control problem

Definition. A weak solution to the McKean-Vlasov control problem is a tuple $(\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}, W, X, \alpha)$ such that:

(i) $(\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ supports $X$, BM $W$, $\alpha$ is $\mathcal{F}$-progress. meas.

(ii) the state equation $dX_t = b_t (X_t, \alpha_t, \mathbb{P} \circ X_t^{-1}) \, dt + dW_t$ holds

(iii) if $(\Omega', (\mathcal{F}'_t)_{t \in [0,T]}, \mathbb{P}', W', X', \alpha')$ is another tuple s.t. (i)-(ii) hold,

$$\mathbb{E}^\mathbb{P} \left[ \int_0^T f_t (X_t, \alpha_t, \mathbb{P} \circ (X_t, \alpha_t)^{-1}) \, dt + g (X_T, \mathbb{P} \circ X_T^{-1}) \right]$$

$$\leq \mathbb{E}^{\mathbb{P}'} \left[ \int_0^T f_t (X'_t, \alpha'_t, \mathbb{P}' \circ (X'_t, \alpha'_t)^{-1}) \, dt + g (X'_T, \mathbb{P}' \circ X'_T^{-1}) \right]$$
We need some **convexity assumptions**.

In the case of linear drift:

\[
dX_t = (c_1^t X_t + c_2^t \alpha_t + c_3^t \mathbb{E}[X_t])dt + dW_t,
\]

\(c_i^t \in \mathbb{R}, \ c_i^2 > 0\), the assumptions reduce to: for all \(x, a, \eta\),

- \(f_t\) is bounded from below uniformly in \(t\)
- \(f_t(x, ., \eta)\) is convex
- \(f_t(x, a, .)\) is \(\ll_{\text{conv}}\)-monotone
Example: Inter-bank systemic risk model

[Carmona-Fouque-Sun 2013]

- Inter-bank borrowing/lending, where the log-monetary reserve of each bank, asymptotically, is governed by the MKV eq.

\[ dX_t = [k(E[X_t] - X_t) + \alpha_t]dt + dW_t, \quad X_0 = 0 \]

- All banks can control their rate of borrowing/lending to a central bank with the same policy \( \alpha \), to minimize the cost

\[ \mathbb{E} \left[ \int_0^T \left( \frac{1}{2} \alpha_t^2 - q\alpha_t(E[X_t] - X_t) + \frac{c}{2}(E[X_t] - X_t)^2 \right) dt + \frac{d}{2}(E[X_T] - X_T)^2 \right] \]

- \( q > 0 \) incentive to borrowing (\( \alpha_t > 0 \)) or lending (\( \alpha_t < 0 \)), \( c, d > 0 \) penalize departure from average
Characterization via non-anticipative optimal transport

- we use transport problems in the path space $C := C_0[0, T]$
- $\gamma$: Wiener measure on $C$, $(\omega, \bar{\omega})$: generic element on $C \times C$
- here for simplicity control = drift

**Theorem**

*Under the above assumptions, the weak MKV problem is equivalent to the variational problem*

$$\inf_{\nu \in \tilde{\mathcal{P}}} \inf_{\pi \in \Pi_c(\gamma, \nu)} \mathbb{E}^\pi \left[ \int_0^T f_t(\omega_t, (\bar{\omega} - \omega)_t, p_t((\omega, \bar{\omega} - \omega)\#\pi)) \, dt + g(\omega_T, \nu_T) \right]$$

where $p_t(\eta) = \eta_t$ for $\eta \in \mathcal{P}(C)$, and

$$\tilde{\mathcal{P}} = \{\nu \in \mathcal{P}(C) : \nu\text{-a.s. pathwise quadr.var. } \exists \text{ and } \langle \omega \rangle_t = t \forall t\}$$
Characterization via non-anticipative optimal transport

'Equivalence' means that the above variational problem and

$$\inf \mathbb{E}^P \left[ \int_0^T f_t (X_t, \alpha_t, P \circ (X_t, \alpha_t)^{-1}) \, dt + g (X_T, P \circ X_T^{-1}) \right]$$

have the same value, where the infimum is taken over tuples $$(\Omega, (\mathcal{F}_t), P, W, X, \alpha)$$ s.t. $dX_t = b_t (X_t, \alpha_t, P \circ X_t^{-1}) \, dt + dW_t$, and that moreover the optimizers are related via:

- $$\nu^* = \mathcal{L}(X^*)$$
- $$\pi^* \longleftrightarrow \alpha^*$$, with $$\pi = \mathcal{L}(W^*, X^*)$$
Characterization via non-anticipative optimal transport

Weak solutions of MKV control problem given by infimum over tuples $(\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}, W, X, \alpha)$.

**Corollary**

1. The infimum can be taken over tuples s.t. $\alpha$ is $\mathcal{F}^X$-measurable (weak closed loop).
2. If the infimum is attained, then the optimal $\alpha$ is of weak closed loop form.

**Remark.** The outer minimization in VP can be done over $\{\nu \ll \gamma\}$ instead of $\tilde{\mathcal{P}}$, whenever the drift is guaranteed to be square integrable (e.g. drift = control, and $f_t(x, a, \eta) \geq K|a|^2 \quad \forall \ x, \eta$ and for $a$ large).
Example: separable cost

**Separable cost:** when running cost $= f_t(x, a) + \tilde{f}_t(\nu_t, x)$,

$$\inf_{\nu \in \tilde{\mathcal{P}}} \left\{ \text{COT}(\gamma, \nu, c(f)) + P_\tilde{f}(\nu) \right\}, \quad P_\tilde{f}(\nu) \text{ penalty term}$$

↑

standard causal transport (A.-Backhoff-Zalashko 2016)

**Example:** take $k = q = 0$ in the example above, then

- state dynamics: $dX_t = \alpha_t dt + dW_t$
- cost: $\mathbb{E} \left[ \int_0^T \left( \frac{1}{2} \alpha_t^2 + \frac{c}{2} (\mathbb{E}[X_t] - X_t)^2 \right) dt + \frac{d}{2} (\mathbb{E}[X_T] - X_T)^2 \right]$

$\Rightarrow$ COT w.r.t. Cameron-Martin distance (Lassalle 2015):

$$\inf_{\pi \in \Pi_c(\gamma, \nu)} \mathbb{E}^{\pi} [||\omega - \omega||^2_H] = \mathcal{H}(\nu|\gamma),$$

thus

$$\inf_{\nu \ll \gamma} \left\{ \mathcal{H}(\nu|\gamma) + \frac{c}{2} \int_0^T \text{Var}(\nu_t) dt + \frac{d}{2} \text{Var}(\nu_T) \right\}$$
Conclusions

Done so far:

- **connection** of McKean-Vlasov control problems to non-anticipative transport problems
- **characterization** of weak McKean-Vlasov solutions via non-anticipative transport

Work in progress:

- The optimization over $\Pi_c(\gamma, \nu)$ is not a standard causal transport problem $\Rightarrow$ new analysis for **existence/duality**
- Use our characterization theorem in order to find
  - **existence and uniqueness** of weak MKV solutions
  - **explicit formulation** of solutions to MKV control problems
- Time-discretization and numerical scheme

Thank you for your attention!