Trading Strategies Generated by Lyapunov Functions

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Talk at ICERM Workshop, Brown University
June 2017
Back in 1999, Erhard Robert Fernholz introduced a construction that was both
(i) remarkable, and
(ii) remarkably easy to prove.

He showed that for a certain class of so-called “functionally-generated” portfolios, it is possible to express the wealth they generate, discounted by (denominated in terms of) the total market capitalization, solely in terms of the individual companies’ market weights – and to do so in a robust, pathwise, model-free manner, that does not involve stochastic integration.
This fact can be proved by an application of Itô’s rule. Once the result is known, its proof can be assigned as a moderate exercise in a stochastic calculus course.

The discovery paved the way for finding simple, structural conditions on large equity markets – that involve more than one stock, and typically thousands – under which it is possible to outperform the market portfolio (w.p.1).

Put a little differently: conditions under which (strong) arbitrage relative to the market portfolio is possible.

Bob Fernholz showed also how to implement this outperformance by simple portfolios – which can be constructed solely in terms of observable quantities, without any need to estimate parameters of the model or to optimize.
Although well-known, celebrated, and quite easy to prove, Fernholz’s construction has been viewed over the past 18+ years as somewhat “mysterious”.

In this talk, and in the work on which the talk is based, we hope to help make the result a bit more celebrated and perhaps a bit less mysterious, via an interpretation of portfolio-generating functions as Lyapunov functions for the vector process of relative market weights.

We will try to settle then a question about functionally-generated portfolios that has been open for 10 years.
SOME NOTATION

- A probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with a right-continuous filtration \(\mathcal{F}\).
- \(\mathcal{L}(X)\): class of progressively measurable processes, integrable with respect to some given vector semimartingale \(X(\cdot)\).
- \(d \in \mathbb{N}\): number of assets in an equity market, at time zero.
- **Nonnegative** continuous \(\mathbb{P}\)-semimartingales, representing the relative market weights of each asset:

\[
\mu(\cdot) = (\mu_1(\cdot), \cdots, \mu_d(\cdot))'
\]

with \(\mu_1(0) > 0, \cdots, \mu_d(0) > 0\) and taking values in the lateral face of the unit simplex

\[
\Delta^d = \left\{ (x_1, \cdots, x_d)' \in [0, 1]^d : \sum_{i=1}^{d} x_i = 1 \right\}.
\]
Some results below require the notion of a stochastic discount factor ("deflator") for the relative market weight process $\mu(\cdot)$.

A **Deflator** is a continuous, adapted, strictly positive process $Z(\cdot)$ with $Z(0) = 1$, for which

all products $Z(\cdot)\mu_i(\cdot)$, $i = 1, \cdots, d$ are local martingales.

In particular, $Z(\cdot)$ is a local martingale itself.

The existence of such a deflator will be invoked explicitly when needed, and ONLY then.
For any given “number-of-shares” process $\vartheta(\cdot) \in \mathcal{L}(\mu)$, we consider its “value”

$$V^\vartheta(t) = \sum_{i=1}^{d} \vartheta_i(t) \mu_i(t), \quad 0 \leq t < \infty.$$ 

We call such $\vartheta(\cdot)$ a Trading Strategy, if its “defect of self-financibility” is identically equal to zero:

$$Q^\vartheta(T) := V^\vartheta(T) - V^\vartheta(0) - \int_{0}^{T} \langle \vartheta(t), d\mu(t) \rangle \equiv 0, \quad T \geq 0.$$
• If $Q^\vartheta(\cdot) \equiv 0$ fails, then $\varrho(\cdot) \in \mathcal{L}(\mu)$ is not a trading strategy.
• However, for any $C \in \mathbb{R}$, the vector process defined via

$$\varphi_i(\cdot) = \varrho_i(\cdot) - Q^\vartheta(\cdot) + C, \quad i = 1, \cdots, d$$

Is a trading strategy, and its value is given by

$$V^\varphi(\cdot) = V^\varrho(0) + \int_0^\cdot \langle \varrho(t), d\mu(t) \rangle + C.$$
RELATIVE ARBITRAGE

Definition

A trading strategy $\varphi(\cdot)$ outperforms the market (or is relative arbitrage with respect to it) over the time horizon $[0, T]$, if

$$V^\varphi(0) = 1; \quad V^\varphi(\cdot) \geq 0$$

and

$$\mathbb{P}(V^\varphi(T) \geq 1) = 1; \quad \mathbb{P}(V^\varphi(T) > 1) > 0.$$ 

• We say that this relative arbitrage is strong, if

$$\mathbb{P}(V^\varphi(T) > 1) = 1.$$


**REGULAR FUNCTIONS**

**Definition**
A continuous function \( G : \text{supp} (\mu) \to \mathbb{R} \) is said to be **Regular** for the process \( \mu(\cdot) \), if:

1. There exists a measurable function

\[
DG = (D_1 G, \cdots, D_d G)' : \text{supp} (\mu) \to \mathbb{R}^d
\]

such that the “generalized gradient” process \( \vartheta(\cdot) \) with

\[
\vartheta_i(\cdot) = D_i G (\mu(\cdot)), \quad i = 1, \cdots, d
\]

belongs to \( \mathcal{L}(\mu) \).

2. The continuous, adapted process \( \Gamma^G(\cdot) \) below has finite variation on compact intervals:

\[
\Gamma^G(T) := G(\mu(0)) - G(\mu(T)) + \int_0^T \langle \vartheta(t), d\mu(t) \rangle, \quad 0 \leq T < \infty.
\]
Lyapunov Functions

Definition
We say that a regular function $G$ is a Lyapunov function for the process $\mu(\cdot)$, if the finite-variation process

$$\Gamma^G(\cdot) = G(\mu(0)) - G(\mu(\cdot)) + \int_0^\cdot \langle DG(\mu(t)), d\mu(t) \rangle$$

is actually non-decreasing.

Definition
We say that a regular function $G$ is Balanced for $\mu(\cdot)$, if

$$G(\mu(t)) = \sum_{j=1}^d \mu_j(t) D_j G(\mu(t)) , \quad 0 \leq t < \infty.$$ 

The geometric mean $M(x) = (x_1 \cdots x_n)^{1/n}$ is an example.
Remark: On Terminology.
To wrap our minds around this terminology, assume that the vector process \( \vartheta(t) = DG(\mu(t)) \) is locally orthogonal to the random motion of the market weights \( \mu(t) \), in the sense that

\[
\int_0^\cdot \langle \vartheta(t), d\mu(t) \rangle \equiv \int_0^\cdot \langle DG(\mu(t)), d\mu(t) \rangle \equiv 0.
\]

Then the Lyapunov property posits that

\[
G(\mu(t)) = G(\mu(0)) - \Gamma^G(t)
\]

is a decreasing process: the classical definition.

More generally, let us assume that \( Z(\cdot) \) is a deflator, and that \( G \geq 0 \) is a Lyapunov function, for the process \( \mu(t) \). Then \( Z(t)G(\mu(t)) \) is a \( \mathbb{P} \)-supermartingale.
Examples of Regular and Lyapunov functions

Example

If $G$ is of class $C^2$ in a neighborhood of $\Delta^d$, Itô’s formula yields

$$
\Gamma^G(\cdot) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^\cdot \left( - D^2_{ij} G(\mu(t)) \right) d\langle \mu_i, \mu_j \rangle(t)
$$

Therefore, such a function $G$ is regular; if it is also concave, then $G$ becomes a Lyapunov function.

*Significance:* an “aggregate cumulative measure of total variation” for the entire market, with the Hessian ("curvature")

$$
-D^2 G(\mu(t))
$$

acting as the “aggregator” at time $t$. 
Remark: The process $\Gamma^G(\cdot)$:

(i) May, in general, depend on the choice of $DG$; it does NOT, i.e., is uniquely determined, if a deflator $Z(\cdot)$ exists for $\mu(\cdot)$.

(iii) Takes the form of the excess growth rate of the market portfolio, or of “cumulative average relative variation of the market”

$$\Gamma^H(\cdot) = \frac{1}{2} \sum_{j=1}^{d} \int_0^\cdot \mu_j(t) \, d\langle \log \mu_j \rangle(t),$$

when $G = H$ is the Gibbs/Shannon entropy function.

We ran into this quantity several times in yesterday’s talk.
CONCAVE FUNCTIONS ARE LYAPUNOV

Theorem
A continuous function $G : \text{supp} (\mu) \to \mathbb{R}$ is Lyapunov, if it can be extended to a continuous, concave function on the set

1. $\Delta^d_+: = \Delta^d \cap (0, 1)^d$ and
\[
\mathbb{P} (\mu(t) \in \Delta^d_+, \ \forall \ t \geq 0) = 1;
\]

2. \[
\left\{ (x_1, \cdots, x_d)' \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1 \right\}
\]

3. $\Delta^d$, and there exists a deflator $Z(\cdot)$ for $\mu(\cdot)$.

. Some interesting Stochastic Analysis is involved here.

Remark: The existence of a deflator is not needed, if $\mu(\cdot)$ has strictly positive components at all times; it is essential, however, when $\mu(\cdot)$ is “allowed to hit a boundary”. Preservation of semimartingale property...
FUNCTIONS BASED ON RANK

• “Rank operator” \( \mathcal{R} : \Delta^d \rightarrow W^d \), where

\[
W^d = \left\{ (x_1, \cdots, x_d) \in \Delta^d : 1 \geq x_1 \geq x_2 \geq \cdots \geq x_{d-1} \geq x_d \geq 0 \right\}.
\]

• Process of market weights ranked in descending order, namely

\[
\mu(\cdot) = \mathcal{R}(\mu(\cdot)) = (\mu(1)(\cdot), \cdots, \mu(d)(\cdot)).
\]

• Then \( \mu(\cdot) \) can be interpreted again as a market model.
(However, this new process may not admit a deflator, even when
the original one does.)

Theorem
Consider a function \( G : \text{supp}(\mu) \rightarrow \mathbb{R} \), which is regular for the
ranked market weights \( \mu(\cdot) \). Then the composite \( G = G \circ \mathcal{R} \)
is a regular function for the original market weights \( \mu(\cdot) \).
Functionally Generated Strategies (Additive Case)

For a regular function $G$, consider the trading strategy $\varphi(\cdot)$ with

$$\varphi_i(t) = D_i G(\mu(t)) - Q^\vartheta(t) + C, \quad i = 1, \ldots, d, \quad 0 \leq t < \infty$$

where $\vartheta(t) := DG(\mu(t))$ and

$$C := G(\mu(0)) - \sum_{j=1}^d \mu_j(0) D_j G(\mu(0))$$

is the “Defect of Balance” at time $t = 0$.

**Definition**

We say that this trading strategy $\varphi(\cdot)$ is **additively generated** by the regular function $G$. 
Proposition

The components of the trading strategy $\varphi(\cdot)$ with

$$\varphi_i(t) = D_i G(\mu(t)) - Q^\varphi(t) + C$$

from the previous slide, can be written equivalently as

$$\varphi_i(t) = D_i G(\mu(t)) + \Gamma^G(t) + \left( G(\mu(t)) - \sum_{j=1}^{d} \mu_j(t) D_j G(\mu(t)) \right)$$

for $i = 1, \ldots, d$;

and the corresponding value (wealth) process is given by

$$V^\varphi(t) = G(\mu(t)) + \Gamma^G(t), \quad 0 \leq t < \infty.$$
Remark: Not quite a Doob-Meyer decomposition, this

\[ V^\varphi(t) = G(\mu(t)) + \Gamma^G(t), \quad 0 \leq t < \infty, \]

but pretty darn close.

Think of it as an

“Additive Regular (resp., Lyapunov) Decomposition”.

It consists of

(i) a term \( G(\mu(t)) \) with controlled behavior, that depends on each day \( t \) on the prevailing configuration \( \mu(t) \) of market weights and

on nothing else; and of

(ii) an additional “earnings” term, path-dependent and of finite variation (resp., increasing)

\[ \Gamma^G(\cdot) = -\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_{0}^{\cdot} D_{ij}^2 G(\mu(t)) \, d\langle \mu_i, \mu_j \rangle(t). \]
OK, we have derived a trading strategy, additively generated from the function $G$. Its value process is also additively decomposed as

$$V^\varphi(T) = G(\mu(T)) + \Gamma^G(T), \quad 0 \leq T < \infty$$

in terms of “value” and “earnings”.

But how about the multiplicative (log-additive) decomposition of the “Master Equation” type

$$\log V^\psi(T) = \log G(\mu(T)) + \int_0^T \frac{d\Gamma^G(t)}{G(\mu(t))}$$

with

$$\Gamma^G(\cdot) = -\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^T \langle D_{ij}^2 G(\mu(t)) \rangle d\langle \mu_i, \mu_j \rangle(t)$$

from yesterday? Integrating factor....
Functionally Generated Strategies (Multiplicative Case)

For a regular function $G > 0$ such that $1/G(\mu(\cdot))$ is locally bounded, consider the integrand in $\mathcal{L}(\mu)$ given as

$$
\eta_i(\cdot) := \vartheta_i(\cdot) \times \exp\left(\int_0^\cdot \frac{d\Gamma^G(t)}{G(\mu(t))}\right)
$$

$$
= D_i G(\mu(\cdot)) \times \exp\left(\int_0^\cdot \frac{d\Gamma^G(t)}{G(\mu(t))}\right)
$$

and the trading strategy $\psi(\cdot)$ with components

$$
\psi_i(\cdot) = \eta(\cdot) - Q^\eta(\cdot) + C, \quad i = 1, \ldots, d
$$

and with

$$
C = G(\mu(0)) - \sum_{j=1}^d \mu_j(0) D_j G(\mu(0)).
$$
Definition

We say that the trading strategy $\psi(\cdot)$ is **multiplicatively generated** by the regular function $G$.

Proposition (**Fernholz (1999, 2002)**)

The value process of the strategy $\psi(\cdot)$ is given by

$$V^\psi(T) = G(\mu(T)) \exp \left( \int_0^T \frac{d\Gamma^G(t)}{G(\mu(t))} \right) > 0, \quad 0 \leq T < \infty.$$ 

Remark: **Exactly** the “Master Equation” from yesterday, as

$$\Gamma^G(\cdot) = -\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^\cdot D^2_{ij} G(\mu(t)) \, d\langle \mu_i, \mu_j \rangle(t).$$

This is an additive regular (resp., **Lyapunov**) decomposition for the log

$$\log V^\psi(T) = \log G(\mu(T)) + \int_0^T \frac{d\Gamma^G(t)}{G(\mu(t))}.$$
Portfolio Weights

The quantities

\[
\frac{\psi_i(t) \mu_i(t)}{V_\psi(t)} = \frac{\mu_i(t)}{G(\mu(t))} \left[ D_i G(\mu(t)) + G(\mu(t)) - \sum_{j=1}^{d} \mu_j(t) D_j G(\mu(t)) \right]
\]

for \( i = 1, \cdots, d \) are the **portfolio weights** of the multiplicatively generated strategy \( \psi(\cdot) \). (Please note the aspect of “\( G \)–modulated delta hedging”, adjusted for possible “lack of balance”.)

They can be shown to be non-negative, when \( G \) is concave.

A GENERAL REMARK: Implementing functionally-generated portfolios in either their additive or multiplicative form, and evaluating their performance relative to the market, **requires no stochastic integration at all.** (“Robust”, “Pathwise”, “Model-Free”, you name it.)
Theorem
Fix a Lyapunov function $G : \text{supp}(\mu) \to [0, \infty)$ with $G(\mu(0)) = 1$, and suppose that for some real number $T_* > 0$ we have

$$\mathbb{P}(\Gamma^G(T_*) > 1) = 1.$$ 

Then the strategy $\varphi(\cdot)$, additively generated from $G$, strongly out-performs the market over every time-horizon $[0, T]$ with $T \geq T_*$. 

Proof:
$$V^\varphi(T) = G(\mu(T)) + \Gamma^G(T) \geq \Gamma^G(T_*) > 1$$

hold w.p.1.
Functionally Generated Arbitrage (Multiplicative Case)

Theorem
Fix a regular function $G : \text{supp}(\mu) \to [0, \infty)$ satisfying $G(\mu(0)) = 1$, and suppose that for some real constants $T_* > 0$ and $\varepsilon > 0$ we have

$$\mathbb{P}(\Gamma^G(T_*) \geq 1 + \varepsilon) = 1.$$

Then there exists a constant $c > 0$ such that the trading strategy $\psi^{(c)}(\cdot)$, multiplicatively generated as above by the regular function $G^{(c)} = \frac{G + c}{1 + c}$, strongly outperforms the market over the time-horizon $[0, T_*]$.

If in addition $G$ is a Lyapunov function, then this holds also over every time-horizon $[0, T]$ with $T \geq T_*$. 
Theorem

Fix a regular function $G : \text{supp}(\mu) \to [0, \infty)$, and suppose that there exists a constant $\eta > 0$, such that a.s.

$$\Gamma^G(T) \geq \eta T, \quad 0 \leq T < \infty.$$  \hfill (1)

Then strong relative arbitrage is possible with respect to the market portfolio over any time horizon $[0, T]$ of sufficiently long, finite duration, namely

$$T > T_* := \frac{G(\mu(0))}{\eta}.$$ 

Moral: “Initial market configurations with $G(\mu(0))$ very close to zero, are the most propitious for launching strong relative arbitrage”. More about this shortly.
EXAMPLE: ENTROPY FUNCTION

- Consider the (nonnegative) **Gibbs/Shannon entropy**

\[
H(x) = \sum_{j=1}^{d} x_j \log \left( \frac{1}{x_j} \right).
\]

- Assuming either that \(\mu(\cdot) \in \Delta^d_+\), or the existence of a deflator \(Z(\cdot)\), this \(H\) is a **Lyapunov function** with nondecreasing

\[
\Gamma^H(\cdot) = \frac{1}{2} \sum_{j=1}^{d} \int_0^\cdot \mu_j(t) d\langle \log \mu_j \rangle(t)
\]

the so-called **cumulative excess growth of the market**.

- If for some real constant \(\eta > 0\) we have

\[
P\left(\Gamma^H(t) \geq \eta t, \ \forall \ t \geq 0\right) = 1
\]

then strong relative arbitrage with respect to the market exists over any time-horizon \([0, T]\) with \(T > H(\mu(0)) / \eta\).
Cumulative excess growth of the market

Figure: Cumulative Excess Growth $\Gamma^H(\cdot)$ for the U.S. Equity Market, during the period 1926 –1999.
**Sufficient Intrinsic Volatility**

Recall:

\[ \Gamma^H(\cdot) = \frac{1}{2} \sum_{j=1}^{d} \int_{0}^{\cdot} \mu_j(t) d\langle \log \mu_j \rangle(t) ; \]

\[ \mathbb{P} \left( \frac{d}{dt} \Gamma^H(t) \geq \eta, \hspace{1em} \forall \ t \geq 0 \right) = 1. \]

This condition posits that there exists "sufficient intrinsic volatility" in the market, as measured via the average – by capitalization weight – relative local variation rate

\[ \sum_{j=1}^{d} \mu_j(t) \frac{d}{dt} \langle \log \mu_j \rangle(t) \]

of the individual stocks.
• Under this a.s. condition

\[ \sum_{j=1}^{d} \mu_j(t) \frac{d}{dt} \langle \log \mu_j \rangle(t) \geq \eta, \quad \forall \ t \geq 0, \]

relative arbitrage with respect to the market is possible over any time-horizon \([0, T]\) with

\[ T > \frac{H(\mu(0))}{\eta}; \]

and can be realized by a unique (additively generated) trading strategy, the same for all such horizons.
An Old Question

In Fernholz & K. (2005) we asked, whether such relative arbitrage is then possible over arbitrary time horizons. It was then shown that the answer is affirmative in a couple of important special cases ("volatility stabilized" markets, and "diverse" strictly non-degenerate markets).

We know now, that the answer to this question is affirmative, if \( d = 2 \) (two assets); and that the relative arbitrage thus generated is, in fact, strong.

We also know via a host of counterexamples, that already with \( d = 3 \) (three assets) the answer to this question is, in general, NEGATIVE.

. Under appropriate additional conditions, however, the answer turns affirmative again. Let’s discuss some of them.
SHORT-TERM RELATIVE ARBITRAGE

**Theorem (Support):** Suppose that for some Lyapunov function $G$ and real constant $\eta > 0$ we have, not only the non-decrease of the process

$$G^G(T) - \eta T, \quad T \in (0, \infty);$$

(2)

but also, for some real constant $g \geq 0$ with

$$G(\mu(\cdot)) \geq g,$$

the additional “time-homogeneous-support” condition

$$\mathbb{P} \left( G(\mu(\cdot)) \text{ visits } (g, g + \varepsilon) \text{ during } [0, T] \right) > 0, \quad \forall \ (T, \varepsilon) \in (0, \infty)^2.$$

Then relative arbitrage with respect to the market can be realized over ANY time-horizon $[0, T]$ with $T \in (0, \infty)$. \qed
IDEA: If you can arrive “fast” and with positive probability at some point in the state-space which is “propitious” for relative arbitrage, then you already have realized short-term relative arbitrage.

However, this relative arbitrage need not be strong.
Corollary (Failure of Diversity): Suppose that diversity fails for the market with relative weights $\mu(\cdot)$, in the sense that

$$
\mathbb{P}\left( \sup_{t \in [0, T]} \max_{1 \leq i \leq d} \mu_i(t) > 1 - \delta \right) > 0, \quad \forall (T, \delta) \in (0, \infty) \times (0, 1).
$$

Suppose also that, for some regular function $G : \Delta^d \to [0, \infty)$ with

$$
G(e_i) = \min_{x \in \Delta^d} G(x) \quad \text{for each} \quad i = 1, \ldots, d,
$$

the condition in (2) holds for some constant $\eta > 0$: the process

$$
\Gamma^G(T) - \eta T, \quad T \in (0, \infty)
$$

is non-decreasing. Relative arbitrage with respect to the market exists then over every time horizon $[0, T]$ of finite length $T > 0$. 
THEOREM (Strict Non-Degeneracy): Suppose that
(i) the \(d - 1\) largest eigenvalues of the matrix-valued process

\[
\alpha_{ij}(t) := \frac{d \langle \mu_i, \mu_j \rangle(t)}{d(\sum_k \langle \mu_k \rangle(t))}; \quad 1 \leq i, j \leq d, \quad 0 \leq t < \infty
\]

are bounded away from zero, uniformly in \((t, \omega)\);
(ii) a deflator exists for the process \(\mu(\cdot)\) of relative market weights;
(iii) for some regular function \(G\), the process

\[
\Gamma^G(T) - \eta T, \quad T \in (0, \infty)
\]

is non-decreasing.

Relative arbitrage with respect to the market exists then over every time horizon \([0, T]\) of finite length \(T > 0\).

Some quite interesting Probability Theory goes into this proof: support theorem, growth of stochastic integrals. Once again: no strength.
COUNTEREXAMPLES TO THE 2005 QUESTION

THEOREM: There exist time-homogeneous \( \text{Itô} \) diffusions \( \mu(\cdot) \) with values in \( \Delta_+^3 \) and Lipschitz–continuous dispersion matrix, for which the cumulative excess growth process

\[
\Gamma^H(\cdot) := \frac{1}{2} \sum_{i=1}^{3} \int_0^\cdot \frac{\langle \mu_i \rangle(t)}{\mu_i(t)} = \frac{1}{2} \sum_{j=1}^{3} \int_0^\cdot \mu_j(t) d\langle \log \mu_j \rangle(t)
\]

is strictly increasing, with slope uniformly bounded from below by a strictly positive constant \( \eta > 0 \).

But with respect to which arbitrage over sufficiently short time-horizons \([0, T]\), with \(0 < T \leq T_b\) for some real number

\[
T_b \in \left(0, \frac{H(\mu(0))}{\eta}\right]
\]

is NOT possible.
A GAP in our Understanding

We know of course that (strong) arbitrage DOES exist, over all time-horizons \([0, T]\) with

\[ T > \frac{H(\mu(0))}{\eta}. \]

This leaves a GAP for time-horizons \([0, T]\) with

\[ T_b < T \leq \frac{H(\mu(0))}{\eta}. \]

We are now trying to understand what happens for such horizons, and hopefully “close the gap”.
Sketch of the Argument

Consider a strict concave, smooth function $G : \Delta^3_+ \to (0, \infty)$, introduce the “cyclical” functions $\sigma_i(x) = D_{i+1} G(x) - D_{i-1} G(x)$ for $i = 1, 2, 3$ and set

$$L(x) := -(1/2) \sigma'(x) D^2 G(x) \sigma(x).$$

If $G$ has a “navel” $c$, that is, a point with the property

$$D_1 G(c) = D_2 G(c) = D_3 G(c),$$

then this $c$ is also a global maximum. Away from this navel, we start an Itô diffusion $\mu = (\mu_1, \mu_2, \mu_3)$ with dynamics

$$d\mu_i(t) = \frac{\sigma_i(\mu(t))}{\sqrt{L(\mu(t))}} dW(t), \quad i = 1, 2, 3.$$

Here $W(\cdot)$ is a standard, one-dimensional Brownian motion.
This diffusion lives on the lateral face of the unit simplex, and moves along level curves of the function $G$ at unit speed ($\eta = 1$):

$$ G(\mu(t)) = G(\mu(0)) - t, \quad \Gamma^G(t) = t, $$

(at least) up until the first time $D$ one of its components vanishes. It follows that

$$ G(\mu(0)) - g \leq D = G(\mu(0)) - G(\mu(D)) \leq G(\mu(0)), $$

$$ g := \sup_{x \in \Delta^3 \setminus \Delta^3_+} G(x). $$

The components of this market weight process $\mu(\cdot)$ are martingales, so no arbitrage can exist relative to this market on any time-horizon $[0, T]$ with

$$ 0 < T \leq T_b := G(\mu(0)) - g. $$
Of course, strong relative arbitrage IS possible over any time-horizon $[0, T]$ with $T \in (G(\mu(0), \infty)$. Thus the gap in question, is the interval

$$\left( G(\mu(0)) - g, G(\mu(0)) \right]$$

where

$$g := \max_{x \in \Delta^3 \setminus \Delta^3_+} G(x).$$

For instance, with $G = H$ the entropy function, we have

$$\max_{x \in \Delta^3} G(x) = 3 \log 3, \quad \max_{x \in \Delta^3 \setminus \Delta^3_+} G(x) = 2 \log 2.$$

No such gap exists for concave functions $G : \Delta^3 \rightarrow [0, \infty)$ that are strictly positive in the interior of the simplex and vanish on its boundary; e.g., the geometric mean function.


THANK YOU FOR YOUR ATTENTION