Trading Strategies Generated by Lyapunov Functions

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OUTLINE

Back in 1999, Erhard Robert Fernholz introduced
a construction that was both
(i) remarkable, and
(ii) remarkably easy to prove.

He showed that
for a certain class of so-called "functionally- generated" portfolios, it is possible to express the wealth they generate, discounted by (denominated in terms of) the total market capitalization, solely in terms of the individual companies' *market weights*
– and to do so in a robust, pathwise, model-free manner, that *does not involve stochastic integration.*

This fact can be proved by an application of ITÔ's rule. Once the result is known, its proof can be assigned as a moderate exercise in a stochastic calculus course.

The discovery paved the way for finding simple, structural conditions on *large* equity markets – that involve more than one stock, and typically thousands – under which it is possible to outperform the market portfolio (w.p.1).

Put a little differently: conditions under which (strong) arbitrage relative to the market portfolio is possible.

Bob FERNHOLZ showed also how to implement this outperformance by simple portfolios – which can be constructed solely in terms of observable quantities, without *any* need to estimate parameters of the model or to optimize.

Although well-known, celebrated, and quite easy to prove, FERNHOLZ's construction has been viewed over the past 18+ years as somewhat "mysterious".

In this talk, and in the work on which the talk is based, we hope to help make the result a bit more celebrated and perhaps a bit less mysterious, via an interpretation of portfolio-generating functions as LYAPUNOV functions for the vector process of relative market weights.

We will try to settle then a question about functionally-generated portfolios that has been open for 10 years.

SOME NOTATION

- A probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a right-continuous filtration $\mathfrak{F}$.
- $\mathcal{L}(X)$: class of progressively measurable processes, integrable with respect to some given vector semimartingale $X(\cdot)$.

- $d \in \mathbb{N}$: number of assets in an equity market, at time zero.
- **Nonnegative** continuous $\mathbb{P}$–semimartingales, representing the relative market weights of each asset:

$$\mu(\cdot) \;=\; \big(\mu_1(\cdot), \cdots, \mu_d(\cdot)\big)'$$

with $\mu_1(0) > 0, \cdots, \mu_d(0) > 0$ and taking values in the lateral face of the unit simplex

$$\mathbf{\Delta}^d = \Big\{ \big(x_1, \cdots, x_d\big)' \in [0,1]^d \;:\; \sum_{i=1}^{d} x_i = 1 \Big\}.$$

STOCHASTIC DISCOUNT FACTORS

- Some results below require the notion of a stochastic discount factor ("deflator") for the relative market weight process $\mu(\cdot)$.
- A **Deflator** is a continuous, adapted, strictly positive process $Z(\cdot)$ with $Z(0) = 1$, for which

  all products $\quad Z(\cdot)\, \mu_i(\cdot)\,, \quad i = 1, \cdots, d \quad$ are local martingales.

In particular, $Z(\cdot)$ is a local martingale itself.

• The existence of such a deflator will be invoked explicitly when needed, and ONLY then.

FROM INTEGRANDS TO TRADING STRATEGIES

- For any given "number-of-shares" process $\vartheta(\cdot) \in \mathcal{L}(\mu)$, we consider its "value"

$$V^{\vartheta}(t) = \sum_{i=1}^{d} \vartheta_i(t)\,\mu_i(t)\,, \qquad 0 \le t < \infty\,.$$

- We call such $\vartheta(\cdot)$ a **Trading Strategy**, if its "defect of self-financibility" is identically equal to zero:

$$Q^{\vartheta}(T) := V^{\vartheta}(T) - V^{\vartheta}(0) - \int_0^T \langle \vartheta(t), \mathrm{d}\mu(t) \rangle \equiv 0\,, \quad T \ge 0.$$

- If $Q^{\vartheta}(\cdot) \equiv 0$ fails, then $\vartheta(\cdot) \in \mathcal{L}(\mu)$ is not a trading strategy.
- **However**, for any $\boldsymbol{C} \in \mathbb{R}$, the vector process defined via

$$\varphi_i(\cdot) = \vartheta_i(\cdot) - Q^{\vartheta}(\cdot) + \boldsymbol{C}\,, \qquad i = 1, \cdots, d$$

IS a trading strategy, and its value is given by

$$V^{\varphi}(\cdot) \;=\; V^{\vartheta}(0) + \int_0^{\cdot} \langle \vartheta(t), \mathrm{d}\mu(t) \rangle + \boldsymbol{C}\,.$$

RELATIVE ARBITRAGE

Definition

A trading strategy $\varphi(\cdot)$ *outperforms the market* (or is **relative arbitrage** with respect to it) over the time horizon $[0, T]$, if

$$V^{\varphi}(0) = 1; \qquad V^{\varphi}(\cdot) \geq 0$$

and

$$\mathbb{P}\Big(V^{\varphi}(T) \geq 1\Big) = 1; \qquad \mathbb{P}\Big(V^{\varphi}(T) > 1\Big) > 0.$$

- We say that this relative arbitrage is **strong**, if

$$\mathbb{P}\Big(V^{\varphi}(T) > 1\Big) = 1\,.$$

REGULAR FUNCTIONS

Definition

A continuous function $G : \mathbf{supp}\,(\mu) \to \mathbb{R}$ is said to be **Regular** for the process $\mu(\cdot)$, if:

1. There exists a measurable function

$$DG = \big(D_1 G, \cdots, D_d G\big)' : \mathbf{supp}\,(\mu) \to \mathbb{R}^d$$

such that the "generalized gradient" process $\boldsymbol{\vartheta}(\cdot)$ with

$$\boldsymbol{\vartheta}_i(\cdot) = D_i G\big(\mu(\cdot)\big), \qquad i = 1, \cdots, d$$

belongs to $\mathcal{L}(\mu)$.

2. The continuous, adapted process $\Gamma^G(\cdot)$ below has finite variation on compact intervals:

$$\Gamma^G(T) := G\big(\mu(0)\big) - G\big(\mu(T)\big) + \int_0^T \langle \boldsymbol{\vartheta}(t), \mathrm{d}\mu(t) \rangle, \quad 0 \le T < \infty\,.$$

LYAPUNOV Functions

Definition

We say that a regular function G is a **Lyapunov function** for the process $\mu(\cdot)$, if the finite-variation process

$$\Gamma^G(\cdot) = G\big(\mu(0)\big) - G\big(\mu(\cdot)\big) + \int_0^{\cdot} \big\langle DG\big(\mu(t)\big), \mathrm{d}\mu(t)\big\rangle$$

is actually non-decreasing.

Definition

We say that a regular function G is **Balanced** for $\mu(\cdot)$, if

$$G\big(\mu(t)\big) = \sum_{j=1}^{d} \mu_j(t)\, D_j G\big(\mu(t)\big)\,, \qquad 0 \le t < \infty.$$

The geometric mean $M(x) = \big(x_1 \cdots x_n\big)^{1/n}$ is an example.

Remark: On Terminology.
To wrap our minds around this terminology, assume that the vector process $\boldsymbol{\vartheta}(\cdot) = DG(\mu(\cdot))$ is *locally orthogonal to the random motion of the market weights* $\mu(\cdot)$, in the sense that

$$\int_0^{\cdot} \big\langle \boldsymbol{\vartheta}(t), \mathrm{d}\mu(t) \big\rangle \equiv \int_0^{\cdot} \big\langle DG(\mu(t)), \mathrm{d}\mu(t) \big\rangle \equiv 0\,.$$

Then the LYAPUNOV property posits that

$$G\big(\mu(\cdot)\big) = G\big(\mu(0)\big) - \Gamma^G(\cdot)$$

is a decreasing process: the classical definition.

. More generally, let us assume that $Z(\cdot)$ is a deflator, and that $G \geq 0$ is a LYAPUNOV function, for the process $\mu(\cdot)$. Then $Z(\cdot)G(\mu(\cdot))$ is a $\mathbb{P}$–supermartingale.

Examples of Regular and Lyapunov functions

Example

If G is **of class** $\mathcal{C}^2$ in a neighborhood of $\mathbf{\Delta}^d$, Itô's formula yields

$$\Gamma^G(\cdot) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^{\cdot} \Big(- D_{ij}^2 G\big(\mu(t)\big)\Big) \, \mathrm{d}\langle \mu_i, \mu_j \rangle(t)$$

Therefore, such a function G is regular;
if it is also **concave**, then G becomes a Lyapunov function.

Significance: an "aggregate cumulative measure of total variation" for the entire market, with the Hessian ("curvature")

$$-D^2 G(\mu(t))$$

acting as the "aggregator" at time t.

Remark: The process $\Gamma^G(\cdot)$:

(i) May, in general, depend on the choice of DG; it does NOT, i.e., is *uniquely determined,* if a deflator $Z(\cdot)$ exists for $\mu(\cdot)$.

(iii) Takes the form of the excess growth rate of the market portfolio, or of "cumulative average relative variation of the market"

$$\Gamma^H(\cdot) = \frac{1}{2} \sum_{j=1}^{d} \int_0^{\cdot} \mu_j(t) \, \mathrm{d}\langle \log \mu_j \rangle(t) \,,$$

when $G = H$ is the Gibbs/Shannon entropy function.

We ran into this quantity several times in yesterday's talk.

CONCAVE FUNCTIONS ARE LYAPUNOV

Theorem

A continuous function $G : \mathbf{supp}(\mu) \to \mathbb{R}$ *is* LYAPUNOV, *if it can be extended to a continuous, concave function on the set*

1. $\mathbf{\Delta}^d_+ := \mathbf{\Delta}^d \cap (0,1)^d$ *and*

$$\mathbb{P}\left(\mu(t) \in \mathbf{\Delta}^d_+ , \quad \forall\ t \geq 0\right) \,=\, 1;$$

2. $\left\{ \left(x_1, \cdots, x_d\right)' \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1 \right\}$
3. $\mathbf{\Delta}^d$, *and there exists a deflator* $Z(\cdot)$ *for* $\mu(\cdot)$.

. Some interesting Stochastic Analysis is involved here.

Remark: The existence of a deflator is not needed, if $\mu(\cdot)$ has strictly positive components at all times; it is essential, however, when $\mu(\cdot)$ is "allowed to hit a boundary". Preservation of semimartingale property...

FUNCTIONS BASED ON RANK

- "Rank operator" $\mathfrak{R} : \mathbf{\Delta}^d \to \mathbb{W}^d$, where

$$\mathbb{W}^d = \Big\{ \big(x_1, \cdots, x_d\big)' \in \mathbf{\Delta}^d : 1 \geq x_1 \geq x_2 \geq \cdots \geq x_{d-1} \geq x_d \geq 0 \Big\}.$$

- Process of market weights ranked in descending order, namely

$$\boldsymbol{\mu}(\cdot) = \mathfrak{R}(\mu(\cdot)) = \big(\mu_{(1)}(\cdot), \cdots, \mu_{(d)}(\cdot)\big).$$

- Then $\boldsymbol{\mu}(\cdot)$ can be interpreted again as a market model. (However, this new process may not admit a deflator, even when the original one does.)

Theorem

Consider a function $\boldsymbol{G} : \mathbf{supp}\,(\boldsymbol{\mu}) \to \mathbb{R}$, which is regular for the ranked market weights $\boldsymbol{\mu}(\cdot)$. Then the composite $G = \boldsymbol{G} \circ \mathfrak{R}$ is a regular function for the original market weights $\mu(\cdot)$.

. Functionally Generated Strategies (Additive Case)

For a regular function G, consider the trading strategy $\varphi(\cdot)$ with

$$\boxed{\varphi_i(t) = D_i G(\mu(t)) - Q^{\vartheta}(t) + \boldsymbol{C}\,,} \qquad i = 1, \cdots, d, \quad 0 \le t < \infty$$

where $\vartheta(t) := DG(\mu(t))$ and

$$\boldsymbol{C} := G\big(\mu(0)\big) - \sum_{j=1}^{d} \mu_j(0)\, D_j G\big(\mu(0)\big)$$

is the "Defect of Balance" at time $t = 0$.

Definition

We say that this trading strategy $\varphi(\cdot)$ is **additively generated** by the regular function G.

Proposition

The components of the trading strategy $\varphi(\cdot)$ *with*

$$\boxed{\varphi_i(t) = D_i G(\mu(t)) - Q^{\vartheta}(t) + \boldsymbol{C}}$$

from the previous slide, can be written equivalently as

$$\varphi_i(t) = D_i G(\mu(t)) + \Gamma^G(t) + \left(G\big(\mu(t)\big) - \sum_{j=1}^{d} \mu_j(t)\, D_j G\big(\mu(t)\big) \right)$$

for $i = 1, \cdots, d\,;$
and the corresponding value (wealth) process is given by

$$V^{\varphi}(t) = G\big(\mu(t)\big) + \Gamma^G(t)\,, \qquad 0 \le t < \infty.$$

Expressions are completely free of stochastic integrals.

Remark: Not quite a Doob-Meyer decomposition, this

$$V^{\varphi}(t) = G\big(\mu(t)\big) + \Gamma^G(t)\,, \qquad 0 \le t < \infty,$$

but pretty darn close.

Think of it as an
"Additive Regular (resp., Lyapunov) *Decomposition".*

It consists of
(i) a term $G(\mu(t))$ with controlled behavior, that depends on each day t on the prevailing configuration $\mu(t)$ of market weights **and on nothing else;** and of
(ii) an additional "earnings" term, path-dependent and of finite variation (resp., increasing)

$$\Gamma^G(\cdot) = -\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^{\cdot} D^2_{ij}G\big(\mu(t)\big)\,\mathrm{d}\langle\mu_i,\mu_j\rangle(t)\,.$$

OK, we have derived a trading strategy, additively generated from the function G. Its value process is also additively decomposed as

$$V^{\varphi}(T) = G\big(\mu(T)\big) + \Gamma^{G}(T)\,, \qquad 0 \le T < \infty$$

in terms of "value" and "earnings".

. But how about the multiplicative (log-additive) decomposition of the "Master Equation" type

$$\log V^{\psi}(T) = \log G\big(\mu(T)\big) + \int_0^T \frac{\mathrm{d}\Gamma^{G}(t)}{G\big(\mu(t)\big)}$$

with

$$\Gamma^{G}(\cdot) = -\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^{\cdot} D^2_{ij}G\big(\mu(t)\big)\,\mathrm{d}\langle \mu_i, \mu_j\rangle(t)$$

from yesterday? Integrating factor....

. Functionally Generated Strategies (Multiplicative Case)

For a regular function $G > 0$ such that $1/G(\mu(\cdot))$ is locally bounded, consider the integrand in $\mathcal{L}(\mu)$ given as

$$\boldsymbol{\eta}_i(\cdot) := \boldsymbol{\vartheta}_i(\cdot) \times \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^G(t)}{G\big(\mu(t)\big)}\right)$$

$$= D_i G(\mu(\cdot)) \times \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^G(t)}{G\big(\mu(t)\big)}\right)$$

and the trading strategy $\psi(\cdot)$ with components

$$\boxed{\psi_i(\cdot) = \boldsymbol{\eta}(\cdot) - Q^{\eta}(\cdot) + \boldsymbol{C}\,,} \qquad i = 1, \cdots, d$$

and with

$$\boldsymbol{C} = G\big(\mu(0)\big) - \sum_{j=1}^{d} \mu_j(0)\, D_j G\big(\mu(0)\big)\,.$$

Definition

We say that the trading strategy $\psi(\cdot)$ is **multiplicatively generated** by the regular function G. $\square$

Proposition (Fernholz (1999, 2002))

The value process of the strategy $\psi(\cdot)$ is given by

$$V^{\psi}(T) = G\big(\mu(T)\big) \exp\left(\int_0^T \frac{\mathrm{d}\Gamma^G(t)}{G\big(\mu(t)\big)} \right) > 0\,, \qquad 0 \le T < \infty.$$

Remark: **Exactly** the "Master Equation" from yesterday, as

$$\Gamma^G(\cdot) = -\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^{\cdot} D^2_{ij} G\big(\mu(t)\big)\, \mathrm{d}\langle \mu_i, \mu_j \rangle(t)\,.$$

This is an additive regular (resp., Lyapunov) decomposition for the log

$$\log V^{\psi}(T) = \log G\big(\mu(T)\big) + \int_0^T \frac{\mathrm{d}\Gamma^G(t)}{G\big(\mu(t)\big)}\,.$$

Portfolio Weights

The quantities

$$\frac{\psi_i(t)\mu_i(t)}{V^\psi(t)} = \frac{\mu_i(t)}{G(\mu(t))}\left[D_i G(\mu(t)) + G\left(\mu(t)\right) - \sum_{j=1}^{d} \mu_j(t)\, D_j G\left(\mu(t)\right)\right]$$

for $i = 1, \cdots, d$ are the **portfolio weights** of the multiplicatively generated strategy $\psi(\cdot)$. (Please note the aspect of "$G-$modulated delta hedging", adjusted for possible "lack of balance".)

They can be shown to be non-negative, when G is concave.

A GENERAL REMARK: Implementing functionally-generated portfolios in either their additive or multiplicative form, and evaluating their performance relative to the market, *requires no stochastic integration at all.* ("Robust", "Pathwise", "Model-Free", you name it.)

Functionally Generated Relative Arbitrage (Additive Case)

Theorem

Fix a LYAPUNOV *function* $G : \textbf{supp}\,(\mu) \to [0, \infty)$ *with* $G(\mu(0)) = 1$, *and suppose that for some real number* $T_* > 0$ *we have*

$$\mathbb{P}\big(\Gamma^G(T_*) > 1\big) = 1.$$

Then the strategy $\varphi(\cdot)$, *additively generated from* G, *strongly outperforms the market over* **every** *time-horizon* $[0, T]$ *with* $T \geq T_*$.

Proof:

$$V^\varphi(T) = G\big(\mu(T)\big) + \Gamma^G(T) \geq \Gamma^G(T_*) > 1$$

hold w.p.1. □

Functionally Generated Arbitrage (Multiplicative Case)

Theorem

Fix a regular function $G : \mathbf{supp}\,(\mu) \to [0, \infty)$ *satisfying* $G(\mu(0)) = 1$, *and suppose that for some real constants* $T_* > 0$ *and* $\varepsilon > 0$ *we have*

$$\mathbb{P}\big(\Gamma^G(T_*) \geq 1 + \varepsilon\big) = 1\,.$$

Then there exists a constant $c > 0$ *such that the trading strategy* $\psi^{(c)}(\cdot)$, *multiplicatively generated as above by the regular function*

$$G^{(c)} = \frac{G + c}{1 + c}\,,$$

strongly outperforms the market over the time-horizon $[0, T_*]$.

. *If in addition* G *is a* Lyapunov *function, then this holds also over every time-horizon* $[0, T]$ *with* $T \geq T_*$.

Theorem

Fix a regular function $G : \mathbf{supp}\,(\mu) \to [0, \infty)$*, and suppose that there exists a constant* $\eta > 0$ *, such that a.s.*

$$\Gamma^{G}(T) \ \geq \ \eta \, T \, , \qquad 0 \leq T < \infty \, . \tag{1}$$

Then strong relative arbitrage is possible with respect to the market portfolio over any time horizon $[0, T]$ *of sufficiently long, finite duration, namely*

$$T \ > \ T_* \ := \ \frac{G\,(\mu(0))}{\eta} \, .$$

Moral: "Initial market configurations with $G(\mu(0))$ very close to zero, are the most propitious for launching strong relative arbitrage". More about this shortly.

EXAMPLE: ENTROPY FUNCTION

- Consider the (nonnegative) GIBBS/SHANNON *entropy*

$$H(x) = \sum_{j=1}^{d} x_j \log \left(\frac{1}{x_j} \right).$$

- Assuming either that $\mu(\cdot) \in \mathbf{\Delta}_+^d$, or the existence of a deflator $Z(\cdot)$, this H is a LYAPUNOV function with nondecreasing

$$\Gamma^H(\cdot) = \frac{1}{2} \sum_{j=1}^{d} \int_0^{\cdot} \mu_j(t)\, \mathrm{d}\langle \log \mu_j \rangle(t)$$

  the so-called *cumulative excess growth of the market.*

- If for some real constant $\eta > 0$ we have

$$\mathbb{P}\left(\Gamma^H(t) \geq \eta\, t\,,\ \ \forall\ t \geq 0 \right) = 1$$

  then strong relative arbitrage with respect to the market exists over any time-horizon $[0, T]$ with $T > H(\mu(0))\ /\eta$.

Cumulative excess growth of the market

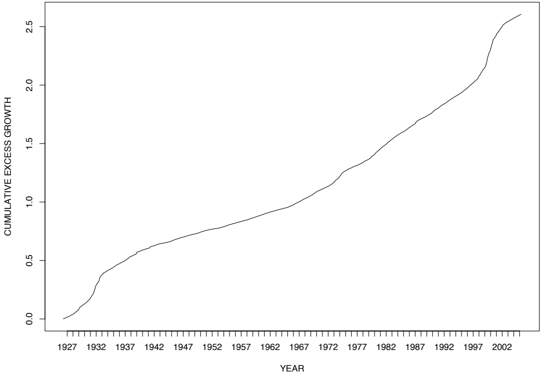

Figure: Cumulative Excess Growth $\Gamma^H(\cdot)$ for the U.S. Equity Market, during the period 1926 –1999.

Sufficient Intrinsic Volatility

Recall:

$$\Gamma^H(\cdot) = \frac{1}{2} \sum_{j=1}^{d} \int_0^{\cdot} \mu_j(t) \, \mathrm{d}\langle \log \mu_j \rangle(t) \, ;$$

$$\mathbb{P}\left(\frac{\mathrm{d}}{\mathrm{d}t} \Gamma^H(t) \geq \eta \, , \ \ \forall \ t \geq 0 \right) = 1.$$

This condition posits that there exists "sufficient intrinsic volatility" in the market, as measured via the average – by capitalization weight – relative local variation rate

$$\sum_{j=1}^{d} \mu_j(t) \, \frac{\mathrm{d}}{\mathrm{d}t} \langle \log \mu_j \rangle(t)$$

of the individual stocks.

- Under this a.s. condition

$$\sum_{j=1}^{d} \mu_j(t) \frac{\mathrm{d}}{\mathrm{d}t} \langle \log \mu_j \rangle(t) \geq \eta, \qquad \forall\ t \geq 0,$$

relative arbitrage with respect to the market is possible over *any* time-horizon $[0, T]$ with

$$T > \frac{H(\mu(0))}{\eta};$$

and can be realized by a unique (additively generated) trading strategy, the same for all such horizons.

An Old Question

In FERNHOLZ & K. (2005) we asked, whether such relative arbitrage is then possible over *arbitrary* time horizons.
It was then shown that the answer is affirmative in a couple of important special cases ("volatility stabilized" markets, and "diverse" strictly non-degenerate markets).

We know now, that the answer to this question is affirmative, if $d = 2$ (two assets); and that the relative arbitrage thus generated is, in fact, strong.

We also know via a host of counterexamples, that already with $d = 3$ (three assets) the answer to this question is, in general, NEGATIVE.

. Under appropriate **additional** conditions, however, the answer turns affirmative again. Let's discuss some of them.

SHORT-TERM RELATIVE ARBITRAGE

Theorem (Support): *Suppose that for some* LYAPUNOV *function* G *and real constant* $\eta > 0$ *we have, not only the non-decrease of the process*

$$\Gamma^G(T) - \eta T, \quad T \in (0, \infty); \tag{2}$$

but also, for some real constant $g \geq 0$ *with*

$$G(\mu(\cdot)) \geq g,$$

the additional "time-homogeneous-support" condition

$$\mathbb{P}\Big(G(\mu(\cdot)) \text{ visits } (g, g+\varepsilon) \text{ during } [0, T] \Big) > 0, \quad \forall (T, \varepsilon) \in (0, \infty)^2.$$

Then relative arbitrage with respect to the market can be realized over ANY time-horizon $[0, T]$ *with* $T \in (0, \infty)$. □

IDEA: If you can arrive *"fast" and with positive probability* at some point in the state-space which is "propitious" for relative arbitrage, then you already have realized short-term relative arbitrage.

However, this relative arbitrage need not be strong.

Corollary (Failure of Diversity): *Suppose that diversity fails for the market with relative weights $\mu(\cdot)$, in the sense that*

$$\mathbb{P}\left(\sup_{t\in[0,T)} \max_{1\le i\le d} \mu_i(t) > 1-\delta \right) > 0\,, \quad \forall\ (T,\delta)\in(0,\infty)\times(0,1).$$

Suppose also that, for some regular function $\boldsymbol{G}:\boldsymbol{\Delta}^d \to [0,\infty)$ with

$$\boldsymbol{G}(\mathfrak{e}_i) = \min_{x\in\boldsymbol{\Delta}^d} \boldsymbol{G}(x) \qquad \textit{for each} \quad i=1,\cdots,d,$$

the condition in (2) holds for some constant $\eta>0$: the process

$$\Gamma^G(T) - \eta\, T\,, \quad T\in(0,\infty)$$

is non-decreasing. Relative arbitrage with respect to the market exists then over every time horizon $[0,T]$ of finite length $T>0$.

THEOREM (Strict Non-Degeneracy): *Suppose that*
(i) the $d-1$ largest eigenvalues of the matrix-valued process

$$\alpha_{ij}(t) := \frac{\mathrm{d}\langle \mu_i, \mu_j \rangle(t)}{\mathrm{d}\big(\sum_k \langle \mu_k \rangle(t) \big)}; \qquad 1 \le i, j \le d\,, \quad 0 \le t < \infty$$

are bounded away from zero, uniformly in (t, ω);
(ii) a deflator exists for the process $\mu(\cdot)$ of relative market weights;
(iii) for some regular function $\boldsymbol{G}$, the process

$$\Gamma^{G}(T) - \eta\, T\,, \quad T \in (0, \infty)$$

is non-decreasing.
Relative arbitrage with respect to the market exists then over every time horizon $[0, T]$ of finite length $T > 0$.

. Some quite interesting Probability Theory goes into this proof: support theorem, growth of stochastic integrals. Once again: no strength.

COUNTEREXAMPLES TO THE 2005 QUESTION

THEOREM: *There exist time-homogeneous* ITÔ *diffusions* $\mu(\cdot)$ *with values in* $\mathbf{\Delta}^3_+$ *and* LIPSCHITZ*–continuous dispersion matrix, for which the cumulative excess growth process*

$$\Gamma^H(\cdot) := \frac{1}{2}\sum_{i=1}^{3}\int_0^{\cdot} \frac{\mathrm{d}\langle \mu_i \rangle(t)}{\mu_i(t)} = \frac{1}{2}\sum_{j=1}^{3}\int_0^{\cdot} \mu_j(t)\,\mathrm{d}\langle \log \mu_j \rangle(t)$$

is strictly increasing, with slope uniformly bounded from below by a strictly positive constant $\eta > 0$.

. *But with respect to which arbitrage over sufficiently short time-horizons* $[0, T]$, *with* $0 < T \leq T_\flat$ *for some real number*

$$T_\flat \in \left(0, \frac{H(\mu(0))}{\eta}\right],$$

is NOT possible. □

A GAP in our Understanding

We know of course that (strong) arbitrage DOES exist, over all time-horizons $[0, T]$ with

$$T > \frac{H(\mu(0))}{\eta}.$$

This leaves a GAP for time-horizons $[0, T]$ with

$$T_\flat < T \leq \frac{H(\mu(0))}{\eta}.$$

We are now trying to understand what happens for such horizons, and hopefully "close the gap".

Sketch of the Argument

Consider a strict concave, smooth function $G : \mathbf{\Delta}^3_+ \to (0, \infty)$, introduce the "cyclical" functions $\sigma_i(x) = D_{i+1}G(x) - D_{i-1}G(x)$ for $i = 1, 2, 3$ and set

$$L(x) := -(1/2)\, \sigma'(x)\, D^2 G(x)\, \sigma(x)\,.$$

If G has a "navel" c, that is, a point with the property

$$D_1 G(c) = D_2 G(c) = D_3 G(c),$$

then this c is also a global maximum. Away from this navel, we start an Itô diffusion $\mu = (\mu_1, \mu_2, \mu_3)$ with dynamics

$$\mathrm{d}\mu_i(t) = \frac{\sigma_i(\mu(t))}{\sqrt{L(\mu(t))}}\, \mathrm{d}W(t)\,, \qquad i = 1, 2, 3\,.$$

Here $W(\cdot)$ is a standard, one-dimensional Brownian motion.

This diffusion lives on the lateral face of the unit simplex, and moves along level curves of the function G at unit speed ($\eta = 1$):

$$G\big(\mu(t)\big) = G\big(\mu(0)\big) - t\,, \qquad \Gamma^G(t) = t\,,$$

(at least) up until the first time $\mathcal{D}$ one of its components vanishes. It follows that

$$G\big(\mu(0)\big) - \mathrm{g} \le \mathcal{D} = G\big(\mu(0)\big) - G\big(\mu(\mathcal{D})\big) \le G\big(\mu(0)\big)\,,$$

$$\mathrm{g} := \sup_{x \in \mathbf{\Delta}^3 \setminus \mathbf{\Delta}^3_+} G(x).$$

The components of this market weight process $\mu(\cdot)$ are martingales, so no arbitrage can exist relative to this market on any time-horizon $[0, T]$ with

$$0 < T \le T_\flat := G\big(\mu(0)\big) - \mathrm{g}\,.$$

Of course, *strong* relative arbitrage IS possible over any time-horizon $[0, T]$ with $T \in \big(G(\mu(0), \infty\big)$.

Thus the gap in question, is the interval

$$\Big(G\big(\mu(0)\big) - \mathrm{g}, G\big(\mu(0)\big)\Big]$$

where

$$\mathrm{g} := \max_{x \in \mathbf{\Delta}^3 \setminus \mathbf{\Delta}^3_+} G(x).$$

For instance, with $G = H$ the entropy function, we have

$$\max_{x \in \mathbf{\Delta}^3} G(x) = 3 \log 3, \qquad \max_{x \in \mathbf{\Delta}^3 \setminus \mathbf{\Delta}^3_+} G(x) = 2 \log 2\,.$$

No such gap exists for concave functions $G : \mathbf{\Delta}^3 \to [0, \infty)$ that are strictly positive in the interior of the simplex and vanish on its boundary; e.g., the geometric mean function.

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THANK YOU FOR YOUR ATTENTION