OVERVIEW OF THE STOCHASTIC THEORY OF PORTFOLIOS

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SYNOPSIS

The purpose of these lectures is to offer an overview of **Stochastic Portfolio Theory**, a rich and flexible framework introduced by E.R. Fernholz (2002) for analyzing portfolio behavior and equity market structure.

This theory is descriptive as opposed to normative, is consistent with observable characteristics of actual markets and portfolios, and provides a theoretical tool which is useful for practical applications.

As a theoretical tool, this framework provides fresh insights into questions of market structure and arbitrage, and can be used to construct portfolios with controlled behavior. Most importantly, it does this in a *model-free, robust and pathwise* manner, whose end results eschew stochastic integration.

As a practical tool, Stochastic Portfolio Theory has been applied to the analysis and optimization of portfolio performance, and has been the theoretical underpinning of successful investment strategies for close to 30 years.

More importantly, SPT explains under what conditions it becomes possible to **outperform a capitalization-weighted benchmark index** – and then, exactly how to do this by means of *simple* investment rules.

These typically take the form of adjusting systematically the capitalization weights of an index portfolio to more efficient combinations.

They do it by exploiting the natural *volatilities* of stock prices, and need no forecasts of mean rates of return (which are notoriously harder to estimate).

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  General semimartingales, as opposed to the ITÔ-process framework discussed here.

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1. THE FRAMEWORK

Equity market framework (BACHELIER, SAMUELSON...)

$$\mathrm{d}B(t) = B(t)r(t)\,\mathrm{d}t, \qquad B(0) = 1, \tag{1}$$

$$\mathrm{d}X_i(t) = X_i(t)\left(b_i(t)\,\mathrm{d}t + \sum_{\nu=1}^{N} \sigma_{i\nu}(t)\,\mathrm{d}W_\nu(t)\right), \qquad i = 1, \ldots, n.$$

Money-market $B(\cdot)$, and n stocks with strictly positive capitalizations $X_1(\cdot), \cdots, X_n(\cdot)$.

Driven by the Brownian motion $W(\cdot) = (W_1(\cdot), \cdots, W_N(\cdot))'$ with $N \geq n$. Probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

All processes are assumed to be measurable, and adapted to a filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ which represents the "flow of information" in the market. *Not much needs to be assumed at this point about it... .*

We shall take $r(\cdot) \equiv 0$ **until further notice:** investing in the money-market will amount to hoarding, whereas borrowing from the money-market will incur no interest.

Arithmetic Mean Rates of Return $b(\cdot) = (b_1(\cdot), \ldots, b_n(\cdot))'$ and *Variation Rates* $(\alpha_{ii}(\cdot))_{1 \le i \le n}$ satisfy for every $T \in (0, \infty)$ the integrability condition

$$\sum_{i=1}^{n} \int_0^T \big(\big| b_i(t) \big| + \alpha_{ii}(t) \big) \, \mathrm{d}t < \infty \,, \qquad \text{a.s.}$$

Here $\sigma(\cdot) = (\sigma_{ij}(\cdot))_{1 \le i \le n,\, 1 \le j \le N}$ is the $(n \times N)-$matrix of local volatility rates, and $\alpha(\cdot) = \sigma(\cdot)\sigma'(\cdot)$ is the $(n \times n)-$matrix of *Variation/Covariation rates*

$$\alpha_{ij}(t) := \sum_{\nu=1}^{N} \sigma_{i\nu}(t)\sigma_{j\nu}(t) = \frac{1}{X_i(t)X_j(t)} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \langle X_i, X_j \rangle(t) \,.$$

2. STRATEGIES and PORTFOLIOS

A **small investor** (whose actions cannot affect market prices) decides, at each time t and for every $1 \le i \le n$, which proportion $\pi_i(t)$ of his current wealth $V(t)$ to invest in the i^{th} stock.

We require that each $\pi_i(t)$ be $\mathcal{F}(t)-$measurable. The proportion $1 - \sum_{i=1}^{n} \pi_i(t)$ gets invested in the money market.

The wealth $V(\cdot) \equiv V^{v,\pi}(\cdot)$ corresponding to an initial capital $v \in (0, \infty)$ and a portfolio $\pi(\cdot) = \big(\pi_1(\cdot), \cdots, \pi_n(\cdot)\big)'$ satisfies $V(0) = v$ and the MARKOWITZ equation

$$\frac{\mathrm{d}\, V(t)}{V(t)} = \sum_{i=1}^{n} \pi_i(t)\, \frac{\mathrm{d}X_i(t)}{X_i(t)} + \left(1 - \sum_{i=1}^{n} \pi_i(t)\right) \frac{\mathrm{d}B(t)}{B(t)}.$$

To wit: The portfolio's arithmetic return is the "weighted average", according to its weights $\pi_1(t), \cdots, \pi_n(t)$, of the individual assets' arithmetic returns.

Equivalently,

$$\frac{\mathrm{d}V(t)}{V(t)} = b^{\pi}(t)\mathrm{d}t + \sum_{\nu=1}^{N} \sigma_{\nu}^{\pi}(t)\,\mathrm{d}W_{\nu}(t)$$

where

$$b^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t) b_i(t)\,, \qquad \sigma_{\nu}^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t)\sigma_{i\nu}(t)\,,$$

are, respectively, the portfolio's arithmetic rate-of-return, and the portfolio's volatilities.

- Let us introduce also the portfolio's variation

$$\mathrm{a}^{\pi\pi}(t) := \big(\sigma^{\pi}(t)\big)'\sigma^{\pi}(t) = \sum_{\nu=1}^{N} (\sigma_{\nu}^{\pi}(t))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\alpha_{ij}(t)\pi_j(t)\,.$$

FORMAL DEFINITION:

- We shall call *portfolio* an $\mathbb{F}-$progressively measurable process $\pi : [0, \infty) \times \Omega \to \mathbb{R}^n$ which satisfies, for each $T \in (0, \infty)$, the integrability condition

$$\int_0^T \Big(\big| b^\pi(t) \big| + \mathrm{a}^{\pi\pi}(t) \Big) \, \mathrm{d}t \, < \, \infty \, , \quad \text{a.s.}$$

The collection of all portfolios will be denoted by $\mathbf{\Pi}$.

- The wealth process corresponding to a portfolio $\pi(\cdot) \in \mathbf{\Pi}$ and an initial capital $v > 0$ is strictly positive:

$$V^{v,\pi}(t) \, = \, v \, \exp \left\{ \int_0^t \gamma^\pi(s) \, \mathrm{d}s + \int_0^t \big(\sigma^\pi(s) \big)' \mathrm{d}W(s) \right\} > 0 \, .$$

Here the portfolio's "instantaneous growth rate" is given as

$$\gamma^\pi(t) \, := \, b^\pi(t) - \frac{1}{2} \, \mathrm{a}^{\pi\pi}(t).$$

(You cannot go broke if you invest *reasonable* proportions of your wealth across assets. Here, "reasonable" reflects the integrability condition.)

- The portfolio $\kappa(\cdot) \equiv 0$ with $\kappa_1(\cdot) \equiv \cdots \equiv \kappa_n(\cdot) \equiv 0$ never invests in the stock market (keeps all wealth in cash: $V^{v,\kappa}(\cdot) \equiv v, \kappa_0(\cdot) \equiv 1$).

- A portfolio $\pi(\cdot) \in \mathbf{\Pi}$ with

$$\sum_{i=1}^{n} \pi_i(t) \, = \, 1 \, , \qquad \forall \;\; 0 \leq t < \infty$$

almost surely, will be called *stock portfolio.*

A stock portfolio never invests in the money market, and never borrows from it.

. The collection of all stock portfolios will be denoted by by $\mathfrak{P}$.

. We shall say that a portfolio $\pi(\cdot)$ is *bounded*, if $\| \pi(t, \omega)\| \le \mathcal{K}_\pi$ holds for all $(t, \omega) \in [0, \infty) \times \Omega$ and some real constant $\mathcal{K}_\pi > 0$.

. We shall call a portfolio $\pi(\cdot) \in \mathbf{\Pi}$ *long-only*, if it satisfies almost surely

$$\pi_1(t) \ \ge \ 0 \, , \ \cdots \ , \pi_n(t) \ \ge \ 0 \, , \quad \sum_{i=1}^{n} \pi_i(t) \le 1 \, , \qquad \forall \ \ 0 \le t < \infty \, .$$

Every long-only portfolio is also bounded.

3. THE MARKET PORTFOLIO

Consider now the *market portfolio* $\mu(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_n(\cdot)\big)'$ given by

$$\mu_i(t) := \frac{X_i(t)}{X(t)}\,, \qquad i = 1, \ldots, n\,,$$

where

$$X(t) := X_1(t) + \ldots + X_n(t)\,.$$

This invests in all stocks in proportion to their relative capitalization weights. Accomplishes this by buying a fixed number of shares in each stock at time $t = 0$ – the same for all stocks – and holding on to these shares afterwards (the ultimate "buy and hold" strategy). Corresponds to the S&P 500 index.

Such an investment amounts to "owning the entire market": the wealth process becomes

$$V^{v,\mu}(\cdot) = v\,X(\cdot)/X(0)\,.$$

4. RELATIVE ARBITRAGE

Given a real number $T > 0$ and any two portfolios $\pi(\cdot) \in \mathbf{\Pi}$ and $\varrho(\cdot) \in \mathbf{\Pi}$, we shall say that $\pi(\cdot)$ is a *relative arbitrage with respect to* $\varrho(\cdot)$ over $[0, T]$, if we have

$$\mathbb{P}\left(V^{1,\pi}(T) \geq V^{1,\varrho}(T)\right) = 1 \quad \text{and} \quad \mathbb{P}\left(V^{1,\pi}(T) > V^{1,\varrho}(T)\right) > 0 \,.$$

✠ *Strong* relative arbitrage: $\mathbb{P}\left(V^{1,\pi}(T) > V^{1,\varrho}(T)\right) = 1 \,.$

A different terminology one can use here, is to say that $\pi(\cdot)$ *outperforms*, or *dominates*, $\varrho(\cdot)$. The classical paper of Merton (1973) actually introduces this latter terminology in an abstract setting, but does not give examples. More on this presently... .

• With $\varrho(\cdot) \equiv \kappa(\cdot) \equiv 0$, this definition becomes the standard definition of arbitrage relative to cash.

• *Simple Exercise:* No relative arbitrage is possible with respect to a portfolio $\varrho^*(\cdot) \in \mathbf{\Pi}$ that has the so-called *"supermartingale numéraire property":*

$$V^{1,\pi}(\cdot) \,/\, V^{1,\varrho^*}(\cdot) \quad \text{is a supermartingale, for every } \pi(\cdot) \in \mathbf{\Pi}.$$

In fact, it suffices that this property hold under some equivalent probability measure.

4.a: Market Price of Risk (Optional)

Suppose for a moment that there exists a *market price of risk* (or "relative risk") $\vartheta : [0, \infty) \times \Omega \to \mathbb{R}^N$: an $\mathbb{F}-$adapted process that satisfies for each $T \in (0, \infty)$ the requirements

$$\sigma(t)\vartheta(t) = b(t)\,, \quad \forall \quad 0 \le t \le T \qquad \text{and} \qquad \int_0^T \|\vartheta(t)\|^2 \, \mathrm{d}t < \infty\,.$$

• Whenever it exists, such a process $\vartheta(\cdot)$ allows us to introduce a corresponding "deflator" $Z^\vartheta(\cdot)$. This is an exponential local martingale and supermartingale

$$Z^\vartheta(t) := \exp\left\{ -\int_0^t \vartheta'(s) \, \mathrm{d}W(s) - \frac{1}{2} \int_0^t \|\vartheta(s)\|^2 \, \mathrm{d}s \right\}, \quad 0 \le t < \infty\,.$$

A martingale, if and only if $\mathbb{E}\big(Z^\vartheta(T)\big) = 1\,, \ \forall \ T \in (0, \infty)$.

• It has the property that $Z^\vartheta(\cdot) V^{v,\pi}(\cdot)$ is also a local martingale (and supermartingale), for every $\pi(\cdot) \in \mathbf{\Pi}$, $v \ge 0$.

In the presence of a market-price-of-risk process $\vartheta(\cdot)$ we have also

$$\frac{\mathrm{d}\, V^{v,\pi}(t)}{V^{v,\pi}(t)} \,=\, \pi'(t)\sigma(t)\big[\,\mathrm{d}W(t) + \vartheta(t)\mathrm{d}t\,\big]\,.$$

Let us pair this with the equation

$$\mathrm{d}Z^{\vartheta}(t) \,=\, -Z^{\vartheta}(t)\,\vartheta'(t)\mathrm{d}W(t)$$

for the corresponding deflator $Z^{\vartheta}(\cdot)$ we introduced in the last slide

$$Z^{\vartheta}(\cdot) \,=\, \exp\left\{-\int_0^{\cdot} \vartheta'(t)\,\mathrm{d}W(t) - \frac{1}{2}\int_0^{\cdot} \|\vartheta(t)\|^2\,\mathrm{d}t\right\}$$

. Simple stochastic calculus shows that the "deflated wealth process" $Z^{\vartheta}(\cdot)V^{v,\pi}(\cdot)$ is also a positive local martingale and a supermartingale for every $\pi(\cdot) \in \mathbf{\Pi}$, $v > 0$, namely

$$Z^{\vartheta}(t)V^{v,\pi}(t) \,=\, v + \int_0^t Z^{\vartheta}(s)V^{v,\pi}(s)\,\big(\sigma'(s)\pi(s) - \vartheta(s)\big)'\,\mathrm{d}W(s)\,.$$

4.b: Strict Local Martingales (Optional)

The existence of such a deflator proscribes *scalable* (or *egregious*, or *immediate*, or *of the first kind*) arbitrage opportunities, a.k.a. UP's BR (*Unbounded Profits with Bounded Risk*).

- For our purposes, it will be very important to allow $Z^{\vartheta}(\cdot)$ to be a *strict local martingale;* i.e., not to exclude the possibility

$$\mathbb{E}\big(Z^{\vartheta}(T)\big) < 1$$

for some horizons $T \in (0, \infty)$.
This means, we still keep the door open to the existence of relative arbitrage opportunities that cannot be scaled (in a somewhat colloquial manner, the existence of some *Small Profits with Bounded Risk*).

• Suppose that the covariation matrix-valued process $\alpha(\cdot)$ satisfies, for some $L \in (0, \infty)$, the a.s. boundedness condition

$$\xi' \alpha(t) \xi = \xi' \sigma(t) \sigma'(t) \xi \le L \|\xi\|^2, \quad \forall \;\; t \in [0, \infty)\,, \;\; \xi \in \mathbb{R}^n\,. \qquad (2)$$

If $\pi(\cdot)$ is arbitrage relative to $\rho(\cdot)$ and both are **bounded portfolios**, then $Z^\vartheta(\cdot)$ and $Z^\vartheta(\cdot) V^{v,\rho}(\cdot)$ are *strict local martingales:*

$$\mathbb{E}\big[Z^\vartheta(T)\big] < 1\,, \qquad \mathbb{E}\,[\, Z^\vartheta(T) V^{v,\rho}(T)\,] < v\,.$$

NO EMM CAN THEN EXIST !

• In particular, if there exists a bounded portfolio $\pi(\cdot)$ which is arbitrage relative to $\mu(\cdot)$, we have

$$\mathbb{E}\big[Z^\vartheta(T)\big] < 1, \quad \mathbb{E}\,[\, Z^\vartheta(T) X(T)\,] < X(0), \quad \mathbb{E}\,[\, Z^\vartheta(T) X_i(T)\,] < X_i(0)\,.$$

Relative arbitrage becomes then a “machine” for generating strict local martingales.

5. REMARKS and PREVIEW (Optional)

- Suppose there exists a real constant $h > 0$ for which we have

$$\boxed{\sum_{i=1}^{n} \mu_i(t)\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i(t)\alpha_{ij}(t)\mu_j(t) \;\geq\; h\,,} \quad \forall\ 0 \leq t < \infty\,. \tag{3}$$

. Under this condition we shall see that, for a sufficiently large real constant $c = c(T) > 0$, the long-only *modified entropic portfolio*

$$\mathfrak{E}_i^{(c)}(t) \;=\; \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\sum_{j=1}^{n} \mu_j(t)\big(c - \log \mu_j(t)\big)}\,, \qquad i = 1, \cdots, n \tag{4}$$

is strong relative arbitrage with respect to the market portfolio $\mu(\cdot)$ over any given time-horizon $[0, T]$ with

$$T > (2 \log n)/h\,.$$

- It was an open question for 10 years, whether such relative arbitrage can be constructed over *arbitrary* time-horizons, under

$$\sum_{i=1}^{n} \mu_i(t)\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{i=1}^{n} \mu_i(t)\alpha_{ij}(t)\mu_j(t) \geq h, \quad \forall \ 0 \leq t < \infty,$$

the condition of (3).

This question has now been settled – and the answer is negative. But with some very interesting twists and turns (to come).

• • Another condition guaranteeing the existence of relative arbitrage with respect to the market is, as we shall see, that there exist a real constant $h > 0$ with

$$\boxed{\big(\mu_1(t)\cdots\mu_n(t)\big)^{1/n}\left[\sum_{i=1}^{n}\alpha_{ii}(t)-\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(t)\right]\geq h,}\quad \forall\ t\geq 0. \tag{5}$$

Then with $\mathfrak{m}(t) := (\mu_1(t)\cdots\mu_n(t))^{1/n}$ and for $c = c(T) > 0$ large enough, the long-only *modified equally-weighted portfolio*

$$\varphi_i^{(c)}(t) = \frac{c}{c+\mathfrak{m}(t)}\cdot\frac{1}{n}+\frac{\mathfrak{m}(t)}{c+\mathfrak{m}(t)}\cdot\mu_i(t), \quad i=1,\cdots,n, \tag{6}$$

a convex combination of equal-weighting and the market, is strong arbitrage relative to the market portfolio $\mu(\cdot)$, over any given time horizon $[0, T]$ with

$$T > (2\,n^{1-(1/n)})/h.$$

• • • Consider now the a.s. strong non-degeneracy condition

$$\xi' \alpha(t) \xi = \xi' \sigma(t) \sigma'(t) \xi \geq \varepsilon \|\xi\|^2, \quad \forall \ \ t \in [0, \infty) \,, \quad \xi \in \mathbb{R}^n \quad (7)$$

for some real number $\varepsilon > 0$, on the covariation process $\alpha(\cdot)$. (Compared to the condition (3), this requirement is quite severe.)

. Suppose that the condition (7) holds; and that (2) and (3), namely

$$\xi' \alpha(t) \xi = \xi' \sigma(t) \sigma'(t) \xi \leq L \|\xi\|^2, \quad \forall \ \ t \in [0, \infty) \,, \quad \xi \in \mathbb{R}^n \,,$$

$$\sum_{i=1}^{n} \mu_i(t) \alpha_{ii}(t) - \sum_{i=1}^{n} \sum_{i=1}^{n} \mu_i(t) \alpha_{ij}(t) \mu_j(t) \ \geq \ h \,, \quad \forall \ \ 0 \leq t < \infty \,,$$

hold as well.

In the presence of the first two requirements, the third amounts to a “diversity” condition; more on this in a moment.

Then, as we shall see, for any given constant $p \in (0,1)$, the long-only *diversity-weighted portfolio*

$$\boxed{\mathfrak{D}_i^{(p)}(t) = \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}}, \qquad i = 1, \cdots, n \qquad (8)$$

is again a strong relative arbitrage with respect to the market portfolio, over sufficiently long time-horizons.

- **Appropriate modifications of this diversity-weighted portfolio do yield such relative arbitrage over any time-horizon** $[0,T]$.

This takes some work to prove. And the shorter the time-horizon, the bigger the amount of initial capital that is required to achieve the extra basis point's worth of outperformance:

$$v \geq v(T) \equiv \frac{q(T)}{(\mu_1(0))^{q(T)}} - 1, \quad q(T) := 1 + \big(2/\varepsilon\,\delta\,T\big)\log\big(1/\mu_1(0)\big).$$

• *Please note that these long-only stock portfolios (entropic, equally-weighted, modified equally-weighted, diversity-weighted) are determined entirely from the market weights* $\mu_1(t), \cdots, \mu_n(t)$ *.*

These market weights are perfectly easy to observe and to measure.

• Construction of these portfolios does not assume *any* knowledge about the exact structure of market parameters, such as the mean rates of return $b_i(\cdot)$'s, or the local covariation rates $\alpha_{ij}(\cdot)$'s.

To put it a bit more colloquially: *does not require us to take these particular features of the model "too seriously".* Only as a general "framework"... so that we are able to formulate notions such as the covariations and growth rates for various assets. *Forthcoming.*

. In the parlance of finance practice: *these portfolios are completely "passive"* (their construction requires neither estimation nor optimization).

6. GROWTH RATES

- An equivalent way of representing the positive ITÔ process $X_i(\cdot)$ of equation (1), namely,

$$\mathrm{d}X_i(t) \;=\; X_i(t)\left(b_i(t)\,\mathrm{d}t + \sum_{\nu=1}^{N}\sigma_{i\nu}(t)\,\mathrm{d}W_\nu(t)\right), \quad i = 1, \ldots, n\,,$$

is in the form

$$X_i(t) \;=\; X_i(0)\,\exp\left\{\int_0^t \gamma_i(s)\,\mathrm{d}s + \int_0^t \sum_{\nu=1}^{N}\sigma_{i\nu}(s)\,\mathrm{d}W_\nu(s)\right\} > 0\,,$$

$$\underbrace{\mathrm{d}\big(\log X_i(t)\big) \;=\; \gamma_i(t)\,\mathrm{d}t + \sum_{\nu=1}^{N}\sigma_{i\nu}(t)\,\mathrm{d}W_\nu(t)\,,}$$

with the *logarithmic mean rate of return* for the i^{th} stock

$$\boxed{\gamma_i(t) \;:=\; b_i(t) - \frac{1}{2}\,\alpha_{ii}(t)}\,.$$

EXAMPLE

Stock XYZ doubles in good years (+100%) and halves in bad years (-50%). Years good and bad alternate independently and equally likely (to wit, with probability 0.50), thus

$$b = \frac{1}{2}(+100\%) + \frac{1}{2}(-50\%) = \frac{1}{2} - \frac{1}{4} = 0.25\,,$$

$$\gamma = \frac{1}{2}(\log 2) + \frac{1}{2}\left(\log \frac{1}{2}\right) = 0\,.$$

On the other hand, $\log 2 \simeq 0.7$, so the variance is

$$\alpha = \sigma^2 = \frac{1}{2}(0.7)^2 + \frac{1}{2}(-0.7)^2 \simeq 0.50\,,$$

and indeed

$$(0.25) = 0 + (1/2)(0.50) \qquad \text{or} \qquad b = \gamma + (1/2)\,\alpha\,.$$

- This logarithmic rate of return can be interpreted also as a *growth-rate*, in the sense that

$$\lim_{T \to \infty} \frac{1}{T} \left(\log X_i(t) - \int_0^T \gamma_i(t) \mathrm{d}t \right) = 0 \quad \text{a.s.}$$

holds, under the assumption $\alpha_{ii}(\cdot) \leq L < \infty$ on the variation of the stock; recall

$$\gamma_i(t) := b_i(t) - \frac{1}{2} \alpha_{ii}(t) .$$

A bit more generally, under the condition

$$\lim_{T \to \infty} \left(\frac{\log \log T}{T^2} \int_0^T \alpha_{ii}(t) \mathrm{d}t \right) = 0 , \qquad \text{a.s.}$$

- Similarly, the solution of the linear equation

$$\frac{\mathrm{d}\,V(t)}{V(t)} = \sum_{i=1}^{n} \pi_i(t)\,\frac{\mathrm{d}X_i(t)}{X_i(t)} + \left(1 - \sum_{i=1}^{n} \pi_i(t)\right)\,\frac{\mathrm{d}B(t)}{B(t)}$$

$$= \pi'(t)\big[\,b(t)\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t)\,\big]$$

for the wealth $V(\cdot) \equiv V^{v,\pi}(\cdot)$ corresponding to an initial capital $v \in (0,\infty)$ and portfolio $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot))'$, is given as

$$V^{v,\pi}(t) = v\,\exp\left\{\int_0^t \gamma^\pi(s)\,\mathrm{d}s + \int_0^t \left(\sigma^\pi(s)\right)'\mathrm{d}W(s)\right\} > 0\,,$$

or equivalently

$$\boxed{\mathrm{d}\left(\log V^{v,\pi}(t)\right) = \gamma^\pi(t)\,\mathrm{d}t + \sum_{\nu=1}^{N} \sigma^\pi_\nu(t)\,\mathrm{d}W_\nu(t)\,.} \qquad (9)$$

Stock Portfolio growth-rate and volatilities

$$\gamma^\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_*^\pi(t)\,, \qquad \sigma_\nu^\pi(t) = \sum_{i=1}^{n} \pi_i(t)\sigma_{i\nu}(t).$$

Stock Portfolio excess growth-rate

$$\underbrace{\gamma_*^\pi(t) := \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\,\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\,\alpha_{ij}(t)\,\pi_j(t)\right)}.$$

Stock Portfolio variation

$$\mathrm{a}^{\pi\pi}(t) = \sum_{\nu=1}^{N}(\sigma_\nu^\pi(t))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\alpha_{ij}(t)\pi_j(t)\,.$$

7. RELATIVE COVARIATION STRUCTURE

• *Variation/Covariation Processes*, not in absolute terms, but relative to the stock portfolio $\pi(\cdot)$:

$$\mathfrak{A}^{\pi}_{ij}(t) := \sum_{\nu=1}^{N} \left(\sigma_{i\nu}(t) - \sigma^{\pi}_{\nu}(t)\right)\left(\sigma_{j\nu}(t) - \sigma^{\pi}_{\nu}(t)\right), \qquad 1 \le i,j \le n$$

where $\sigma^{\pi}_{\nu}(t) = \sum_{i=1}^{n} \pi_i(t)\sigma_{i\nu}(t)$. If the covariation matrix $\alpha(t)$ with entries

$$\alpha_{ij}(t) = \sum_{\nu=1}^{N} \sigma_{i\nu}(t)\,\sigma_{j\nu}(t)\,, \qquad 1 \le i,j \le n$$

is positive-definite, then the relative covariation matrix

$$\mathfrak{A}^{\pi}(t) = \{\mathfrak{A}^{\pi}_{ij}(t)\}_{1 \le i,j \le n}$$

has rank $n-1$ and its null space is spanned by the vector $\pi(t)$.

- The excess growth-rate

$$\gamma_*^\pi(t) := \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\,\alpha_{ii}(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\,\alpha_{ij}(t)\pi_j(t)\right)$$

has, for *any* two stock portfolios $\pi(\cdot) \in \mathfrak{P}$, $\rho(\cdot) \in \mathfrak{P}$, the **invariance property**

$$\gamma_*^\pi(t) = \tfrac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\,\mathfrak{A}_{ii}^\rho(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\,\mathfrak{A}_{ij}^\rho(t)\,\pi_j(t)\right).$$

Consequently, reading the above with $\rho(\cdot) \equiv \pi(\cdot)$ and recalling that the null space of the relative covariation matrix $\{\mathfrak{A}_{ij}^\pi(t)\}_{1\le i,j\le n}$ is spanned by $\pi(t)$, we obtain

$$\boxed{\gamma_*^\pi(t) = \frac{1}{2}\sum_{i=1}^n \pi_i(t)\,\mathfrak{A}_{ii}^\pi(t)\,.}$$

In particular, we have $\gamma_*^\pi(\cdot) \ge 0$ for a long-only stock portfolio.

- Now let us consider the market portfolio $\pi \equiv \mu$. The excess growth rate

$$\gamma_*^\mu(t) = \frac{1}{2}\Big(\sum_{i=1}^n \mu_i(t)\alpha_{ii}(t) - \sum_{i=1}^n\sum_{i=1}^n \mu_i(t)\alpha_{ij}(t)\mu_j(t)\Big)$$

of the market portfolio can then be interpreted as a measure of *intrinsic variation* available in the market:

$$\boxed{\gamma_*^\mu(t) = \frac{1}{2}\sum_{i=1}^n \mu_i(t)\,\mathfrak{A}_{ii}^\mu(t)}\,,$$

where

$$\mu_i(t) := \frac{X_i(t)}{X(t)}\,, \qquad \sigma_\nu^\mu(t) := \sum_{i=1}^n \mu_i(t)\sigma_{i\nu}(t)\,,$$

$$\mathfrak{A}_{ij}^\mu(t) := \sum_{\nu=1}^N \big(\sigma_{i\nu}(t) - \sigma_\nu^\mu(t)\big)\big(\sigma_{j\nu}(t) - \sigma_\nu^\mu(t)\big) = \frac{\mathrm{d}\langle\mu_i,\mu_j\rangle(t)}{\mu_i(t)\mu_j(t)\mathrm{d}t}\,.$$

Thus the excess growth rate of the market portfolio

$$\gamma_*^{\mu}(t) \;=\; \frac{1}{2}\sum_{i=1}^{n}\mu_i(t)\,\mathfrak{A}_{ii}^{\mu}(t)$$

is also a weighted average, according to market capitalization, of the local variation rates

$$\mathfrak{A}_{ii}^{\mu}(t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\,\langle \log \mu_i \rangle(t)$$

of individual stocks – not in absolute terms, but *relative to the market*.

This quantity will be very important in what follows.
It is a much more meaningful measure of "market volatility" than some commonly used as such, in my opinion.

- (OPTIONAL) Related to the dynamics of the log-market-weights

$$\mathrm{d}\log\mu_i(t) = \big(\gamma_i(t) - \gamma^\mu(t)\big)\,\mathrm{d}t + \sum_{\nu=1}^{N}\big(\sigma_{i\nu}(t) - \sigma_\nu^\mu(t)\big)\,\mathrm{d}W_\nu(t)$$

for all stocks $i = 1, \ldots, n$. Equivalently, in arithmetic terms

$$\frac{\mathrm{d}\mu_i(t)}{\mu_i(t)} = \Big(\gamma_i(t) - \gamma^\mu(t) + \frac{1}{2}\,\mathfrak{A}_{ii}^\mu(t)\Big)\,\mathrm{d}t + \sum_{\nu=1}^{N}\big(\sigma_{i\nu}(t) - \sigma_\nu^\mu(t)\big)\mathrm{d}W_\nu(t)\,. \tag{10}$$

It is now clear from this, that

$$\frac{\mathrm{d}\langle\mu_i,\mu_j\rangle(t)}{\mu_i(t)\mu_j(t)\mathrm{d}t} = \sum_{\nu=1}^{N}\big(\sigma_{i\nu}(t) - \sigma_\nu^\mu(t)\big)\big(\sigma_{j\nu}(t) - \sigma_\nu^\mu(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\,\langle\log\mu_i,\log\mu_j\rangle(t) = \mathfrak{A}_{ij}^\mu(t)\,.$$

THE PARABLE OF TWO STOCKS

Suppose there are only two, perfectly negatively correlated, stocks A and B. We toss a fair coin, independently from day to day; when the toss comes up heads, stock A doubles and stock B halves in price (and vice-versa, if the toss comes up tails).

Clearly, each stock has a growth rate of zero: holding any one of them produces nothing in the long term.

- What happens if we hold both stocks? Suppose we invest $100 in each; one of them will rise to $200 and the other fall to $50, for a guaranteed total of $250, representing a net gain of 25%; the winner has gained more than the loser has lost.

If we **rebalance** to $125 in each stock (so as to maintain the equal proportions we started with), and keep doing this day after day, we lock in a long-term growth rate of 25%.

Indeed, taking $n = 2$ and

$$\gamma_1 = \gamma_2 = 0\,, \qquad \alpha_{11} = \alpha_{22} = -\alpha_{12} = -\alpha_{21} = 0.50$$

from our earlier computations, and

$$\pi_1 \;=\; \pi_2 \;=\; 0.50$$

in

$$\gamma^{\pi} \;=\; \sum_{i=1}^{n} \pi_i\, \gamma_i + \frac{1}{2} \left(\sum_{i=1}^{n} \pi_i\, \alpha_{ii} - \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i\, \alpha_{ij}\, \pi_j \right)$$

$$=\; \frac{1}{2} \Big(\pi_1 \big(1 - \pi_1 \big) \alpha_{11} + \pi_2 \big(1 - \pi_2 \big) \alpha_{22} \Big) - \pi_1 \pi_2 \alpha_{12}$$

we get the same growth rate that we computed a moment ago:

$$\gamma^{\pi} \;=\; \gamma^{\pi}_{*} \;=\; 0.25\,.$$

A POSSIBLE MORAL OF THIS PARABLE

• In the presence of "sufficient intrinsic variation (volatility)", setting target weights and rebalancing to them, can **capture this volatility** and **turn it into growth**.

(And this can occur even if carried out relatively naively, without precise estimates of model parameters and without refined optimization.)

We have encountered several variations on this parable already, and will encounter a few more below. In particular, we shall quantify what "sufficient intrinsic volatility" means.

8. PORTFOLIO DIVERSIFICATION AND MARKET VOLATILITY AS DRIVERS OF GROWTH

Now let us suppose that, for some real number $\varepsilon > 0$, condition (7) holds:

$$\xi' \alpha(t) \xi = \xi' \sigma(t) \sigma'(t) \xi \geq \varepsilon \|\xi\|^2, \quad \forall \;\; t \in [0, \infty), \quad \xi \in \mathbb{R}^n .$$

That is, we have a strictly nondegenerate covariation structure. Then an elementary computation shows

$$\gamma^\pi(t) - \sum_{i=1}^n \pi_i(t) \gamma_i(t) = \gamma_*^\pi(t) \geq (\varepsilon/2) \Big(1 - \max_{1 \leq i \leq n} \pi_i(t) \Big) \geq (\varepsilon/2) \eta > 0 ,$$

as long as for some $\eta \in (0,1)$ we have

$$\max_{1 \leq i \leq n} \pi_i(t) \leq 1 - \eta .$$

To wit, such a stock portfolio's growth rate $\gamma^{\pi}(t)$ will dominate, and strictly, the average growth rate of the constituent assets

$$\sum_{i=1}^{n} \pi_i(t)\,\gamma_i(t)$$

(Fernholz & Shay, *Journal of Finance* (1982)):

$$\gamma^{\pi}(t) \;\geq\; \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \big(\varepsilon/2\big)\eta\,.$$

In words: Under the above condition of "sufficient volatility", even *the slightest bit of portfolio diversification can not only decrease the portfolio's variation, as is well known, but also enhance its growth.*

We shall see below additional – and actually quite more realistic – incarnations of this principle.

¶ To see just how significant such an enhancement can be, consider any fixed-proportion, long-only stock portfolio $\pi(\cdot) \equiv \mathfrak{p}$, for some vector $\mathfrak{p} \in \Delta^n$ with

$$1 - \max_{1 \le i \le n} \mathfrak{p}_i =: \eta > 0 \,,$$

and with

$$\Delta^n := \big\{ (p_1, \cdots, p_n) \,:\, p_1 \ge 0, \cdots, p_n \ge 0 \,, \quad p_1 + \cdots + p_n = 1 \big\} \,.$$

For any stock portfolio $\pi(\cdot)$ and $T \in (0, \infty)$, we have the identity

$$\log \left(\frac{V^{1,\pi}(T)}{V^{1,\mu}(T)} \right) = \int_0^T \gamma_*^\pi(t)\, \mathrm{d}t + \sum_{i=1}^n \int_0^T \pi_i(t)\, \mathrm{d} \log \mu_i(t). \tag{11}$$

At least in principle, a way to keep track of the performance of $\pi(\cdot)$ relative to the market. This is a simple consequence of (9), slide 30:

$$\mathrm{d} \big(\log V^{v,\pi}(t) \big) = \gamma^\pi(t)\, \mathrm{d}t + \sum_{\nu=1}^N \sigma_\nu^\pi(t)\, \mathrm{d}W_\nu(t).$$

From the equation

$$\boxed{\log\left(\frac{V^{1,\pi}(T)}{V^{1,\mu}(T)}\right) = \int_0^T \gamma_*^{\pi}(t)\,\mathrm{d}t + \sum_{i=1}^n \int_0^T \pi_i(t)\,\mathrm{d}\log\mu_i(t)\,,}$$

of the previous slide, we get for a constant-proportion stock portfolio the a.s. comparisons

$$\frac{1}{T}\,\log\left(\frac{V^{1,\mathfrak{p}}(T)}{V^{1,\mu}(T)}\right) - \sum_{i=1}^n \frac{\mathfrak{p}_i}{T}\,\log\left(\frac{\mu_i(T)}{\mu_i(0)}\right) =$$

$$= \frac{1}{T}\int_0^T \gamma_*^{\mathfrak{p}}(t)\,\mathrm{d}t \geq \frac{\varepsilon\eta}{2} > 0\,.$$

Suppose now that the market is *coherent*, meaning that no individual stock crashes relative to the rest of the market:

$$\lim_{T\to\infty} \frac{1}{T} \log \mu_i(T) = 0 \,, \quad \forall \;\; i = 1, \cdots, n \,.$$

Then passing to the limit as $T \to \infty$ in

$$\frac{1}{T} \log \left(\frac{V^{1,\mathfrak{p}}(T)}{V^{1,\mu}(T)} \right) - \sum_{i=1}^{n} \frac{\mathfrak{p}_i}{T} \log \left(\frac{\mu_i(T)}{\mu_i(0)} \right) \geq \frac{\varepsilon \eta}{2} > 0$$

we see that the wealth corresponding to any such fixed-proportion, long-only portfolio, grows exponentially at a rate **strictly higher** than that of the overall market:

$$\liminf_{T\to\infty} \frac{1}{T} \log \left(\frac{V^{1,\mathfrak{p}}(T)}{V^{1,\mu}(T)} \right) \geq \frac{\varepsilon \eta}{2} > 0 \,, \qquad \text{a.s.}$$

Remark: Optional.

Tom Cover's (1991) "universal portfolio"

$$\Pi_i(t) := \frac{\int_{\Delta^n} \mathfrak{p}_i \, V^{1,\mathfrak{p}}(t)\,\mathrm{d}\mathfrak{p}}{\int_{\Delta^n} V^{1,\mathfrak{p}}(t)\,\mathrm{d}\mathfrak{p}}\,, \quad i = 1, \cdots, n$$

has value

$$V^{1,\Pi}(t) = \frac{\int_{\Delta^n} V^{1,\mathfrak{p}}(t)\,\mathrm{d}\mathfrak{p}}{\int_{\Delta^n} \mathrm{d}\mathfrak{p}} \sim \max_{\mathfrak{p}\in\Delta^n} V^{1,\mathfrak{p}}(t)\,.$$

Please note the "total agnosticism" of this portfolio regarding the details of the underlying model; and check out the recent work of Cuchiero, Schachermayer & Wong (2017) regarding this portfolio.

✠ Up to now we have not even tried to select portfolios in an "optimal" fashion. Here a few Portfolio Optimization problems; some of them are classical, while for others very little is known.

9. PORTFOLIO OPTIMIZATION

Problem #1: Quadratic criterion, linear constraint (Markowitz, 1952). Minimize the portfolio variation

$$\mathrm{a}^{\pi\pi}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t)\alpha_{ij}(t)\pi_j(t) \tag{12}$$

among all stock portfolios $\pi(\cdot) \in \mathfrak{P}$ that keep the rate-of-return at least equal to a given constant:

$$b^{\pi}(t) = \sum_{i=1}^{n} \pi_i(t) b_i(t) \geq \beta .$$

Problem #2: Quadratic criterion, quadratic constraint. Minimize the portfolio variation $\mathrm{a}^{\pi\pi}(t)$ of (12) among all stock portfolios $\pi(\cdot) \in \mathfrak{P}$ with growth-rate at least equal to a given constant γ_0 :

$$\sum_{i=1}^{n} \pi_i(t) b_i(t) \geq \gamma_0 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t)\alpha_{ij}(t)\pi_j(t) .$$

Problem #3: *Maximize over stock portfolios the probability of reaching a given "ceiling" $\mathfrak{c}$ before reaching a given "floor" $\mathfrak{f}$, with $0 < \mathfrak{f} < 1 < \mathfrak{c} < \infty$.*

More specifically, maximize over $\pi(\cdot) \in \mathfrak{P}$ the probability

$$\mathbb{P}\,[\,\mathfrak{T}_{\mathfrak{c}}^{\pi} < \mathfrak{T}_{\mathfrak{f}}^{\pi}\,], \qquad \text{with} \qquad \mathfrak{T}_{c}^{\pi} := \inf\{\, t \geq 0 \,:\, X^{1,\pi}(t) = c \,\}.$$

. In the case of constant coëfficients γ_i and α_{ij}, and with

Γ_n the collection of vectors $p \in \mathbb{R}^n$ with $p_1 + \cdots + p_n = 1$,

the solution to this problem is given by the vector $\pi \in \Gamma_n$ that maximizes the mean-variance, or *signal-to-noise*, ratio:

$$\frac{\gamma^{\pi}}{\mathrm{a}^{\pi\pi}} = \frac{\sum_{i=1}^{n} \pi_i \big(\gamma_i + \frac{1}{2}\alpha_{ii}\big)}{\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \alpha_{ij} \pi_j} - \frac{1}{2}$$

(Pestien & Sudderth, *Mathematics of Operations Research* 1985). Open Question: *How about (more) general coëfficients?*

Problem #4: *Maximize over stock portfolios the probability*

$$\mathbb{P}\left[\, \mathfrak{T}^{\pi}_{\mathfrak{c}} < T \wedge \mathfrak{T}^{\pi}_{\mathfrak{f}} \,\right]$$

of reaching a given "ceiling" $\mathfrak{c}$ *before reaching a given "floor"* $\mathfrak{f}$ *with* $0 < \mathfrak{f} < 1 < \mathfrak{c} < \infty$ *, by a given "deadline"* $T \in (0, \infty)$.

Always with constant coëfficients, suppose there is a vector $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)' \in \Gamma_n$ that maximizes **both** the signal-to-noise ratio **and** the variance,

$$\frac{\gamma^p}{\mathrm{a}^{pp}} = \frac{\sum_{i=1}^n p_i(\gamma_i + \frac{1}{2}\alpha_{ii})}{\sum_{i=1}^n \sum_{j=1}^n p_i\, \alpha_{ij}\, p_j} - \frac{1}{2} \qquad \text{and} \qquad \mathrm{a}^{pp} = \sum_{i=1}^n \sum_{j=1}^n p_i\, \alpha_{ij}\, p_j \,,$$

over all $p = (p_1, \cdots, p_n)' \in \mathbb{R}^n$ with $\sum_{i=1}^n p_i = 1$.

Then the constant-proportion portfolio $\hat{p}$ is optimal for the above criterion (SUDDERTH & WEERASINGHE, *Mathematics of Operations Research*, 1989).

This is a huge assumption; it is satisfied, for instance, under the (very stringent) condition that, for some $\beta \le 0$, we have

$$b_i \, = \, \gamma_i + \tfrac{1}{2}\,\alpha_{ii} \, = \, \beta\,, \quad \text{for all } \; i = 1, \ldots, n\,.$$

Open Question: *As far as I can tell, nobody seems to know the solution to this problem when such "simultaneous maximization" is not possible.*

Problem #5: *Minimize over stock portfolios* $\pi(\cdot)$ *the expected time* $\mathbb{E}\,[\,\mathfrak{T}^{\pi}_{\mathfrak{c}}\,]$ *until a given "ceiling"* $\mathfrak{c} \in (1, \infty)$ *is reached.*

Again with constant coëfficients, it turns out that it is enough to maximize, over all vectors $\pi \in \mathbb{R}^n$ with $\sum_{i=1}^n \pi_i = 1$, the drift in the equation for $\log X^{\pi}(\cdot)$, namely the portfolio growth-rate

$$\gamma^{\pi} = \sum_{i=1}^{n} \pi_i \left(\gamma_i + \tfrac{1}{2}\alpha_{ii}\right) - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \alpha_{ij} \pi_j \,.$$

(See Heath, Orey, Pestien & Sudderth, *SIAM Journal on Control & Optimization*, 1987.)

Again, how about (more) general coëfficients?
Partial answer: Kardaras & Platen, *SIAM Journal on Control & Optimization* (2010).

Problem #6: Growth Optimality, Relative Log-Optimality, and the Supermartingale Numéraire Property:

Suppose we can find a portfolio $\varrho^*(\cdot) \in \mathbf{\Pi}$ which maximizes, over vectors $p \in \mathbb{R}^n$, the drift in the equation for $\log X^\pi(\cdot)$, namely the growth-rate

$$\sum_{i=1}^{n} p_i \left(\gamma_i(t) + \tfrac{1}{2}\alpha_{ii}(t)\right) - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i \, \alpha_{ij}(t) \, p_j$$

(just as we ended up doing in the previous problem). Then for every portfolio $\pi(\cdot) \in \mathbf{\Pi}$ we have the supermartingale numéraire property

$$V^{1,\pi}(\cdot) \,/\, V^{1,\varrho^*}(\cdot) \quad \text{is a supermartingale,}$$

as well as

$$\limsup_{T\to\infty} \frac{1}{T} \log \left(\frac{V^{1,\pi}(T)}{V^{1,\varrho^*}(T)} \right) \le 0 \,, \quad \text{a.s.,}$$

$$\mathbb{E}\left[\log \left(\frac{V^{1,\pi}(T)}{V^{1,\varrho^*}(T)} \right) \right] \le 1 \,, \quad \forall \;\; T \in (0,\infty) \,.$$

• As Constantinos KARDARAS showed in his dissertation, the solvability of very general hedging / utility maximization problems only needs the existence of a portfolio $\varrho^*(\cdot)$ with the supermartingale numéraire property
(equivalently, the growth-optimality property;
equivalently, the relative-log-optimality property;
equivalently, the existence of a supermartingale numéraire).

In fact, the entire mathematical theory of Finance can be re-cast, and generalized, in terms of the existence of this portfolio $\varrho^*(\cdot)$ with the supermartingale numéraire property
(rather than requiring the existence of an EMM – TOO MUCH!).

Subject of Book in Preparation, with Kostas.

- Now then, every portfolio $\varrho^*(\cdot)$ with

$$\beta(\cdot) = \alpha(\cdot)\,\varrho^*(\cdot)$$

has all the above properties; leads to a market-price-of-risk

$$\vartheta(\cdot) = \sigma'(\cdot)\,\varrho^*(\cdot)$$

and thence to a deflator $Z^{\vartheta}(\cdot)$; and its wealth process $V^{\varrho^*}(\cdot)$ is uniquely determined.

The market is then "viable", in the sense that it becomes impossible to finance *something* (a non-negative contingent claim which is strictly positive with positive probability) for *next to nothing* (i.e., starting with initial capital arbitrarily close to zero but positive).

These are some of the ingredients of a **new, very general FTAP** (and quite simple to prove), in which EMM's play no rôle whatsoever. They are replaced by supermartingale numéraires.

Problem # 7: Enhanced Indexing. Consider a long-only stock portfolio $\rho(\cdot)$, which plays the role of a *benchmark index*. Typical case is $\rho(\cdot) \equiv \mu(\cdot)$.

We want to construct a long-only stock portfolio $\pi(\cdot)$ that minimizes the relative variation (square of the tracking error)

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t)\, \mathfrak{A}_{ij}^{\rho}(t)\, \pi_j(t)$$

with respect to $\rho(\cdot)$, subject to the constraint

$$\gamma^{\pi}(t) \geq \gamma$$

for some given constant γ, namely

$$\sum_{i=1}^{n} \pi_i(t)\, \gamma_i(t) + \frac{1}{2} \left(\sum_{i=1}^{n} \pi_i(t)\, \mathfrak{A}_{ii}^{\rho}(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t)\, \mathfrak{A}_{ij}^{\rho}(t)\, \pi_j(t) \right) \geq \gamma$$

and of course subject to

$$\pi_1(t) \geq 0\,, \cdots\,, \pi_n(t) \geq 0\,, \quad \pi_1(t)+\ldots+\pi_n(t) = 1 \qquad \text{for all } t \geq 0\,.$$

Now the quadratic term in

$$\sum_{i=1}^{n} \pi_i(t)\,\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\,\mathfrak{A}^{\rho}_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\,\mathfrak{A}^{\rho}_{ij}(t)\,\pi_j(t)\right) \geq \gamma$$

is just the relative variation (square of the tracking error) we are trying to minimize.

Rough Approximation: If the tracking error is to be held, as is usual, to about 2% per year or less, this quadratic term is no more than 0.02% per year, thus negligible, and we can use the modified constraint

$$\gamma^{\pi}(t) \simeq \sum_{i=1}^{n} \pi_i(t)\left(\gamma_i(t) + \frac{1}{2}\,\mathfrak{A}^{\rho}_{ii}(t)\right) \geq \gamma\,,$$

which is linear.

Still, however, we need to estimate the $\gamma_i(t)$'s ... .

Problem # 8: Enhanced Large-Cap Indexing. Assume now that the long-only benchmark portfolio $\rho(\cdot)$ is a *large-cap index*, consisting of assets with the *same growth rate* $\gamma_i(\cdot) \equiv \gamma(\cdot)$.

We want to construct a long-only stock portfolio $\pi(\cdot)$ that minimizes the relative variation (square of the tracking error) with respect to $\rho(\cdot)$, namely

$$(\rho\text{-Tracking Error})^2 \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\,\mathfrak{A}^{\rho}_{ij}(t)\,\pi_j(t)\,,$$

subject to the constraint

$$\gamma^{\pi}(t) \;\geq\; \gamma^{\rho}(t) + g\,, \qquad \text{for all} \quad t \geq 0\,,$$

for some constant g , and subject to

$$\pi_1(t) \geq 0\,, \cdots\,,\; \pi_n(t) \geq 0\,, \quad \pi_1(t) + \ldots + \pi_n(t) \;=\; 1 \qquad \text{for all} \;\; t \geq 0\,.$$

Under the assumption of equal growth rates,

$$\gamma^{\pi}(t) \geq \gamma^{\rho}(t) + \gamma\,, \qquad \text{for all} \quad t \geq 0\,,$$

becomes

$$\gamma^{\pi}_{*}(t) \geq \gamma^{\rho}_{*}(t) + \gamma\,, \qquad \text{for all} \quad t \geq 0\,.$$

But from the invariance property we have

$$2\,\gamma^{\pi}_{*}(t) = \sum_{i=1}^{n} \pi_i(t)\,\mathfrak{A}^{\rho}_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\,\mathfrak{A}^{\rho}_{ij}(t)\,\pi_j(t)\,,$$

$$2\,\gamma^{\rho}_{*}(t) = \sum_{i=1}^{n} \rho_i(t)\,\mathfrak{A}^{\rho}_{ii}(t)$$

and the constraint $\gamma^{\pi}(t) \geq \gamma^{\rho}(t) + g$ becomes

$$\sum_{i=1}^{n} (\pi_i(t) - \rho_i(t))\,\mathfrak{A}^{\rho}_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\,\mathfrak{A}^{\rho}_{ij}(t)\,\pi_j(t) \geq 2\,g\,.$$

Please note that there is no need any longer to estimate any growth rates.

Discussion:

In none of these problems did we need to assume the existence of an equivalent martingale measure – or even of a deflator $Z(\cdot)$, in most of the cases.

In most of them, we needed to "take our model quite seriously", to the extent that the solution assumed knowledge of both the covariation structure of the market and of the assets' growth rates. Whereas in some (rather special) such problems, the solution only needs estimates of the covariation structure of the market – not a trivial task, but much easier than estimating growth rates of individual assets.

FUNCTIONALLY-GENERATED PORTFOLIOS

Let us recall the expression

$$\log\left(\frac{V^{1,\pi}(T)}{V^{1,\mu}(T)}\right) = \int_0^T \gamma_*^\pi(t)\,\mathrm{d}t + \sum_{i=1}^n \int_0^T \pi_i(t)\,\mathrm{d}\log\mu_i(t)$$

of (11) for the relative performance of an arbitrary stock portfolio $\pi(\cdot)$ with respect to the market.

In conjunction with the dynamics of the log-market-weights

$$\mathrm{d}(\log\mu_i(t)) = \big(\gamma_i(t) - \gamma^\mu(t)\big)\,\mathrm{d}t + \sum_{\nu=1}^N \big(\sigma_{i\nu}(t) - \sigma_\nu^\mu(t)\big)\,\mathrm{d}W_\nu(t)$$

that we have also seen, this leads to the decomposition of the log-relative-performance for the portfolio $\pi(\cdot)$ with respect to the market.

In general, it is VERY difficult to get any useful information, regarding the relative performance of a portfolio $\pi(\cdot)$ with respect to the market, from this decomposition

$$\mathrm{d}(\log \mu_i(t)) = \big(\gamma_i(t) - \gamma^{\mu}(t)\big)\,\mathrm{d}t + \sum_{\nu=1}^{N} \big(\sigma_{i\nu}(t) - \sigma^{\mu}_{\nu}(t)\big)\,\mathrm{d}W_{\nu}(t).$$

HOWEVER: There is a class of very special portfolios $\pi(\cdot)$ – described solely in terms of the market weights $\mu_1(\cdot), \ldots, \mu_n(\cdot)$, *and nothing else* – for which the stochastic integrals disappear completely from the right-hand side of the above decomposition. Whereas the remaining (LEBESGUE) integrals also depend solely on market weights, and are monotone increasing.

. This allows for pathwise comparisons of relative performance; or, to put it a bit differently, for the construction of arbitrage relative to the market, under appropriate conditions.

We start with a smooth function $\mathbf{S} : \Delta^n_+ \to \mathbb{R}_+$, and consider *the stock portfolio* $\pi^{\mathbf{S}}(\cdot)$ *generated by it:*

$$\boxed{\frac{\pi_i^{\mathbf{S}}(t)}{\mu_i(t)} := D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) \cdot D_j \log \mathbf{S}(\mu(t))\,.} \tag{13}$$

(Blue term: familiar "delta hedging". The remaining terms on the RHS are there to ensure the resulting portfolio is fully invested.)

Then an application of ITÔ's rule gives the "Master Equation"

$$\boxed{\log\left(\frac{V^{1,\pi^{\mathbf{S}}}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{\mathbf{S}(\mu(T))}{\mathbf{S}(\mu(0))}\right) + \int_0^T \mathfrak{g}(t)\,dt\,.} \tag{14}$$

Here, thanks to our assumptions, the quantity $\mathfrak{g}(\cdot)$ is nonnegative:

$$\mathfrak{g}(t) := \frac{-1}{\mathbf{S}(\mu(t))} \sum_i \sum_j D^2_{ij}\mathbf{S}(\mu(t)) \cdot \mu_i(t)\mu_j(t)\,\mathfrak{A}^{\mu}_{ij}(t)\,. \tag{15}$$

$$\pi_i^{\mathbf{S}}(t) := \mu_i(t)\left(D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) \cdot D_j \log \mathbf{S}(\mu(t)) \right)$$

$$\mathfrak{g}(t) := \frac{-1}{\mathbf{S}(\mu(t))} \sum_i \sum_j D_{ij}^2 \mathbf{S}(\mu(t)) \cdot \frac{\mathrm{d}\langle \mu_i, \mu_j \rangle(t)}{\mu_i(t)\mu_j(t)\mathrm{d}t}$$

✠ Please note that, when the smooth function $\mathbf{S} : \Delta_+^n \to \mathbb{R}_+$ is concave, the above process $\mathfrak{g}(\cdot)$ is non-negative, and thus its indefinite integral an increasing process.

In this case, it can also be shown that the generated portfolio $\pi^{\mathbf{S}}$ is long-only.

SIGNIFICANCE: *Stochastic integrals have been excised in (14)*, i.e.,

$$\log \left(\frac{V^{1,\pi^{\mathbf{S}}}(T)}{V^{1,\mu}(T)} \right) = \log \left(\frac{\mathbf{S}(\mu(T))}{\mathbf{S}(\mu(0))} \right) + \int_0^T \mathfrak{g}(t)\, dt \ ,$$

and we can begin to make comparisons that are valid with probability one (a.s.)...

Equally significantly: The first term on the right-hand side has controlled behavior, and is usually bounded. Thus, the growth of this expression as T increases, is determined by the second (LEBESGUE integral) term on the right-hand side.

Proof of the "Master Equation" (14): To ease notation we set

$$h_i(t) := D_i \log \mathbf{S}(\mu(t)) \qquad \text{and} \qquad N(t) := \sum_{j=1}^{n} \mu_j(t)\, h_j(t)\,,$$

so (13), that is

$$\pi_i(t) \,=\, \mu_i(t) \left(D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) \cdot D_j \log \mathbf{S}(\mu(t)) \right) \,,$$

reads:

$$\pi_i(t) = \big(h_i(t) + N(t)\big)\mu_i(t)\,, \qquad i = 1, \cdots n\,.$$

Then the terms on the right-hand side of

$$\mathrm{d}\log\left(\frac{V^{1,\pi}(t)}{V^{1,\mu}(t)}\right) = \sum_{i=1}^{n}\frac{\pi_i(t)}{\mu_i(t)}\,\mathrm{d}\mu_i(t) - \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i(t)\pi_j(t)\mathfrak{A}^{\mu}_{ij}(t)\right)\mathrm{d}t\,,$$

an equivalent version of

$$\log\left(\frac{V^{1,\pi}(t)}{V^{1,\mu}(t)}\right) = \int_0^T \gamma_*^{\pi}(t)\,\mathrm{d}t + \sum_{i=1}^{n}\int_0^T \pi_i(t)\,\mathrm{d}\log\mu_i(t)$$

in (11), become

$$\sum_{i=1}^{n}\frac{\pi_i(t)}{\mu_i(t)}\,\mathrm{d}\mu_i(t) = \sum_{i=1}^{n} h_i(t)\,\mathrm{d}\mu_i(t) + N(t)\cdot\mathrm{d}\Big(\sum_{i=1}^{n}\mu_i(t)\Big)$$

$$= \sum_{i=1}^{n} h_i(t)\,\mathrm{d}\mu_i(t)\,,$$

whereas $\sum_{i=1}^n \sum_{j=1}^n \pi_i(t)\pi_j(t)\mathfrak{A}^{\mu}_{ij}(t)$ becomes

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\big(h_i(t)+N(t)\big)\big(h_j(t)+N(t)\big)\mu_i(t)\mu_j(t)\mathfrak{A}^{\mu}_{ij}(t)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} h_i(t)h_j(t)\mu_i(t)\mu_j(t)\mathfrak{A}^{\mu}_{ij}(t)\,.$$

(Again, because $\mu(t)$ spans the null subspace of $\{\mathfrak{A}^{\mu}_{ij}(t)\}_{1\le i,j\le n}$.) Thus, using the dynamics of market weights in (10), the above equation gives

$$\mathrm{d}\log\left(\frac{V^{\pi}(t)}{V^{\mu}(t)}\right) = \sum_{i=1}^{n} h_i(t)\,\mathrm{d}\mu_i(t)$$
$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} h_i(t)h_j(t)\mu_i(t)\mu_j(t)\mathfrak{A}^{\mu}_{ij}(t)\,\mathrm{d}t\,. \qquad (16)$$

On the other hand, we have

$$D^2_{ij} \log \mathbf{S}(x) \;=\; \big(D^2_{ij}\mathbf{S}(x)/\mathbf{S}(x)\big) - D_i \log \mathbf{S}(x) \cdot D_j \log \mathbf{S}(x)\,,$$

so we get

$$\mathrm{d} \log \mathbf{S}(\mu(t)) = \sum_{i=1}^{n} h_i(t)\,\mathrm{d}\mu_i(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} D^2_{ij} \log \mathbf{S}(\mu(t))\,\mathrm{d}\langle \mu_i, \mu_j\rangle(t)$$

$$= \sum_{i=1}^{n} h_i(t)\mathrm{d}\mu_i(t)$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{D^2_{ij}\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(t))} - h_i(t)h_j(t)\Big)\mu_i(t)\mu_j(t)\mathfrak{A}^{\mu}_{ij}(t)\mathrm{d}t$$

by Itô's rule. Comparing this last expression with (16) and recalling the notation of (15), we deduce (14), namely:

$$\mathrm{d} \log \mathbf{S}(\mu(t) = \mathrm{d} \log \big(V^{\pi}(t)/V^{\mu}(t)\big) - \mathfrak{g}(t)\mathrm{d}t\,.$$

For instance: PASSIVE INVESTMENTS.

- $\mathbf{S}(\cdot) \equiv w$, a positive constant, generates the *market* portfolio.
- The function

$$\mathbf{S}(m) = w_1 m_1 + \cdots + w_n m_n \,, \qquad m = (m_1, \cdots, m_n)' \in \Delta_+^n$$

generates the *passive* portfolio that buys at time $t = 0$, and holds up until time $t = T$, a fixed number of shares w_i in each asset $i = 1, \cdots, n$.

(The market portfolio corresponds to the special case

$$w_1 = \cdots = w_n = w$$

of equal numbers of shares across assets.)

- The geometric mean

$$\mathbf{S}(m) \equiv \mathbf{G}(m) \ := \ (m_1 \cdots m_n)^{1/n}$$

generates the *equal-weighted* portfolio

$$\varphi_i(\cdot) \equiv 1/n \,, \qquad i = 1, \cdots, n \,,$$

with drift equal to the excess growth rate:

$$\mathfrak{g}^{\varphi}(\cdot) \equiv \gamma^*_{\varphi}(\cdot) \ = \ \frac{1}{2n} \left(\sum_{i=1}^{n} \alpha_{ii}(\cdot) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}(\cdot) \right) .$$

The resulting portfolio corresponds to the so-called "Value-Line Index".

Discussion on Equal Weighting:

The equal-weighted portfolio $\varphi(\cdot)$ maintains the same weights in all stocks at all times; it accomplishes this by selling those stocks whose price rises relative to the rest, and by buying stocks whose price falls relative to the others.

. Because of this built-in aspect of *"buying-low-and-selling-high"*, equal-weighting can be used as a simple prototype for studying systematically the performance of *statistical arbitrage* strategies in equity markets; see FERNHOLZ & MAGUIRE (2006) for details.

It has been observed empirically, that such a portfolio can outperform the market (we shall see a rigorous result along these lines in a short while). Of course, implementing such a strategy necessitates very frequent trading and can incur substantial transaction costs for an investor who is not a broker/dealer.

It can also involve considerable risk: whereas the second term on the right-hand side of

$$\log V^{1,\varphi}(T) = \frac{1}{n} \log \left(\frac{X_1(T) \cdots X_n(T)}{X_1(0) \cdots X_n(0)} \right) + \int_0^T \gamma_\varphi^*(t)\, dt \,,$$

or of

$$\log \left(\frac{V^{1,\varphi}(T)}{V^{1,\mu}(T)} \right) = \frac{1}{n} \log \left(\frac{\mu_1(T) \cdots \mu_n(T)}{\mu_1(0) \cdots \mu_n(0)} \right) + \int_0^T \gamma_\varphi^*(t)\, dt \,,$$

is increasing it T, the first terms on the right-hand sides of these expressions can fluctuate quite a bit.

- The **diversity-weighted portfolio** $\mathfrak{D}^{(p)}(\cdot)$ of

$$\mathfrak{D}_i^{(p)}(t) = \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}, \qquad i = 1, \cdots, n$$

with $0 < p < 1$, stands between these two extremes, of
. capitalization weighting (as in the S&P 500 Idex), and of
. equal weighting (as in the Value-Line Index).
It is generated by the concave function

$$\mathbf{S}^{(p)}(m) := \left(m_1^p + \cdots + m_n^p\right)^{1/p},$$

and has drift proportional to the excess growth rate:

$$\mathfrak{g}(\cdot) \equiv (1-p)\,\gamma_*^{\mathfrak{D}^{(p)}}(\cdot).$$

$$\mathfrak{D}_i^{(p)}(t) = \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}, \qquad i = 1, \cdots, n$$

With $p = 0$ this becomes equal weighting $\varphi_i(\cdot) \equiv 1/n, \; 1 \leq i \leq n$.
With $p = 1$ we get the market portfolio $\mu(\cdot)$.
Think of it as a way to "interpolate" between the two extremes.

This portfolio *over-weighs the small-cap stocks and under-weighs the large-cap stocks,* relative to the market weights.

. It tries to capture some of the "buy-low/sell-high" characteristics of equal weighting, but without deviating too much from market capitalizations—and also without incurring a lot of trading costs or excessive risk.

It can be viewed as an "enhanced market portfolio" or "enhanced capitalization index", in this sense.

- Another way to "interpolate" between the extremes of equal-weighting and capitalization-weighting, goes as follows. Consider the *geometric mean*

$$\mathbf{G}(m) := \big(m_1 \cdots m_n\big)^{1/n}$$

and, for any given $c \in (0, \infty)$, its modification

$\mathbf{G}_c(m) := c + \mathbf{G}(m)$, which satisfies: $c < \mathbf{G}_c(m) \le c + (1/n)$.

This modified geometric mean function generates the *modified equally-weighted portfolio*

$$\varphi_i^{(c)}(t) = \frac{c}{c + \mathbf{G}(\mu(t))} \cdot \frac{1}{n} + \frac{\mathbf{G}(\mu(t))}{c + \mathbf{G}(\mu(t))} \cdot \mu_i(t),$$

for $i = 1, \cdots, n$ that we saw already in (6).
These weights are convex combination of the equal-weighted and market portfolios; and

$$\mathfrak{g}^{\varphi^{(c)}}(t) = \frac{\mathbf{G}(\mu(t))}{c + \mathbf{G}(\mu(t))} \, \gamma_\varphi^*(t).$$

- In a similar spirit, consider the *entropy function*

$$\mathbf{H}(m) := -\sum_{i=1}^{n} m_i \log m_i \,, \quad m \in \Delta_+^n \,.$$

This entropy function generates the *entropic* portfolio $\mathfrak{E}(\cdot)$, with weights

$$\mathfrak{E}_i(t) = \frac{-\mu_i(t) \, \log \mu_i(t)}{\mathbf{H}(\mu(t))} \,, \qquad i = 1, \cdots, n$$

and drift-process

$$\mathfrak{g}^{\mathfrak{E}}(t) = \frac{\gamma_\mu^*(t)}{\mathbf{H}(\mu(t))} \,.$$

- Now take again the *entropy function*

$$\mathbf{H}(m) = -\sum_{i=1}^{n} m_i \log m_i \,, \quad m \in \Delta_+^n$$

and, for any given $c \in (0, \infty)$, look at its modification

$\mathbf{S}_c(m) := c + \mathbf{H}(m)$, which satisfies: $c < \mathbf{S}_c(m) \leq c + \log n\,.$

This modified entropy function generates the *modified entropic* portfolio $\mathfrak{E}^{(c)}(\cdot)$ of (4), with weights

$$\mathfrak{E}_i^{(c)}(t) = \frac{\mu_i(t)\left(c - \log \mu_i(t)\right)}{c + \mathbf{H}(\mu(t))}\,, \qquad i = 1, \cdots, n$$

and drift-process given by

$$\mathfrak{g}^{\mathfrak{E}^{(c)}}(t) = \frac{\gamma_\mu^*(t)}{c + \mathbf{H}(\mu(t))}\,.$$

11. SUFFICIENT INTRINSIC VOLATILITY LEADS TO ARBITRAGE RELATIVE TO THE MARKET

Principle: *Sufficient volatility creates growth opportunities in a financial market.*

We have already encountered an instance of this principle in section 8: we saw there that, in the presence of a strong non-degeneracy condition on the market's covariation structure, "reasonably diversified" long-only portfolios with constant weights can represent superior long-term growth opportunities relative to the overall market.

We shall examine in Proposition 1 below another instance of this phenomenon.

More precisely, we shall try again to put the above intuition on a precise quantitative basis, by identifying the *excess growth rate*

$$\gamma_*^{\mu}(t) = \frac{1}{2} \sum_{i=1}^{n} \mu_i(t)\, \mathfrak{A}_{ii}^{\mu}(t)$$

of the market portfolio – which also measures the market's *intrinsic volatility* – as a *driver of growth*.
To wit, as a quantity whose "availability" or "sufficiency" (boundedness away from zero) can lead to opportunities for strong arbitrage and for superior long-term growth, relative to the market.

Proposition 1: *Assume that over* $[0, T]$ *there is "sufficient intrinsic volatility" (excess growth):*

$$\int_0^T \gamma_*^\mu(t)dt \geq hT\,, \quad \text{or} \quad \gamma_*^\mu(t) \geq h\,, \quad 0 \leq t \leq T$$

holds a.s., for some constant $h > 0$. *Take*

$$T > T_* := \frac{\mathbf{H}(\mu(0))}{h}\,, \qquad \text{and} \qquad \mathbf{H}(x) := -\sum_{i=1}^{n} x_i \log x_i$$

the entropy function. Then the modified entropic stock portfolio (from a couple of slides ago)

$$\mathfrak{E}_i^{(c)}(t) := \frac{\mu_i(t)\,(c - \log\mu_i(t))}{\sum_{j=1}^{n} \mu_j(t)\,(c - \log\mu_j(t))}\,, \quad i = 1, \cdots, n$$

is generated by the function

$$\mathbf{H}_c(m) := c + \mathbf{H}(m)$$

on Δ_+^n ; *and for* $c = c(T) > 0$ *sufficiently large, it effects strong arbitrage relative to the market.*

- *Sketch of Argument for Proposition 1:* Note that the function $\mathbf{H}_c(\cdot) := c + \mathbf{H}(\cdot)$ is bounded both from above and below:

$$0 < c < \mathbf{H}_c(m) \leq c + \log n \,, \qquad m \in \Delta^n_+ \,.$$

The master equation now shows that

$$\log\left(\frac{V^{1,\mathfrak{E}^{(c)}}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{c + \mathbf{H}(\mu(T))}{c + \mathbf{H}(\mu(0))}\right) + \int_0^T \mathfrak{g}^{\mathfrak{E}^{(c)}}(t)\,\mathrm{d}t$$

is strictly positive, provided

$$T > \frac{1}{h}(c + \log n)\log\left(1 + \frac{\log n}{c}\right) \longrightarrow \frac{\log n}{h}$$

as $c \to \infty$.

This is because the first term on the right-hand side of

$$\log\left(\frac{V^{1,\mathfrak{E}^{(c)}}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{c+\mathbf{H}(\mu(T))}{c+\mathbf{H}(\mu(0))}\right) + \int_0^T \mathfrak{g}^{\mathfrak{E}^{(c)}}(t)\,\mathrm{d}t$$

dominates

$$-\log\left(\frac{c+\log n}{c}\right)$$

and, under the conditions of the proposition, the second term

$$\int_0^T \mathfrak{g}^{\mathfrak{E}^{(c)}}(t)\,\mathrm{d}t = \cdots = \int_0^T \frac{\gamma_*^{\mu}(\cdot)}{c+\mathbf{H}(\cdot)}\,\mathrm{d}t \geq \int_0^T \frac{\gamma_*^{\mu}(\cdot)}{c+\log n}\,\mathrm{d}t$$

dominates $hT\,/\,(c+\log n)$.

To put it a bit differently: *if you have a constant wind on your back, sooner all later you'll overtake any obstacle* – e.g., the constant $\log\big((c+\log n)/c\big)$.

This leads to strong relative arbitrage with respect to the market, for sufficiently large $T > \log n/h$; indeed to

$$\mathbb{P}\Big(V^{1,\mathfrak{E}^{(c)}}(T) > V^{1,\mu}(T) \Big) = 1 .$$

(Intuition, as before: you can generate such relative arbitrage if there is "enough intrinsic variation (volatility)" in the market... .)

Major Question (Stayed Open for 10 Years): Is such relative arbitrage possible over arbitrary time-horizons, under the conditions of Proposition 1 ?

We shall discuss below two special cases, where the answer to this question is known – and is affirmative.

Johannes RUF showed in 2015, with a very interesting example, that the answer to this question is, in general, NEGATIVE.

Then a few months later, Bob FERNHOLZ provided a host of simpler examples, some of them quite amazing.

Johannes and Bob also proved general theorems to the effect that, under some ADDITIONAL conditions, the answer to the question does become affirmative. Those theorems cover the special cases described in Propositions 1 (above) and 2 (below).

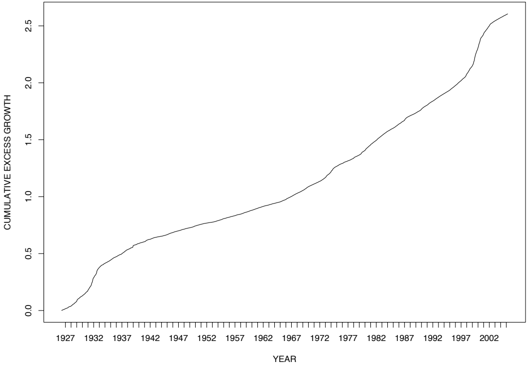

Figure 1: Cumulative Excess Growth $\int_0^{\cdot} \gamma_*^{\mu}(t)\,\mathrm{d}t$ for the U.S. Stock Market during the period 1926-1999.

The previous figure plots the cumulative excess growth $\int_0^{\cdot} \gamma_\mu^*(t)\,\mathrm{d}t$ for the U.S. equities market over most of the twentieth century. Note the conspicuous bumps in the curve, first in the Great Depression period in the early 1930s, then again in the "irrational exuberance" period at the end of the century. The data used for this chart come from the monthly stock database of the Center for Research in Securities Prices (CRSP) at the University of Chicago.

The market we construct consists of the stocks traded on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX) and the NASDAQ Stock Market, after the removal of all REITs, all closed-end funds, and those ADRs not included in the S&P 500 Index. Until 1962, the CRSP data included only NYSE stocks. The AMEX stocks were included after July 1962, and the NASDAQ stocks were included at the beginning of 1973. The number of stocks in this market varies from a few hundred in 1927 to about 7500 in 2005.

Proposition 2: *Introduce the "modified intrinsic volatility"*

$$\zeta_*(t) := \big(\mu_1(t)\cdots\mu_n(t)\big)^{1/n} \left[\sum_{i=1}^n \alpha_{ii}(t) - \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \alpha_{ij}(t) \right]$$

and assume that over the given horizon $[0, T]$ *we have a.s.:*

$$\int_0^T \zeta_*(t)\mathrm{d}t \geq h\,T\,, \quad \text{or} \quad \boxed{\zeta_*(t) \geq h\,, \quad 0 \leq t \leq T}$$

for some constant $h > 0$. *Then, with* $\mathfrak{m}(t) := (\mu_1(t)\cdots\mu_n(t))^{1/n}$ *and for sufficiently large* $c > 0$, *the modified equally-weighted portfolio of (6)*

$$\varphi_i^{(c)}(t) = \frac{c}{c+\mathfrak{m}(t)}\cdot\frac{1}{n} + \frac{\mathfrak{m}(t)}{c+\mathfrak{m}(t)}\cdot\mu_i(t)\,, \qquad i = 1, \cdots, n\,,$$

is arbitrage relative to the market over $[0, T]$*, provided* $T > (2n^{1-(1/n)})/h$.

The proof is similar to that of Proposition 1. The modified-equal-weighted stock-portfolio is generated by $c + (m_1 \cdots m_n)^{1/n}$, and

Major Question (Was open for 10 Years): Is such relative arbitrage possible over arbitrary time-horizons, under the conditions

$$\int_0^T \gamma_*^\mu(t)\,\mathrm{d}t \geq hT\,, \quad \text{or} \quad \gamma_*^\mu(t) \geq h\,, \quad 0 \leq t \leq T$$

of Proposition 1 ?

Partial Answer #1: YES, if the variation/covariation matrix $\alpha(\cdot) = \sigma(\cdot)\sigma'(\cdot)$ has all its eigenvalues bounded away from zero and infinity: to wit, if we have (a.s.)

$$\kappa||\,\xi\,||^2 \;\leq\; \xi'\alpha(t)\xi \;\leq\; \mathcal{K}||\,\xi\,||^2\,, \qquad \forall\; t \geq 0\,, \quad \xi \in \mathbb{R}^d \tag{17}$$

for suitable constants $0 < \kappa < \mathcal{K} < \infty$.

In this case one can show (Bob FERNHOLZ, Kostas KARDARAS)

$$\boxed{\frac{\kappa}{2}\left(1-\pi_{(1)}(t)\right) \le \gamma_*^{\pi}(t) \le 2\mathcal{K}\left(1-\pi_{(1)}(t)\right)} \qquad (18)$$

for the maximal weight of any long-only portfolio $\pi(\cdot)$, namely

$$\pi_{(1)}(t) := \max_{1 \le i \le n} \pi_i(t) \, .$$

Thus, under the structural assumption of (17), i.e.,

$$\kappa || \, \xi ||^2 \le \xi' \alpha(t) \xi \le \mathcal{K} || \, \xi ||^2 \, , \qquad \forall \; t \ge 0 \, , \;\; \xi \in \mathbb{R}^d \, ,$$

the "sufficient intrinsic volatility" (a.s.) condition of Proposition 1, namely

$$\int_0^T \gamma_*^{\mu}(t)dt \geq hT\,, \qquad \text{or} \qquad \gamma_*^{\mu}(t) \geq h\,, \quad 0 \leq t \leq T\,,$$

is equivalent to the (a.s.) requirement of **Market Diversity**

$$\int_0^T \mu_{(1)}(t)dt \leq (1-\delta)T\,, \qquad \text{or} \qquad \boxed{\max_{0\leq t\leq T} \mu_{(1)}(t) \leq 1-\delta}$$

for some $\delta \in (0,1)$.
(Weak diversity and strong diversity, respectively.)

Remark: The maximal relative capitalization never gets above a certain percentage. In the S&P 500 universe, no company has ever attained more than 15% of the total market capitalization; in the last 40 years, this has been more like 6%.

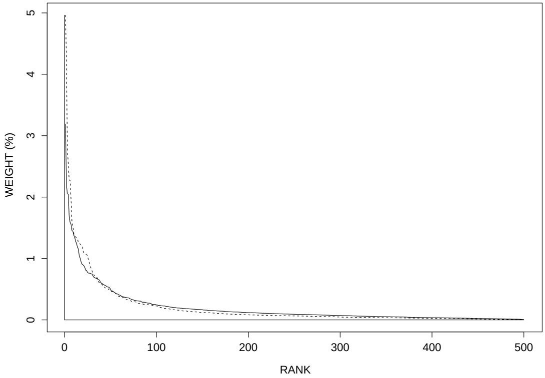

Figure 2: Capital Distribution for the S&P 500 Index. December 30, 1997 (solid line), and December 29, 1999 (broken line).

Proposition 3: *Suppose (weak) diversity prevails, and the lowest eigenvalue of the covariation matrix is bounded away from zero. For fixed* $p \in (0,1)$, *consider the simple "diversity-weighted" portfolio*

$$\mathfrak{D}_i^{(p)}(t) \equiv \mathfrak{D}_i(t) := \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}\,, \quad \forall \;\; i = 1, \ldots, n\,,$$

generated by the concave function

$$\mathbf{S}^{(p)}(m) \equiv \mathbf{S}(m) = \left(m_1^p + \cdots + m_n^p \right)^{1/p}.$$

Then this portfolio leads to arbitrage relative to the market, over sufficiently long time horizons.

With $p = 0$ this becomes equal weighting $\varphi_i(\cdot) \equiv 1/n,\ 1 \le i \le n$.

With $p = 1$ we get the market portfolio $\mu(\cdot)$.

(Recall in this vein the modified equal-weighted portfolio of (6), which "interpolates" between equal-weighting and cap-weighting in a rather different manner.)

With respect to the market portfolio, this "diversity-weighted" portfolio

$$\mathfrak{D}_i^{(p)}(t) \equiv \mathfrak{D}_i(t) := \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}\,, \qquad \forall \;\; i = 1, \ldots, n\,,$$

de-emphasizes the "upper (big cap) end" of the market, and over-emphasizes the "lower (small cap) end" – but observes all relative rankings. It does all this in a completely passive way, without estimating or optimizing anything.

. Appropriate modifications of this rule generate such arbitrage over *arbitrary* time-horizons; for detais, see FKK (2005).

For extensive discussion of the actual performance of this "diversity-weighted portfolio" as well as of the "pure entropic portfolio" (with $c = 0$) we saw before, see FERNHOLZ (2002).

Proof of Proposition 3: For this *"diversity-weighted"* portfolio $\mathfrak{D}^{(p)}(\cdot)$ we have from the "master equation" (14) the formula

$$\log\left(\frac{V^{1,\mathfrak{D}^{(p)}}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{\mathbf{S}^{(p)}(\mu(T))}{\mathbf{S}^{(p)}(\mu(0))}\right) + (1-p)\int_0^T \gamma_*^{\mathfrak{D}^{(p)}}(t)dt \; .$$

- First term on RHS tends to be mean-reverting, and is certainly bounded:

$$1 = \sum_{j=1}^{n} m_j \le \sum_{j=1}^{n} (m_j)^p = \left(\mathbf{S}^{(p)}(m)\right)^p \le n^{1-p} \, .$$

Measure of Diversity: minimum occurs when one company is the entire market, maximum when all companies have equal relative weights.

• We remarked already, that the biggest weight of $\mathfrak{D}^{(p)}(\cdot)$ does not exceed the largest market weight:

$$\mathfrak{D}^{(p)}_{(1)}(t) := \max_{1 \le i \le n} \mathfrak{D}^{(p)}_i(t) = \frac{\big(\mu_{(1)}(t)\big)^p}{\sum_{k=1}^n \big(\mu_{(k)}(t)\big)^p} \le \mu_{(1)}(t)\,.$$

By weak diversity over $[0, T]$, there is a number $\delta \in (0,1)$ for which

$$\int_0^T \big(1 - \mu_{(1)}(t)\big)\,\mathrm{d}t > \delta\, T$$

holds.

From the strict non-degeneracy of the covariation matrix we have

$$\boxed{\frac{\kappa}{2}\left(1-\pi_{(1)}(t)\right) \leq \gamma_*^{\pi}(t)}$$

as in (18), and thus:

$$\frac{2}{\kappa}\int_0^T \gamma_*^{\mathfrak{D}^{(p)}}(t)\,\mathrm{d}t \geq \int_0^T \left(1-\mathfrak{D}_{(1)}^{(p)}(t)\right)\mathrm{d}t \geq \int_0^T \left(1-\mu_{(1)}(t)\right)\mathrm{d}t > \delta T.$$

- From these two observations we get

$$\log\left(\frac{V^{1,\mathfrak{D}^{(p)}}(T)}{V^{1,\mu}(T)}\right) > (1-p)\left[\frac{\kappa T}{2}\cdot\delta - \frac{1}{p}\cdot\log n\right],$$

so for a time-horizon

$$T > T_* := (2\log n)/(p\kappa\delta)$$

sufficiently large, the RHS is strictly positive. $\square$

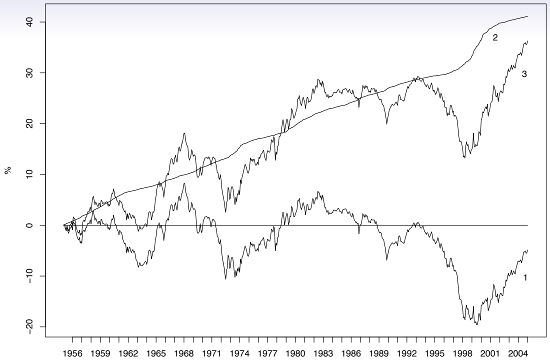

Figure 3: Simulation of a diversity-weighted portfolio, 1956–2005. (1: generating function; 2: drift process; 3: relative return.)

$$\log\left(\frac{V^{1,\mathfrak{D}^{(p)}}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{\mathbf{S}^{(p)}(\mu(T))}{\mathbf{S}^{(p)}(\mu(0))}\right) + (1-p)\int_0^T \gamma_*^{\mathfrak{D}^{(p)}}(t)dt\ .$$

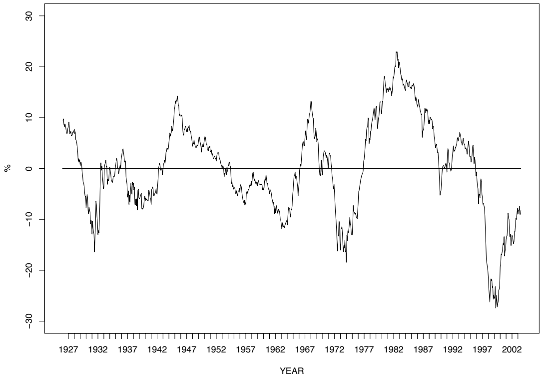

Figure 4: Cumulative Change in Market Diversity, 1927-2004. The mean-reverting character of this quantity is rather apparent.

- **Remark:** *Consider a market that satisfies the strong non-degeneracy condition as in (7):*

$$\xi'\alpha(t)\xi = \xi'\sigma(t)\sigma'(t)\xi \geq \kappa\|\xi\|^2 \,, \quad \forall \;\; t \in [0,\infty) \,, \quad \xi \in \mathbb{R}^n \,.$$

If all its stocks $i = 1, \ldots, n$ *have the same growth-rate* $\gamma_i(\cdot) \equiv \gamma(\cdot)$*, then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma_*^\mu(t)\,\mathrm{d}t \;=\; 0, \quad \text{a.s.}$$

. In particular, such a market *cannot be diverse on long time horizons*: once in a while a single stock dominates such a market, then recedes; sooner or later another stock takes its place as absolutely dominant leader; and so on.

. The same can be seen to be true for a market that satisfies the above strong non-degeneracy condition as in (7) and its assets have *constant,* though not necessarily equal, growth rates.

- Here is a quick argument: from $\gamma_i(\cdot) \equiv \gamma(\cdot)$ and $X(\cdot) = X_1(\cdot) + \cdots X_n(\cdot)$ we have

$$\lim_{T\to\infty} \frac{1}{T} \left(\log X(T) - \int_0^T \gamma^\mu(t)\mathrm{d}t \right) = 0\,,$$

$$\lim_{T\to\infty} \frac{1}{T} \left(\log X_i(T) - \int_0^T \gamma(t)\mathrm{d}t \right) = 0$$

for all $1 \leq i \leq n$. But then

$$\lim_{T\to\infty} \frac{1}{T} \left(\log X_{(1)}(T) - \int_0^T \gamma(t)\mathrm{d}t \right) = 0\,, \quad \text{holds a.s.}$$

for the biggest stock $X_{(1)}(\cdot) := \max_{1\leq i\leq n} X_i(\cdot)$, and we note

$$X_{(1)}(\cdot) \;\leq\; X(\cdot) \;\leq\; n\,X_{(1)}(\cdot)\,.$$

Therefore, from $X_{(1)}(\cdot) \le X(\cdot) \le n\, X_{(1)}(\cdot)$ we deduce

$$\lim_{T\to\infty} \frac{1}{T} \Big(\log X(T) - \log X_{(1)}(T) \Big) = 0\,, \quad \text{thus}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma^{\mu}(t) - \gamma(t)\big)\, \mathrm{d}t \;=\; 0\,.$$

But

$$\gamma^{\mu}(t) = \sum_{i=1}^{n} \mu_i(t)\gamma(t) + \gamma_*^{\mu}(t) = \gamma(t) + \gamma_*^{\mu}(t)\,,$$

because all growth rates are equal. □

Major Open Question (Was open for 10 Years): Is such relative arbitrage possible over arbitrary time-horizons, under the conditions of Proposition 1 ?

$$\int_0^T \gamma_*^\mu(t)dt \geq hT\,, \quad \text{or} \quad \gamma_*^\mu(t) \geq h\,, \;\; 0 \leq t \leq T\,.$$

Partial Answer #2: YES, for the (non-diverse!) so-called *VOLATILITY-STABILIZED* model that we broach now.

Consider the abstract market model

$$\boxed{\mathrm{d}\Big(\log X_i(t)\Big) = \frac{\alpha\,\mathrm{d}t}{2\,\mu_i(t)} + \frac{1}{\sqrt{\mu_i(t)}}\,\mathrm{d}W_i(t)}$$

for $i = 1, \cdots, n$ with $d = n \geq 2$ and $\alpha \geq 0$.

In other words, we assign the largest volatilities and the largest log-drifts to the smallest stocks.

This specification amounts to solving in the positive orthant of $\mathbb{R}^n$ the system of degenerate stochastic differential equations, for $i = 1, \cdots, n$:

$$\mathrm{d}X_i(t) = \frac{1+\alpha}{2} \Big(X_1(t) + \cdots + X_n(t) \Big) \, \mathrm{d}t + \sqrt{X_i(t) \Big(X_1(t) + \cdots + X_n(t) \Big)} \cdot \mathrm{d}W_i(t) \, .$$

General theory: BASS & PERKINS (TAMS 2002). Shows this system has a weak solution, unique in distribution, so the model is well-posed.

Very recent extension of this model in the frameork of *Polynomial Processes,* to allow for co-variations among different stocks, has been carried out by Christa CUCHIERO (2017).

Better still: It is possible to describe this solution fairly explicitly, in terms of Bessel processes.

- Since we have

$$\alpha_{ij}(t) \;=\; \frac{\delta_{ij}}{\mu_i(t)}$$

in this model, an elementary computation gives the quantities

$$\gamma_*^{\mu}(t) \;=\; \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\big(1-\mu_i(t)\big)\,\alpha_{ij}(t) \;=\; \frac{n-1}{2} \;=:\; h > 0\,,$$

$$\mathrm{a}^{\mu\mu}(\cdot) \;\equiv\; 1$$

for the market portfolio $\mu(\cdot)$, and

$$\gamma^{\mu}(\cdot) \;\equiv\; \frac{(1+\alpha)n-1}{2} \;=:\; \gamma > 0\,.$$

Despite the erratic, widely fluctuating behavior of individual stocks, the overall market performance is remarkably stable. In particular, the total market capitalization is

$$X(t) = X_1(t) + \ldots + X_n(t) = x \cdot e^{\gamma t + B(t)},$$

for the scalar Brownian motion

$$B(t) := \sum_{\nu=1}^{n} \int_0^t \sqrt{\mu_\nu(s)}\, \mathrm{d}W_\nu(s), \qquad 0 \le t < \infty.$$

• We call this phenomenon **stabilization by volatility**: the big volatility swings for the smallest stocks, together with the smaller volatility swings for the largest stocks, end up stabilizing the overall market by producing constant, positive overall growth and variation.

(Note $\kappa = 1$ but $\mathcal{K} = \infty$, so

$$\kappa || \xi ||^2 \le \xi' \alpha(t) \xi \le \mathcal{K} || \xi ||^2, \quad \forall\ t \ge 0,\ \xi \in \mathbb{R}^d$$

in (17), slide 85, fails.)

- The condition $\gamma_*^{\mu}(\cdot) \geq h > 0$ of Proposition 1 is satisfied here, with $h = (n-1)/2$. Thus the model admits arbitrage relative to the market, at least on time-horizons $[0, T]$ with

$$T > T_* , \qquad \text{where} \qquad T_* := \frac{2\,\mathbf{H}(\mu(0))}{n-1} < \frac{2\,\log n}{n-1} .$$

The upper estimate $(2\log n)/(n-1)$ is a rather small number if $n = 5000$ as in WILSHIRE 5000.

- This adds plausibility to the earlier claim, that such outperformance is possible over all time-horizons. Proved by A. BANNER and D. FERNHOLZ (2008), not just for the volatility-stabililized model but for quite general growth rates in

$$\mathrm{d}\Big(\log X_i(t)\Big) = \gamma_i(t)\,\mathrm{d}t + \frac{1}{\sqrt{\mu_i(t)}}\,\mathrm{d}W_i(t) , \quad i = 1, \cdots , n$$

that such arbitrage is now possible on *any given* time-horizon.

- On the other hand, the condition

$$\zeta_*(\cdot) \geq h > 0$$

of Proposition 2 (slide 82) is also satisfied here, with $h = n - 1$.
This follows from the geometric mean / harmonic mean inequality

$$\zeta_*(t) = \big(\mu_1(t) \cdots \mu_n(t)\big)^{1/n} \left[\sum_{i=1}^n \alpha_{ii}(t) - \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}(t) \right]$$

$$= \big(\mu_1(t) \cdots \mu_n(t)\big)^{1/n} \cdot \sum_{i=1}^n \left(1 - \frac{1}{n}\right) \alpha_{ii}(t)$$

$$\geq \frac{n}{\frac{1}{\mu_1(t)} + \cdots + \frac{1}{\mu_n(t)}} \cdot \frac{n-1}{n} \sum_{i=1}^n \frac{1}{\mu_i(t)} = n - 1 \,.$$

- *What is the long-term-growth behavior of an* ***individual*** *stock?*

A little bit of Stochastic Analysis provides the Representations

$$X_i(t) \,=\, \Big(\mathfrak{R}_i(\Lambda(t))\Big)^2 , \qquad 0 \le t < \infty\,, \quad i = 1, \cdots, n$$

and

$$X(t) \,=\, X_1(t) + \cdots + X_n(t) = x\, e^{\,\gamma t + B(t)} \,=\, \Big(\mathfrak{R}(\Lambda(t))\Big)^2 .$$

Here

$$4\,\Lambda(t) \,:=\, \int_0^t X(s)\, ds \,=\, x \int_0^t e^{\,\gamma s + B(s)}\, \mathrm{d}s\,,$$

whereas $\mathfrak{R}_1(\cdot), \cdots, \mathfrak{R}_n(\cdot)$ are *independent* BESSEL processes in dimension $m := 2(1+\alpha)$, and

$$\mathfrak{R}(u) \,:=\, \sqrt{\,\big(\mathfrak{R}_1(u)\big)^2 + \,\cdots\, + \big(\mathfrak{R}_n(u)\big)^2\,}\,.$$

That is, with $\widehat{W}_1(\cdot), \cdots, \widehat{W}_n(\cdot)$ independent scalar Brownian motions, we have

$$\mathrm{d}\mathfrak{R}_i(u) = \frac{m-1}{2\,\mathfrak{R}_i(u)}\,\mathrm{d}u + \mathrm{d}\widehat{W}_i(u)\,, \quad i = 1, \cdots, n\,.$$

Finally,

$$\mathfrak{R}(u) := \sqrt{\big(\mathfrak{R}_1(u)\big)^2 + \cdots + \big(\mathfrak{R}_n(u)\big)^2}$$

is BESSEL process in dimension mn.

We are led to the *skew representation* (Irina Goia, Soumik Pal)

$$\boxed{\mathfrak{R}_i^2(u) \;=\; \mathfrak{R}^2(u)\cdot \mu_i\left(4\int_0^u \frac{\mathrm{d}v}{\mathfrak{R}^2(v)}\right), \quad 0 \le u < \infty\,.}$$

Here the vector $\mu(\cdot) = \big(\mu_1(\cdot), \ldots, \mu_n(\cdot)\big)$ of market-weights

$$\mu_i(\cdot) \;=\; \big(\mathfrak{R}_i^2/\mathfrak{R}^2\big)\big(\Lambda(\cdot)\big)$$

is *independent* of the Bessel process $\mathfrak{R}(\cdot)$; thus also of the change-of-clock $\Lambda(\cdot)$ which is defined in terms of this Bessel process $\mathfrak{R}(\cdot)$ via the integral equation

$$4\,\Lambda(\cdot) \;=\; \int_0^{\cdot} \mathfrak{R}^2\big(\Lambda(t)\big)\,\mathrm{d}t\,, \quad \text{equivalently} \quad \Lambda^{-1}(\cdot) \;=\; 4\int_0^{\cdot} \frac{\mathrm{d}v}{\mathfrak{R}^2(v)}\,;$$

and of the total market capitalization $X(\cdot)$.

This vector $\mu(\cdot) = (\mu_i(\cdot))_{i=1}^n$ of market-weights is a so-called vector JACOBI *process* with values in Δ_+^n and the dynamics

$$\mathrm{d}\mu_i(t) = (1+\alpha)\big(1 - n\mu_i(t)\big)\mathrm{d}t + \big(1 - \mu_i(t)\big)\sqrt{\mu_i(t)}\,\mathrm{d}\beta_i(t) \\ -\mu_i(t)\sum_{j\neq i}\sqrt{\mu_j(t)}\,\mathrm{d}\beta_j(t)\,,$$

for $i = 1, \cdots, n$.

Here, $\beta_1(\cdot), \cdots, \beta_n(\cdot)$ are independent, standard Brownian motions.

In particular, the processes $\mu_1(\cdot), \cdots, \mu_n(\cdot)$ have local variations $\mu_i(t)(1 - \mu_i(t))$ and covariations $-\mu_i(t)\,\mu_j(t)$.

This structure suggests that the invariant measure for the Δ_+^n −valued diffusion $\mu(\cdot) = (\mu_i(\cdot))_{i=1}^n$ of market weights, is the distribution of the vector

$$\left(\frac{Q_1}{Q_1 + \cdots + Q_n} , \cdots , \frac{Q_n}{Q_1 + \cdots + Q_n} \right) ,$$

where $Q_1, \cdots , Q_n$ are independent random variables with common distribution

$$\frac{2^{-(1+\alpha)}}{\Gamma(1+\alpha)} \, q^{\alpha} \, e^{-q/2} \, dq \, , \qquad 0 < q < \infty \, ,$$

(chi-square with "$2(1+\alpha)$-degrees-of-freedom").

- From these representations, one obtains the (a.s.) *long-term growth rates of the entire market and of the largest stock*

$$\boxed{\lim_{T\to\infty} \frac{1}{T} \log X(T) = \lim_{T\to\infty} \frac{1}{T} \log \left(\max_{1\le i\le n} X_i(T) \right) = \gamma};$$

the a.s. *long-term growth rates for individual stocks*

$$\lim_{T\to\infty} \frac{1}{T} \log X_i(T) = \gamma\,, \quad i = 1, \cdots, n \tag{19}$$

for $\alpha > 0$;

the a.s. long-term stock variations

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{dt}{\mu_i(t)} = \frac{2\gamma}{\alpha} = n + \frac{n-1}{\alpha}$$

(for $\alpha > 0$, using the BIRKHOFF ergodic theorem);
that this model is not diverse;
and much more...

NOTE: When $\alpha = 0$, the equation

$$\lim_{T\to\infty} \frac{1}{T} \log X_i(T) = \gamma, \quad i = 1, \cdots, n$$

of (19) holds only *in probability;* the (a.s.) limit-superior is γ, whereas the (a.s.) limit-inferior is $-\infty$.

SPITZER's 0-1 law for planar Brownian motion.
Crashes.... Failure of diversity... .

13.a: Some Concluding Remarks

We have exhibited simple conditions, such as "sufficient level of intrinsic volatility" and "diversity", which lead to arbitrages relative to the market.

These conditions are **descriptive** as opposed to normative, and can be tested from the predictable characteristics of the model posited for the market.

In contrast, familiar assumptions, such as the existence of an equivalent martingale measure (EMM), are **normative** in nature, and *cannot* be decided on the basis of predictable characteristics in the model; see example in Karatzas & Kardaras (2007).